What is n-ary associativity?

Carlos Zapata Carratalá

SEMF
Society for Multidisciplinary
and Fundamental Research

Institute for the
Foundations of Computation

What is n-ary associativity?

...or the odd tale of ternary mathematics and the quest for the elusive notion of higher-arity category

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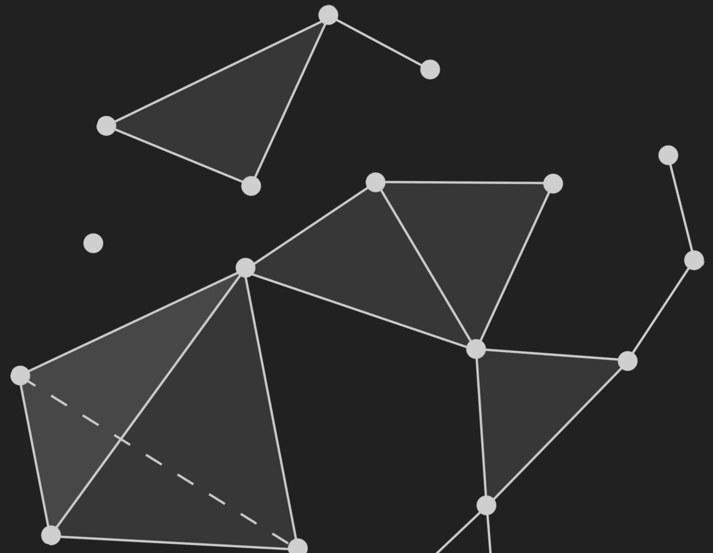

Ternary Algebras

Ternary Algebras

3-Lie Algebras

$(\mathfrak{t}, [\,,\,,\,])$ $\qquad [-,-,-]\colon \Lambda^3 \mathfrak{t} \to \mathfrak{t}$

$$[x,y,[a,b,c]] = [[x,y,a],b,c] + [a,[x,y,b],c] + [a,b,[x,y,c]]$$

3-Lie Functor?

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Lie functor : $(2\text{-})\mathrm{LieAlg} \longrightarrow \mathrm{LieGrp}$

$$(\mathfrak{g}, [\cdot,\cdot]) \longmapsto (G, *) \; : \; T_e G \cong \mathfrak{g}$$

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3-LieAlg $\longrightarrow$?

OPEN PROBLEM!!

Cubic Matrix Algebra

$m_{ijk} \in \mathbb{F}^{N \times M \times L}$

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$$(abc)_{\text{fish}} = \sum_{prq} a_{ijp} \cdot b_{qrp} \cdot c_{qrk}$$

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no other known forms of associativity!!

(Hyper)Graph-Theoretic Approach

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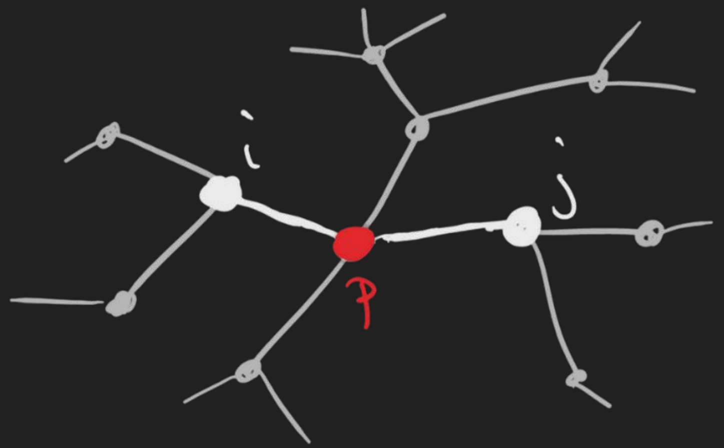
$$\sum_{p} a_{ip} \cdot a_{pj} \sim$$



