Foam, Data, Games

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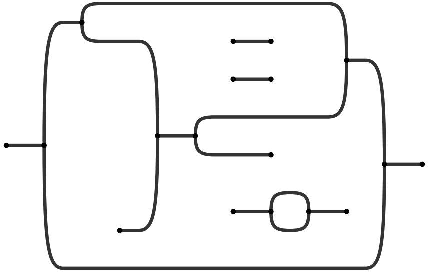
Where are we going?

Claim (Playing with Foam)

The free monoidal category on one object equipped with a frobenius algebra is a category of open two-player games with choice-multiplicity.

Frobenius Foam



A recipe for "open X"; find a monoidal category where the scalars are "X".

The **PRO** `Game`

We can spell out the PRO (no symmetry braidings!) as a finitely-presented $\infty-$category.

The **PRO** `Game`: 0-cells

Just one, which we will call $\circ$.

The **PRO** `Game`: 1-cells

Just one, $\bullet := \circ \to \circ$.

The **PRO** `Game`: 2-cells

(Co)multiplications:

 $:= (\bullet;\bullet) \to \bullet$

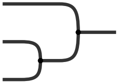
 $:= \bullet \to (\bullet;\bullet)$

(Co)units:

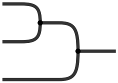
 $:= (\circ = \circ) \to \bullet$

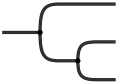
 $:= \bullet \to (\circ = \circ)$

The **PRO** `Game`: 3-cells



$\overset{\text{Ass.}}{=}$





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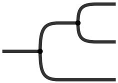
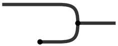
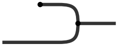
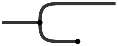
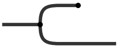
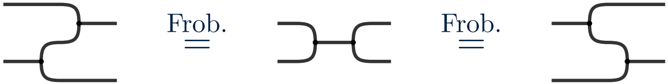


The **PRO** `Game`: 3-cells

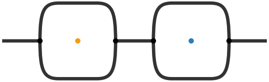
 $\overset{\text{Unit}}{=}$ $\overset{\text{Unit}}{=}$ 

 $\overset{\text{Counit}}{=}$ $\overset{\text{Counit}}{=}$ 

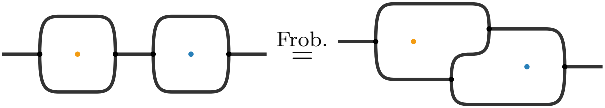
The **PRO** `Game`: 3-cells



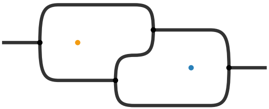
Swapping connected bubbles



Swapping connected bubbles



Swapping connected bubbles



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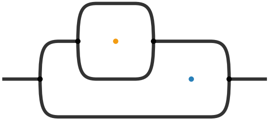
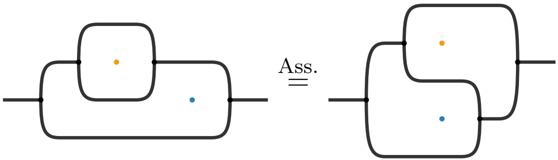
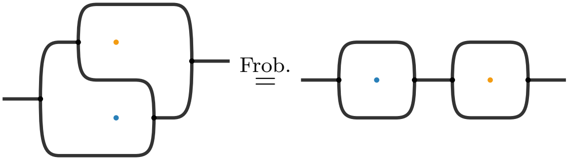


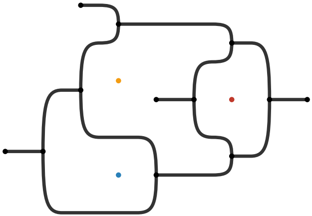
Swapping connected bubbles



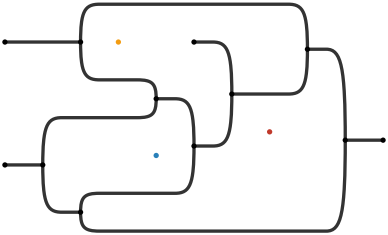
Swapping connected bubbles



Consider the outermost $\otimes$-**non**-separable scalar, which we consider to have K enclosed regions which each may contain some scalars.

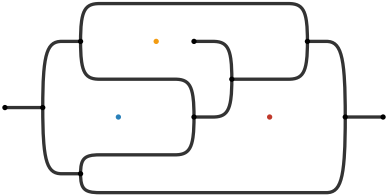


Force all comultiplications to come before all multiplications.



