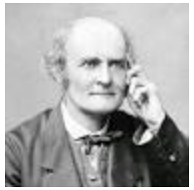
Cayley Monoids

Cayley meets Church



Definition: A Cayley monoid K is a structure (M,*,i,a,b,A,B)
where

(1) (M,*,i) is a monoid

(2) a,b : M

(3) A : M -> M and B : M -> M such that for all x,y,z:M

(i) A(a * B(x)) = A(x)

(ii) A(b * B(x)) = B(x)

(iii) A(i * B(x)) = x

(iv) A(x * y * B(z)) = A(x * B(A(y * B(z)))

Notation: Let K be a Cayley monoid. For each x:M we can define
X : M -> M by X(u) = A(x * B(u)).

Example 1: For any monoid M there is always the trivial Cayley monoid A(x) = B(x) = x, a = b = i. Here X(u) = x*u.

Let O be a subset of M

Definition: The Cayley monoid K is said to have an automonous commutator relative to O if there exists c : M such that whenever x and y are distinct members of O we have

(v)_{O} A(A(c * B(x)) * B(y)) = A(y * B(x)).

Example 1 continued: Let M be the group of quaternions

{p,q,r,-p,-q,-r,i,-i}

We have used the letters 'p','q','r' for the usual 'i','j','k'.

Let O = {p,q,r}.We have the autonomous commutator property for c = -i

Example 2: If g belongs to the group of a monoid M with inverse
g^ then we can set b := g, a := g^ A(x) = a^ * x and B(x) = a * x.
Just as in example 1 we get a Cayley monoid K. K generally does not have an autonomous commutator for K. A concrete example is given
the "almost" isometries .If u,v are vectors in the plane R x R let E(u,v) be the Euclidian distance of v from u. f : R x R -> R x R
is said to be an almost isometry if there exists e in R such that
(1) | E(f(u),f(v)) - E(u,v) | < e, and
(2) For each v there exists u s.t. E(f(u),v) < e.
The almost isometries form a monoid under composition of maps. Now
set b(u) = 2u, a(u)= 1/2 u, B(f) = 2f and A(f) = 1/2 f so we have a Cayley monoid K. But, K has no autonomous commutator.

Example 3: Let M be the monoid of all functions from the set of all non negative reals into itself. As in example 2, let

$g(x) := k + x$ for k a positive integer

and let

$f_{n}(x) := n - e^{-x}$.

We set $O = \{ f_{n} \mid n \text{ a positive integer} \}$. Now c is defined as follows:

Input x

solve $x = n - e^{-y} - k$

solve $y = (m - e^{-z}) + k$

Output

$n - e^{ m - e^{-z} + k } = k$

Definition : In a Cayley monoid K we say that B is an endo if

For all x,y:K we have $B(x*y) = B(x)*B(y)$.

That is, B is a semigroup endomorphism.

Example 4:

The Freyd-Heller monoid has been rediscovered many times
It is the positive part of Thompson's group F,
and can be presented with the infinite set of generators b_{n}
for natural numbers n and the relations

$$b_{n+1} * b_{k} = b_{k} * b_{n}$$

for $k < n$. This is also a presentation of a monoid. Here we
wish to add left inverses a_{n} satisfying $a_{n}*b_{n} = i$
and

$$a_{k} * b_{n+1} = b_{n} * a_{k}$$
$$a_{k} * a_{n+1} = a_{n} * a_{k}$$

for $k < n+1$. This is not yet the group F but another monoid M.

Now define

$$B(b_{n}) = b_{n+1}$$

$$B(a_{n}) = a_{n+1}$$

then B extends to an endomorphism

$$B(x*y) = B(x)*B(y).$$

So, given these relations each element can be written in the normal form

$$B(d) * b^{k} * a^{l}.$$

Proposition: Normal forms are unique.

Corollary 1 : M has a Cayley monoid structure where B is as above and A is defined by

$$A(B(d) * b^{k} * a^{l}) = d.$$

Corollary 2 : In the Freyd-Heller Cayley monoid

$$x * y = A(A(b * B(x)) * B(y))$$

Proof: $A(A(b * B(x)) * B(y)) = A(B(x) * B(y))$

$$= A(B (x * y))$$

$$= x * y.$$

Theorem: Every monoid can be embedded into a Cayley Monoid with an autonomous commutator.

Proof: We embed M into the lambda calculus using the Hindley-Rosen Theorem.