(Hyper)Graph-Theoretic Approach

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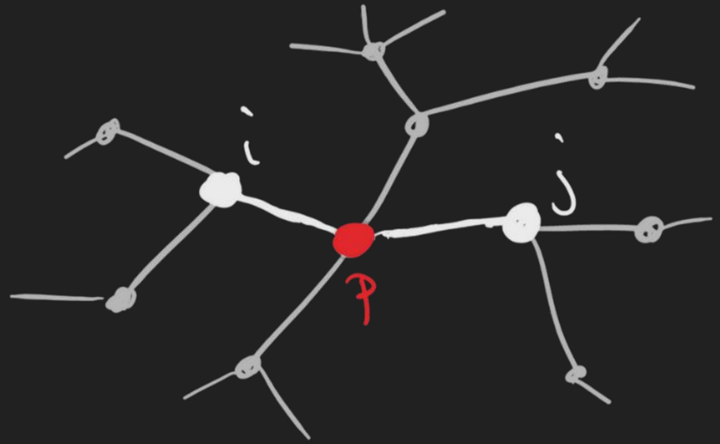

associativity $\sim$ path transitivity

(Hyper)Graph-Theoretic Approach

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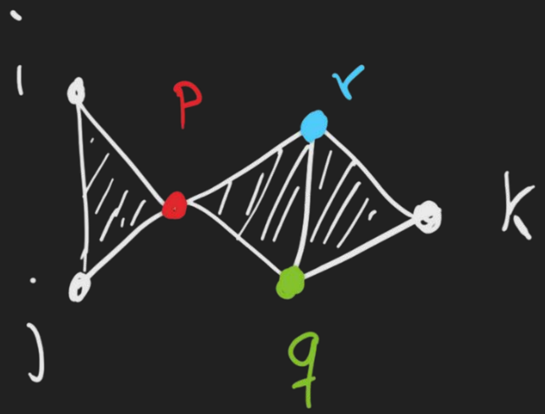
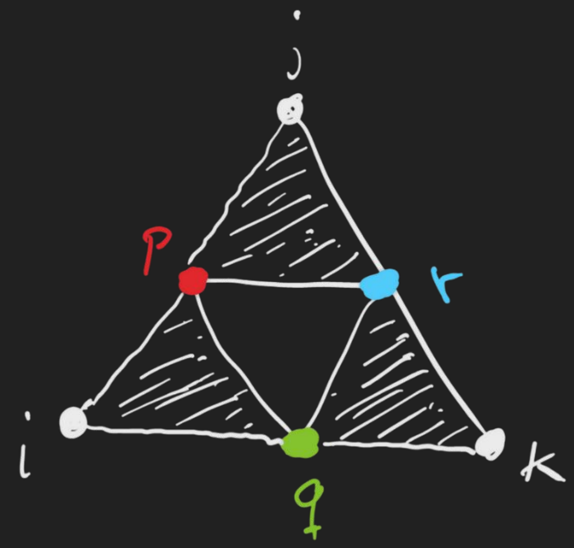


fish

(Hyper)Graph-Theoretic Approach

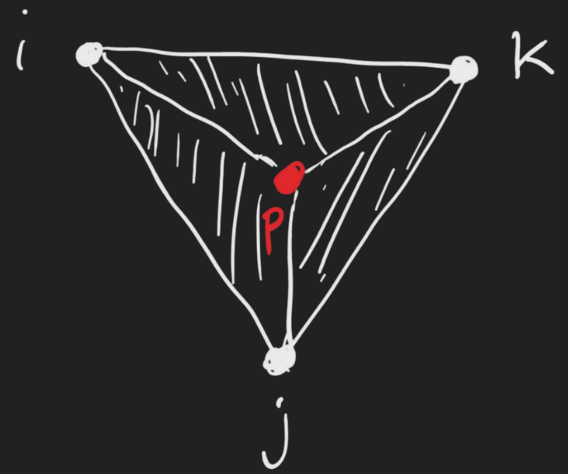
$$\sum_{pqr} a_{ipq} \cdot a_{pjr} \cdot a_{qrk}$$



triforce

(Hyper)Graph-Theoretic Approach

$$\sum_{p} a_{ijp} \cdot a_{ipk} \cdot a_{pjk}$$



Bhattacharya - Mesner

n-ary Categories?