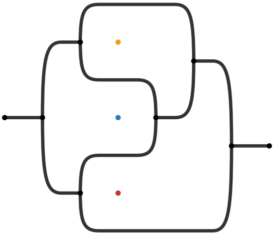
Chain Normal Form

Gather the comultiplications and multiplications such that the outermost scalar starts with a single unit and ends with a single counit.

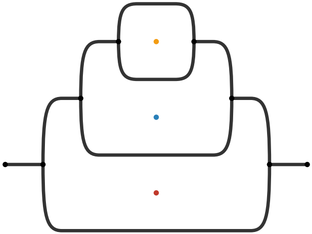


Chain Normal Form

Unit equalities force exactly K multiplications and K comultiplications.

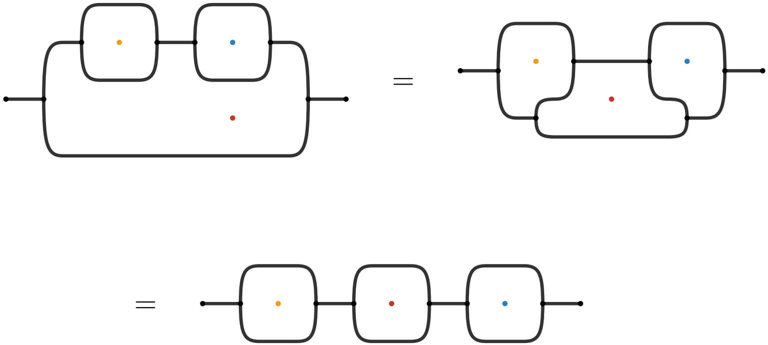


We can then use associativity to get a tower of hanoi.



Chain Normal Form

Which we can reshape into a chain.



- We can swap links in chains: Baglike.
- $\otimes$-collections of scalars: Baglike.

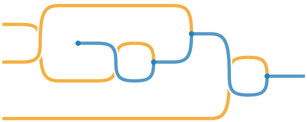
A $\mathbb{F}$oam is either the empty scalar $\circ$, or a bag of bags of foam:

$$\mathbb{F} ::= \circ \mid \texttt{Bag}(\texttt{Bag}(\mathbb{F}))$$

A (finite two-player game with choice-multiplicity) is either the empty game, or a *turn*, where Eloise chooses (from a finite bag of choices) a bag of choices for Abelard, who chooses choices for Eloise, who...

Is there a correspondence between foam and the algebraic data type?

Data structures via PROPs



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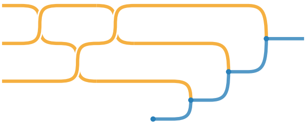


From Backus-Naur to string diagrams

$$\texttt{List}(A) := \texttt{empty} \mid \texttt{Cons}\ A\ \texttt{List}(A) \to \texttt{List}(A)$$

From Backus-Naur to string diagrams

$$\texttt{empty} : 0 \to \texttt{List}(A)$$

$$\texttt{Cons} : A \otimes \texttt{List}(A) \to \texttt{List}(A)$$

From Backus-Naur to string diagrams

`empty` : 

`Cons` : 

From $\texttt{Lists}$ to $\texttt{Bags}$

$$\texttt{List}(A) \stackrel{?}{=} \frac{1}{1-A}$$

$$\texttt{List}(A) \mapsto 1 + A \times \texttt{List}(A)$$

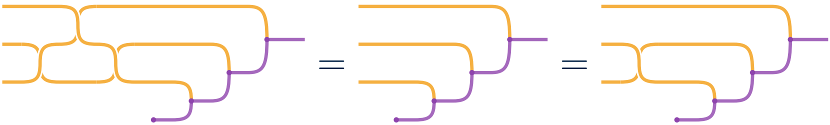
$$\texttt{List}(A) \simeq 1 + A + A^2 + A^3 \cdots \equiv \sum_{i \in \mathbb{N}} A^i$$

$$\texttt{Bag}(A) \stackrel{?}{\simeq} 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \equiv \sum_{i \in \mathbb{N}} \frac{A^i}{i!} := e^A$$

From `Lists` to `Bags`

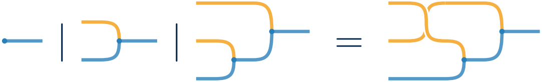
The one difference between a bag and a list is that putting things in bags is order-independent.

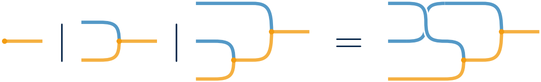
From `Lists` to `Bags`



The denominators of
$1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \equiv \sum_{i \in \mathbb{N}} \frac{A^i}{i!}$ arise from
counting diagrams; we are overcounting braids.

