Higher categories for quantum many-body physics

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Modelling quantum many-body systems

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![Brickwork circuits diagram]

We want to analyze emergent properties of these circuits, and relate them to real systems.
Correlations outside the light cone

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This factorizes, so the correlation is trivial.
Correlation inside light cone

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Then applying a space-time symmetry, the previous proof applies. So correlations inside the light cone are now also trivial!
Dual unitary brickwork circuits

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As toy models of many-body quantum systems, they have many cool, unusual properties:

- **Exact solvability.** Single-site correlation functions can be efficiently computed.
- **Maximal entanglement velocity.** Entanglement spreads at fastest possible rate.
- **Maximally chaotic.** Ergodic behaviour with same statistics as random matrix models.
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He calls them *interaction round-a-face*, but we call them *clockwork circuits*. Prosen gives a new definition of *dual unitarity* for these circuits. He then shows they share all the good properties of dual unitary brickwork circuits! This is surprising — their structure is very different. **How can we understand this?**
Shaded tensor networks

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- Vertices are *controlled* by the wires of adjacent regions
Biunitarity

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U^\dagger U = \lambda U^\ast U = \lambda
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It can also be *horizontally unitary*:

\[
U U^\ast = \lambda \quad \text{and} \quad U^\ast U = \lambda
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Biunitarity

In the world of shaded tensor networks, a 4-valent vertex $U$ can be vertically unitary:

$$U^\dagger U U U^\dagger = \lambda$$

It can also be horizontally unitary:

$$U U^* U^* U = \lambda$$

A 4-valent map is biunitary when it is vertically and horizontally unitary.
Homogeneous biunitary circuits

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So brickwork and clockwork circuits have a unified description using the shaded calculus.

This also recovers Prosen’s definition of dual unitarity for clockwork circuits.
Biunitary circuits — dynamical boundary

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![Dynamical boundary diagram]

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At the boundary we require a new sort of vertex, with two shaded and two unshaded regions.
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The simplest possible structure is a \textit{dynamical boundary}:

This boundary moves left-to-right, separating clockwork and brickwork circuits.

At the boundary we require a new sort of vertex, with two shaded and two unshaded regions.

We can classify these. They are exactly \textit{quantum Latin squares}: grids of elements of a Hilbert space, with every row and column giving an orthonormal basis.
Biunitary circuits — boundary creation

We can create these boundaries dynamically:

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At point $P$ we encounter another new vertex type, with one shaded region.

These are known to correspond to *unitary error bases*, defined as orthogonal and complete families of unitary matrices. (For example, the Pauli matrices.)
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Here a clockwork region contracts to zero width, then expands again.

The central point has a vertex with two non-adjacent shaded regions.

These are known to correspond to Hadamard matrices, unitary matrices where every coefficient has the same absolute value. (This was discovered by Vaughan Jones.)