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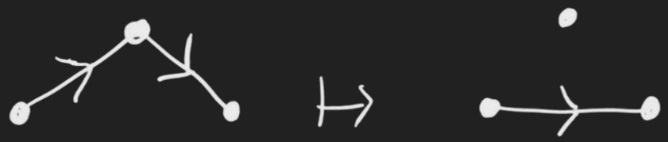

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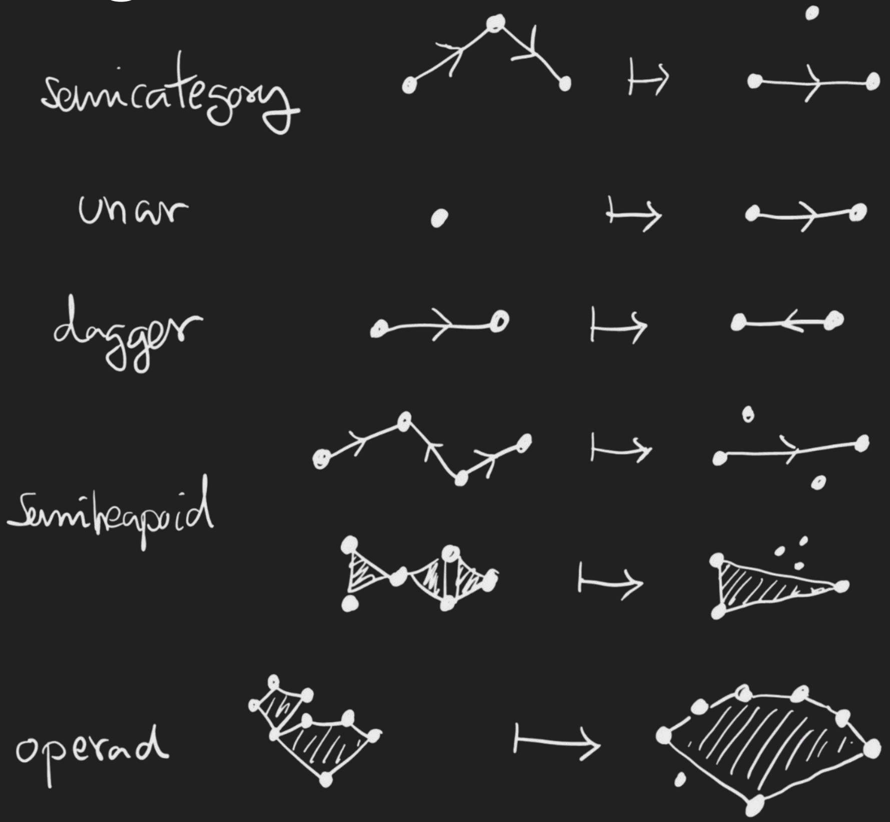

how do we implement axioms?

what is n-ary associativity?

The Principle of Diagrammatic Simplicity

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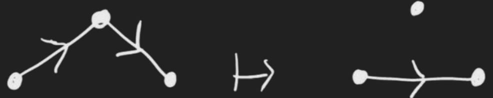
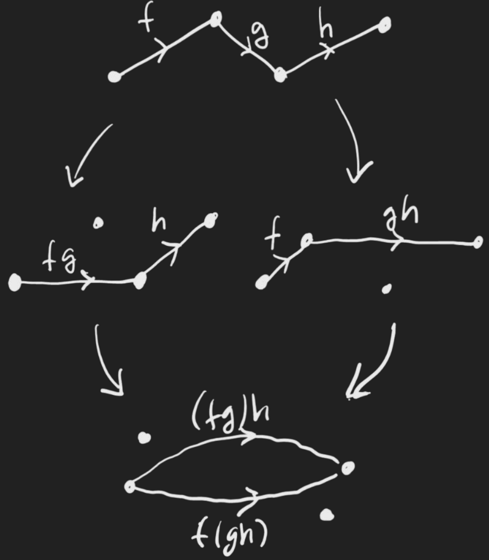
semicategory

The Principle of Diagrammatic Simplicity

semicategory

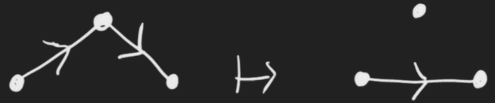




$$(fg)h = f(gh)$$

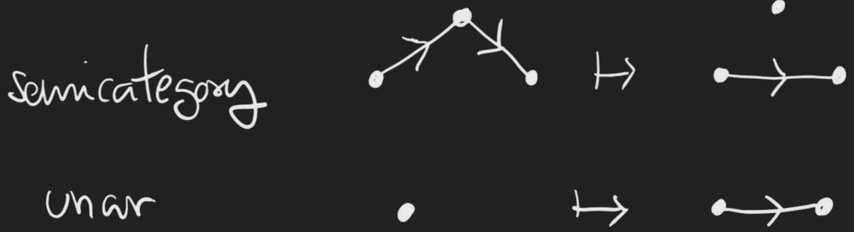
The Principle of Diagrammatic Simplicity