𝔾ames, PROP-algebraically

`Bag(Bag) ::=` 

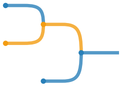
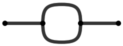
`Bag(Bag) ::=` 

Mutual recursion is easy in PROPs!

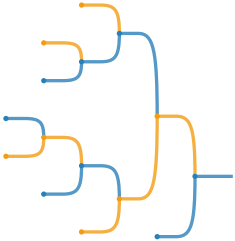
$\mathbb{G}$ames and $\mathbb{F}$oam

 $\leftrightarrow$ $\circ$ (the tensor unit)

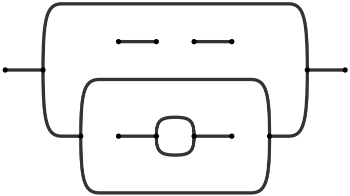
 $\leftrightarrow$ 

 $\leftrightarrow$ 

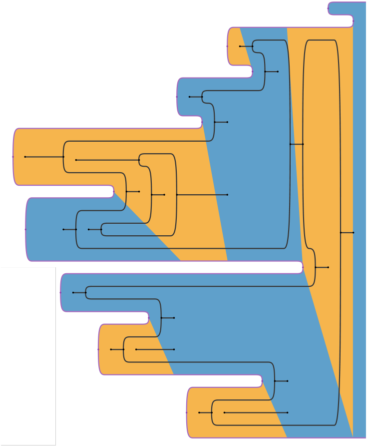
$\mathbb{G}$ames and $\mathbb{F}$oam



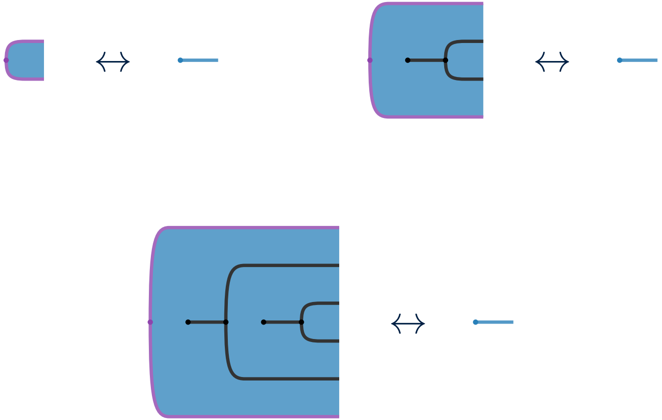
$\leftrightarrow$



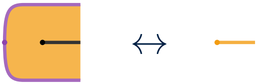
Relating 𝔾ames and 𝔽oam

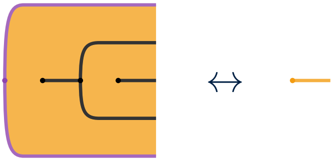


Relating $\mathbb{G}$ames and $\mathbb{F}$oam



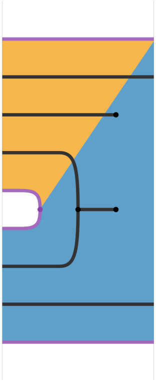
Relating $\mathbb{G}$ames and $\mathbb{F}$oam



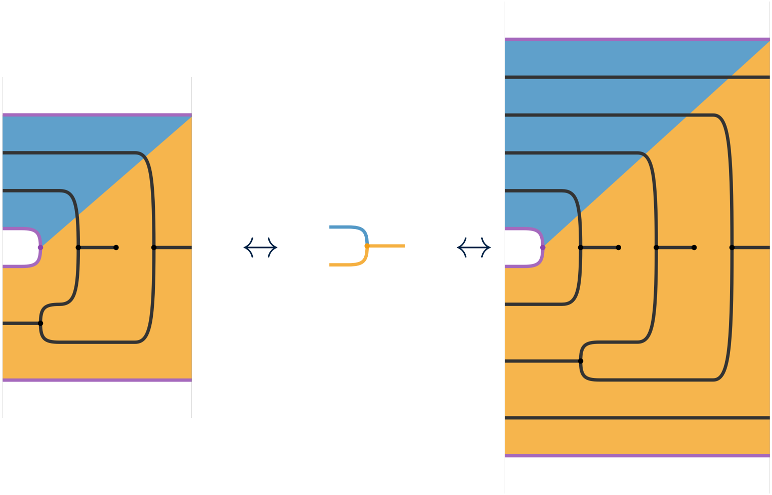


Relating 𝔾ames and 𝔽oam

 $\leftrightarrow$ 

 $\leftrightarrow$ 

Relating $\mathbb{G}$ames and $\mathbb{F}$oam



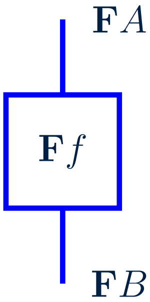
Relating $\mathbb{G}$ames and $\mathbb{F}$oam

What structure relates generators of $\mathbb{G}$ames to context-dependent collections of open $\mathbb{F}$oam?

