A layout algorithm for higher-dimensional string diagrams

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Homotopy.io is a proof assistant for higher category theory. It lets you build terms in finitely-presented $n$-categories. Terms have a direct geometrical representation.
Example of 2-diagram
Corresponding zigzag diagram
Corresponding zigzag diagram
**Poset** admits a mono-epi factorization system:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{m_f} & & \downarrow{e_f} \\
E_f & & \\
\end{array}
\]

where \( E_f = A \sqcup (B \setminus f[A]) \), \( m_f \) is the canonical inclusion, \( e_f = [f, \text{id}] \).
Injectification

Definition
Given a diagram in a category $\mathcal{C}$,

$$X : J \rightarrow \mathcal{C}$$

an *injectification* is defined to be a diagram

$$\hat{X} : J \rightarrow \mathcal{C}_{\text{mono}}$$

equipped with a pointwise epi natural transformation

\[
\begin{array}{ccc}
J & \xrightarrow{X} & \mathcal{C} \\
\downarrow{\hat{X}} & & \uparrow{\epsilon} \\
\mathcal{C}_{\text{mono}} & \xrightarrow{\epsilon} & \mathcal{C}
\end{array}
\]
Injectification

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- If $J \downarrow i$ is empty, then define
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- If not, $\hat{X}$ and $\epsilon$ must already be defined on $J \downarrow i$. 
Injectification

\[ X_{j_1} \rightarrow X_{j_2} \rightarrow X_{j_3} \rightarrow X_i \]

\[ \hat{X}_{j_1} \rightarrow \hat{X}_{j_2} \rightarrow \hat{X}_{j_3} \rightarrow X_i \]
Injectification

\[ X_j \]

\[ X_j_1 \]

\[ X_j_2 \]

\[ X_j_3 \]

\[ \hat{X}_j_1 \]

\[ \hat{X}_j_2 \]

\[ \hat{X}_j_3 \]

\[ E_j_1 \]

\[ E_j_2 \]

\[ E_j_3 \]

\[ X_i \]
Injectification

\[
\begin{align*}
X_j^1 &\rightarrow X_j^2 &\rightarrow X_j^3 \\
&\downarrow &\downarrow \\
\hat{X}_j^1 &\rightarrow \hat{X}_j^2 &\rightarrow \hat{X}_j^3 \\
&\downarrow &\downarrow \\
&\hat{X}_j &\rightarrow \hat{X}_j^2 &\rightarrow \hat{X}_j^3 \\
&\downarrow &\downarrow &\downarrow \\
&\hat{X}_j &\rightarrow E_j^1 &\rightarrow E_j^2 &\rightarrow E_j^3 \\
&\downarrow &\downarrow &\downarrow &\downarrow \\
&\hat{X}_i &\rightarrow \hat{X}_i := \text{colim } E
\end{align*}
\]
Layout
f_1 \rightarrow f_2 \rightarrow f_3 \rightarrow m_1 \rightarrow f_4 \rightarrow f_5 \rightarrow m_2 \rightarrow f_6
Layout
Layout

\[ f_6 \]

\[ f_4 \rightarrow f_5 \]

\[ f_1 \rightarrow f_2 \rightarrow f_3 \]
Distance constraints

\[ f_2 - f_1 \geq 1 \]
\[ f_3 - f_2 \geq 1 \]
\[ f_3 - f_4 \geq 1 \]
\[ f_5 - f_2 \geq 1 \]
\[ f_5 - f_4 \geq 1 \]
Fair averaging constraints (strict)

\[
\frac{1}{2}(f_1 + f_2) - f_4 = 0 \\
\quad f_3 - f_5 = 0 \\
\frac{1}{2}(f_4 + f_5) - f_6 = 0
\]
Fair averaging constraints (weak)

minimize \( c_1 + c_2 + c_3 \)

subject to

\[ |\frac{1}{2}(f_1 + f_2) - f_4| \leq c_1 \]
\[ |f_3 - f_5| \leq c_2 \]
\[ |\frac{1}{2}(f_4 + f_5) - f_6| \leq c_3 \]
New version of the tool

- Available at beta.homotopy.io
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- TikZ export (finally!)