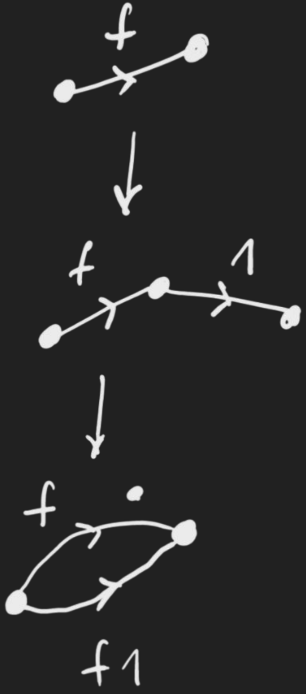
semicategory



unar

The Principle of Diagrammatic Simplicity





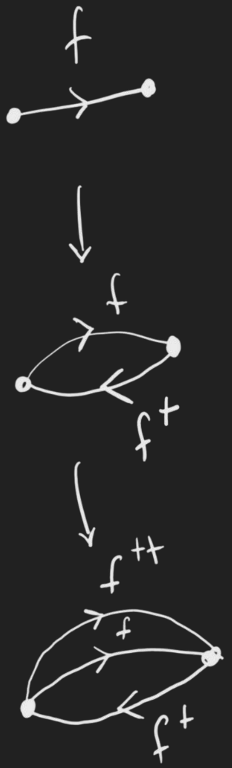
$$f1 = f$$

The Principle of Diagrammatic Simplicity

dagger $\qquad \bullet\!\!\longrightarrow\!\!\bullet \quad \mapsto \quad \bullet\!\!\longleftarrow\!\!\bullet$

The Principle of Diagrammatic Simplicity

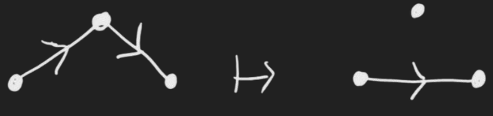
dagger 



$f = f^{\dagger\dagger}$

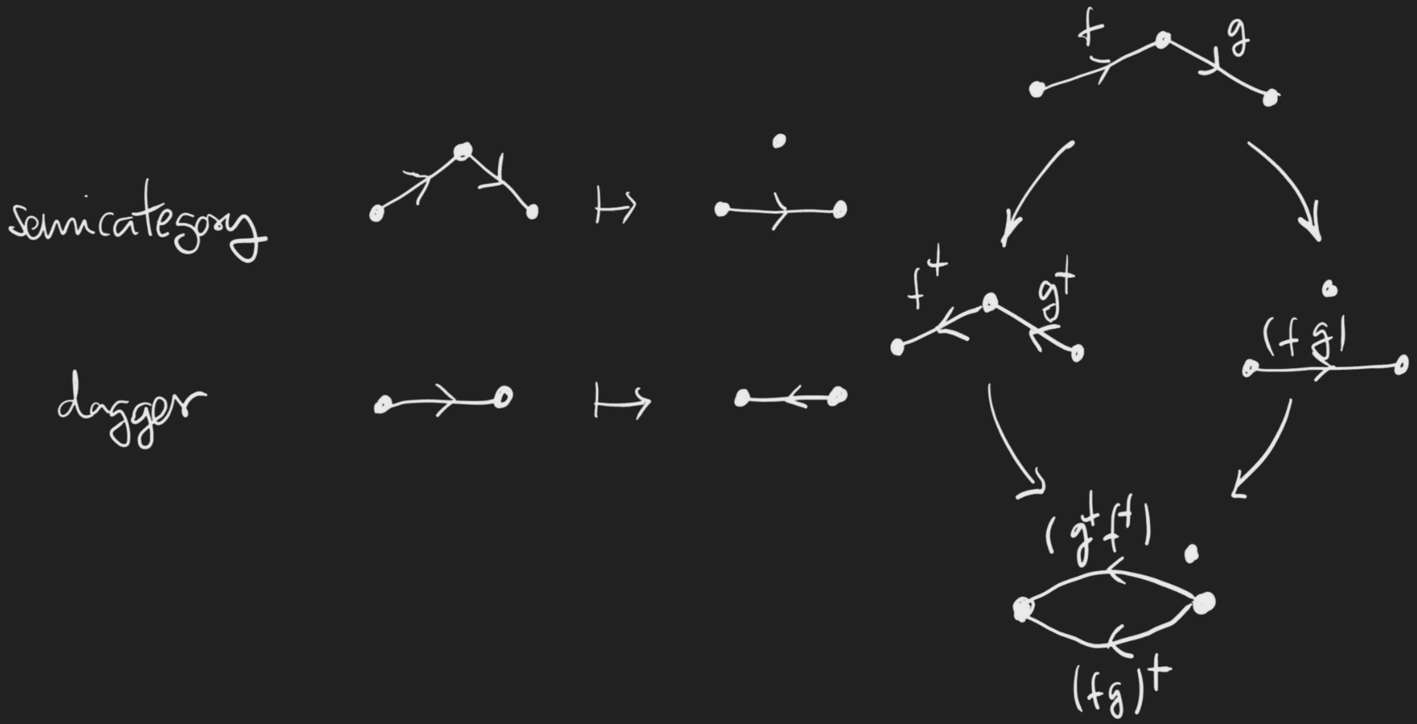
The Principle of Diagrammatic Simplicity