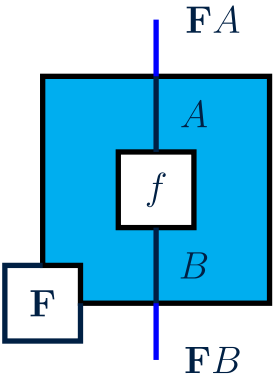
Monoidal Discrete Fibrations

Monoidal Discrete Fibrations

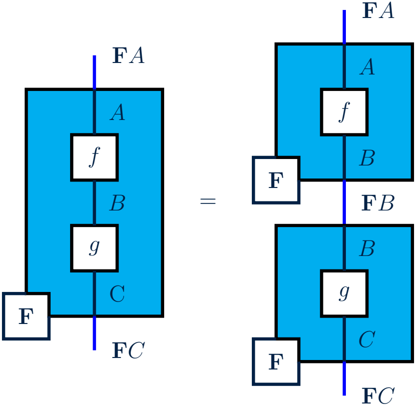
Functor Boxes:



$:=:$

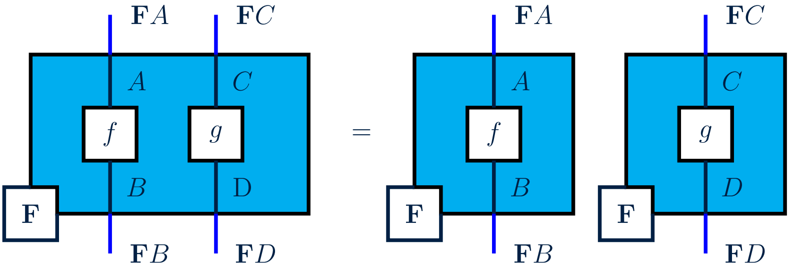


Functor Boxes respect composition and identities



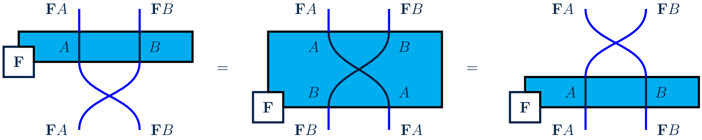
Monoidal Discrete Fibrations

Monoidal Functor Boxes respect parallel composition.

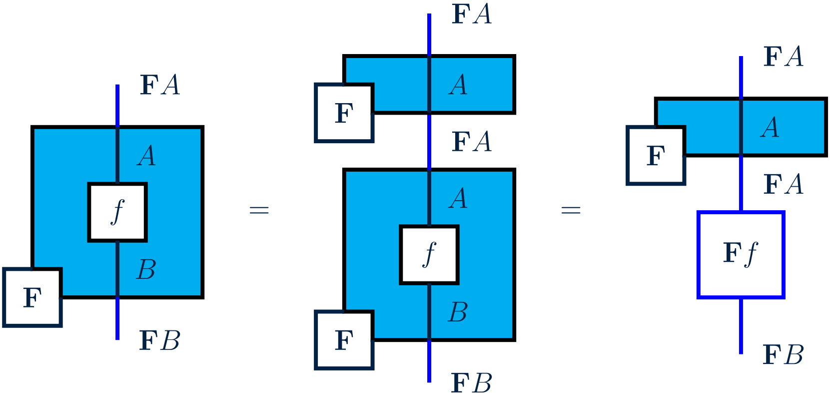


Monoidal Discrete Fibrations

Symmetric Monoidal Functor Boxes also respect braidings.



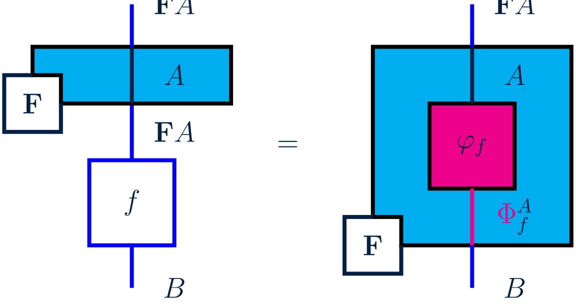
You can always slide the insides of functor boxes out, but in general you can't slide the outsides in.