semicategory



dagger

The Principle of Diagrammatic Simplicity



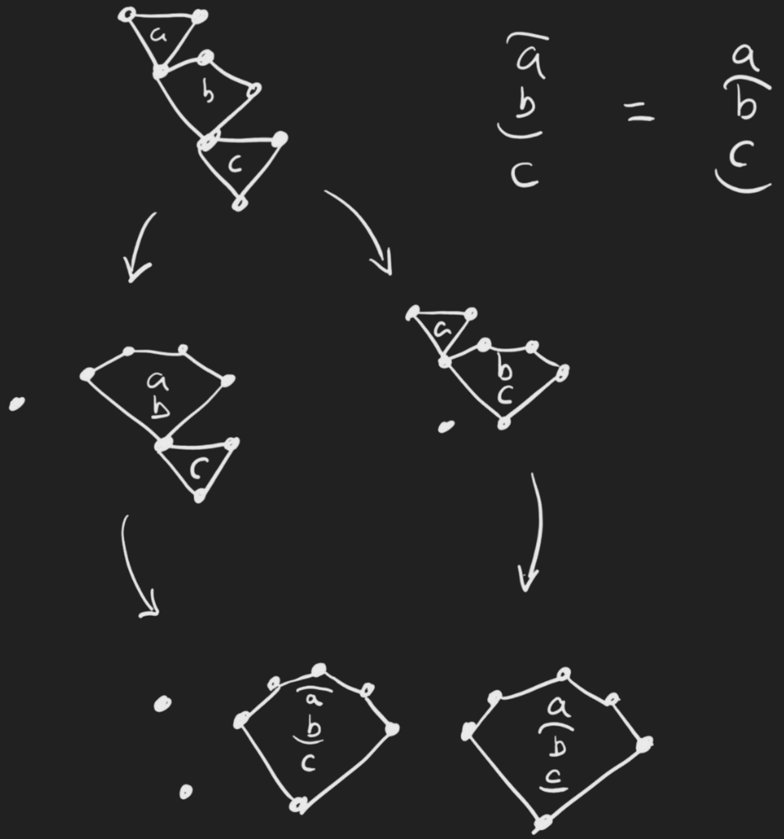
$$(fg)^{\dagger} = g^{\dagger} f^{\dagger}$$

The Principle of Diagrammatic Simplicity

operad

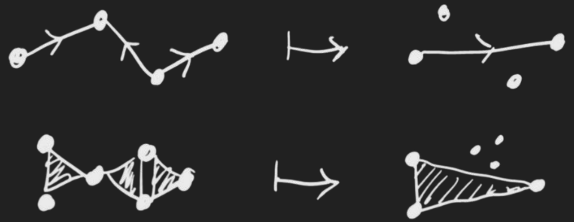
The Principle of Diagrammatic Simplicity

operad



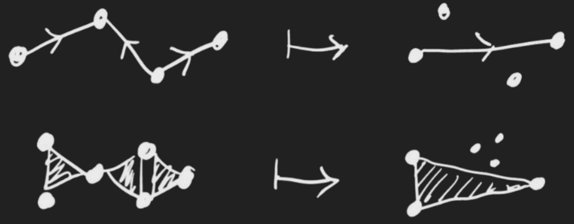
The Principle of Diagrammatic Simplicity

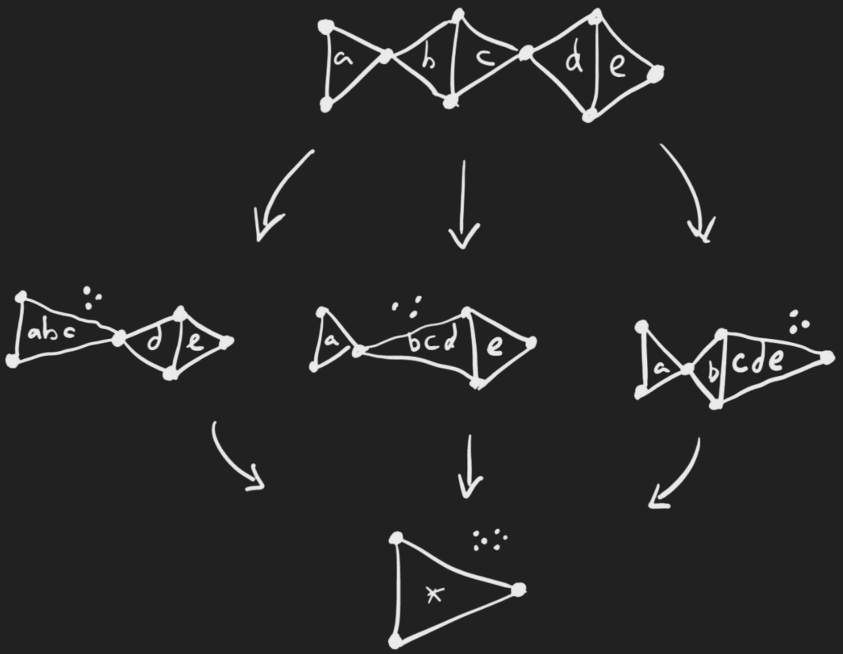
Semitrapoid



The Principle of Diagrammatic Simplicity

Semiheapoid





$(abc)de = a(dcb)e = ab(cde)$

The Principle of Diagrammatic Simplicity

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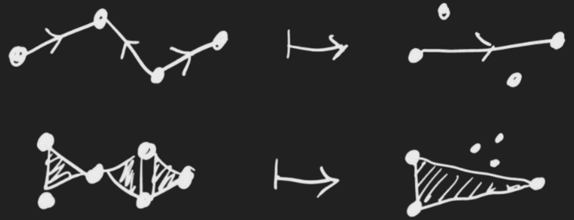

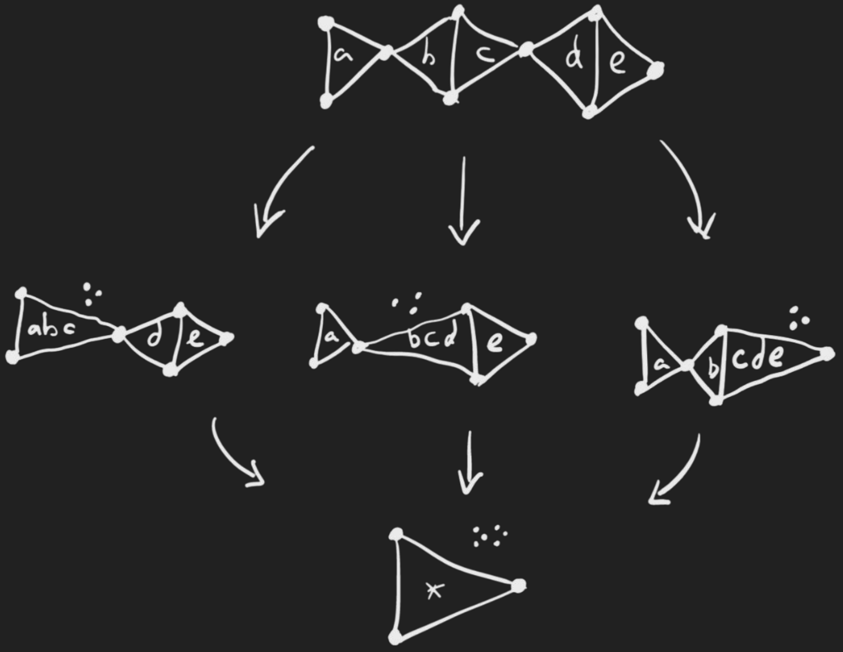

$$(abc)de = a(dcb)e = ab(cde)$$

arXiv > math > arXiv:2205.05456

Mathematics > Rings and Algebras

[Submitted on 1 May 2022]