Monoidal Discrete Fibrations

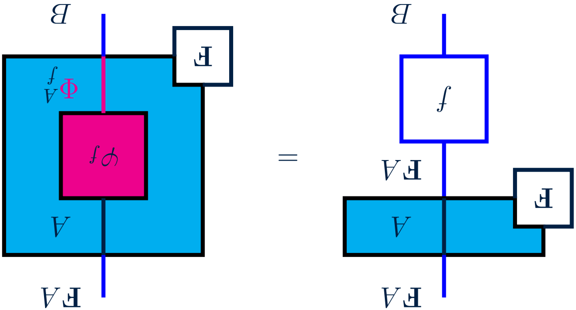
Discrete fibrations are functors that let you slide outsides in from below.

$$\forall f : \mathbf{F}A \to B \in \mathcal{D}$$
$$\exists! \varphi_f : A \to \Phi_f^A \in \mathcal{C}$$



Monoidal Discrete Fibrations

Discrete opfibrations are functors that let you slide outsides in from above.

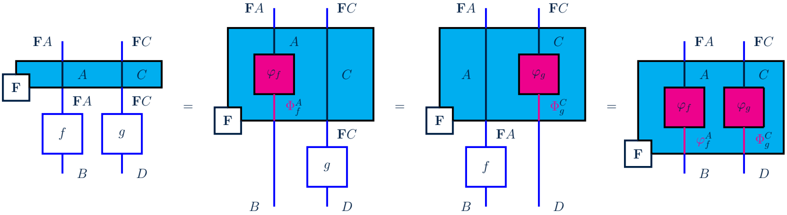


$$\forall f : \mathbf{F}A \to B \in \mathcal{D}$$

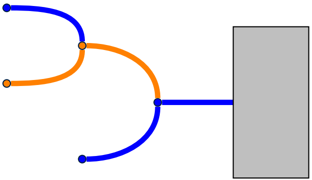
$$\exists! \varphi_f : A \to \Phi^f_A \in \mathcal{C}$$

Monoidal Discrete Fibrations

A Discrete *monoidal* fibration additionally satisfies interchange: the order of sliding-in doesn't matter.



Shapeshifting $\mathbb{G}$ames and $\mathbb{F}$oam using a monoidal discrete opfibration



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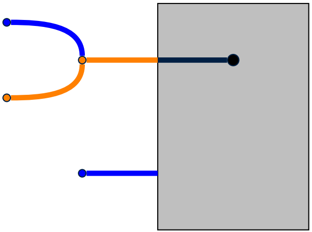

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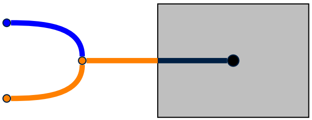
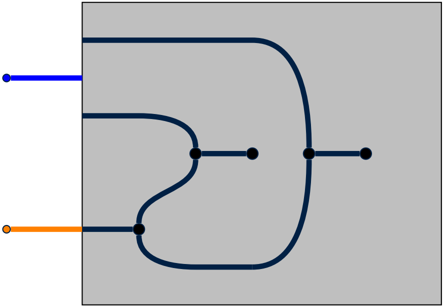


Shapeshifting $\mathbb{G}$ames and $\mathbb{F}$oam using a monoidal discrete opfibration



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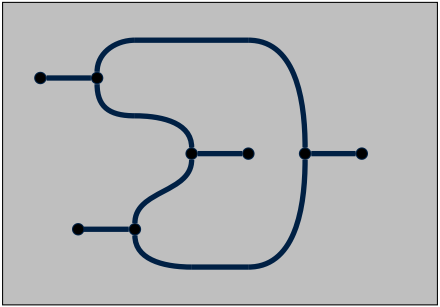

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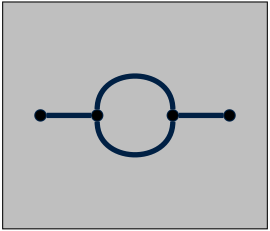

Where did we start again?

Claim (Playing with Foam)

The free monoidal category on one object equipped with a frobenius algebra is a category of open two-player games with choice-multiplicity.

Because the scalars of the category are, up to equality, the domain of a discrete monoidal fibration into a PROP-representation of games as a data structure.

What have we achieved?

Nothing, but we had some fun.