Heaps of Fish: arrays, generalized associativity and heapoids

Carlos Zapata-Carratala, Xerxes D. Arsiwalla, Taliesin Beynon

Chemoids - a higher-arity generalization of categories

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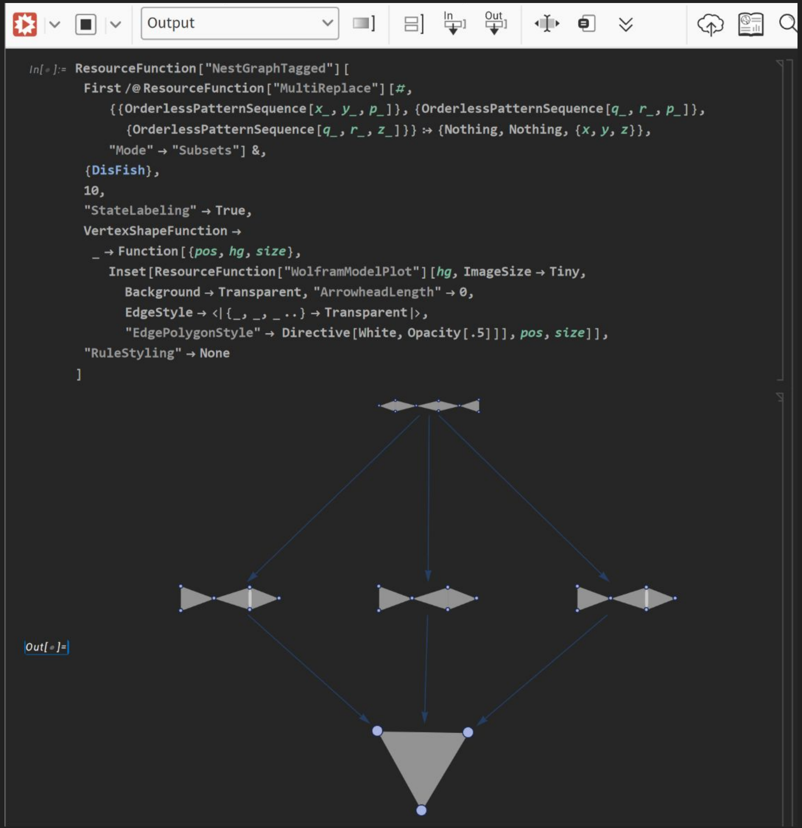
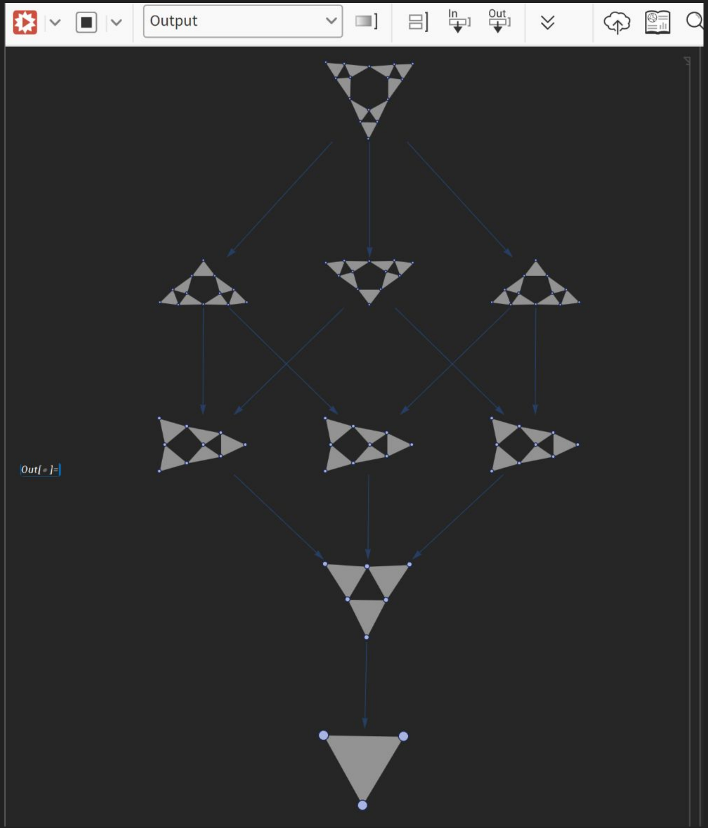
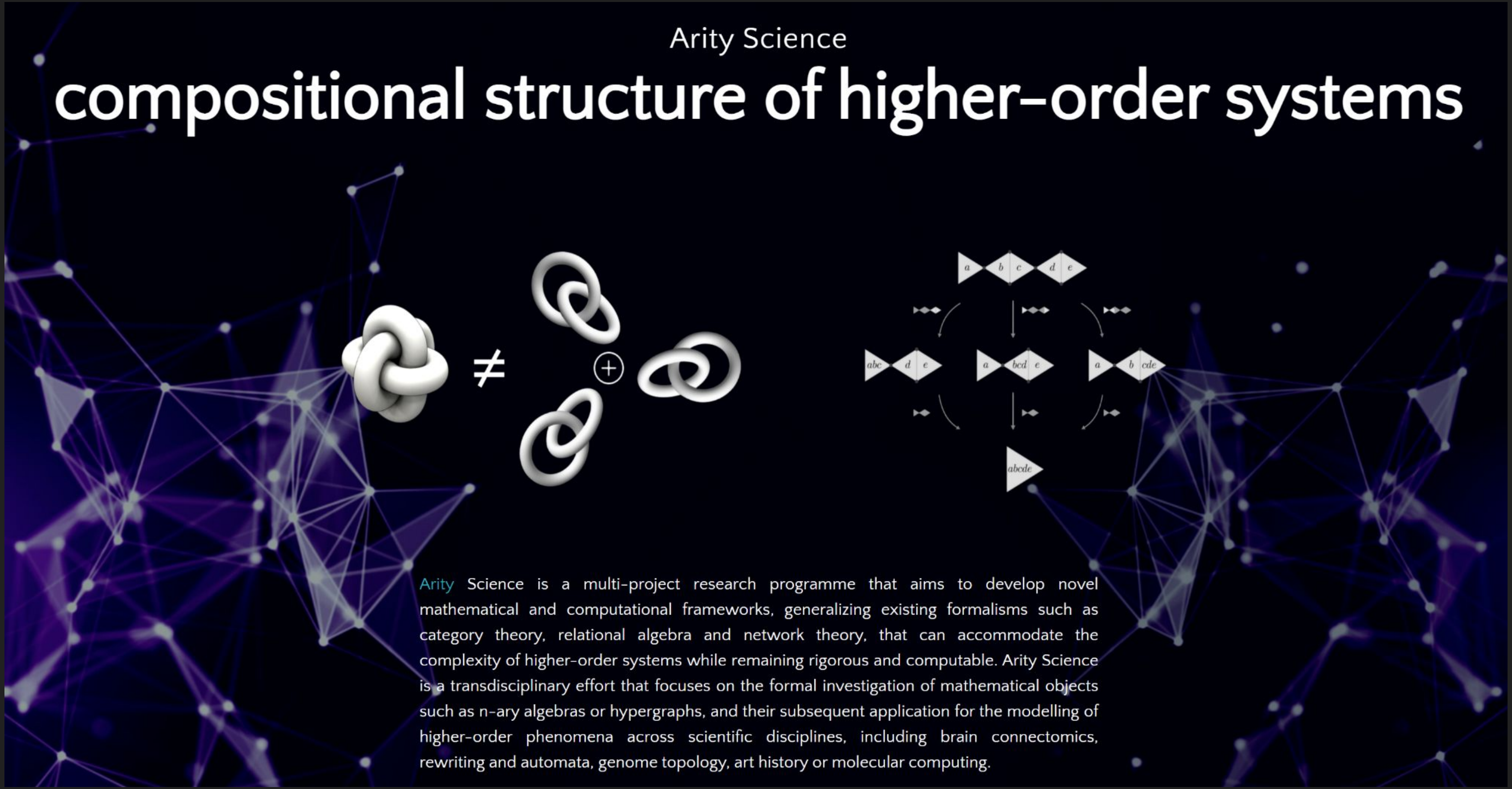
⊕ enforce the Principle of Diagrammatic Simplicity

Chemoids - a higher-arity generalization of categories





https://arity.science/



Get in touch!

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