A Type Theory for Strictly Associative $\infty$-Categories

Alex Rice  Eric Finster  Jamie Vicary

SYCO 10
1 Weak Globular Infinity Categories

2 Type Theories for Infinity Categories

3 Strict Associators
Globular sets are one natural shape of higher categories.
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- For each pair of parallel arrows $f, g$, a set of 2-arrows (or 2-cells) from $f$ to $g$. 

\[
\begin{array}{c}
x \overset{\alpha}{\longrightarrow} y \\
\downarrow \hspace{3cm} \downarrow \\
\alpha \hspace{3cm} \alpha
\end{array}
\]

\[
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...
Composition in Globular Sets

Composition of 1 cells

Commutation of 2 cells

Composition along a 1-boundary:

Composition along a 0-boundary:
Composition in Globular Sets

Composition of 1 cells

\[ f \rightarrow g \]

Composition of 2 cells

Composition along a 1-boundary:

\[ \beta \uparrow \]

\[ \alpha \uparrow \]
Composition in Globular Sets

Composition of 1 cells

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \]

Composition of 2 cells

Composition along a 1-boundary:

\[ \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet \]

Composition along a 0-boundary:

\[ \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \]
In strict category theory, we add equalities between certain arrows.

In higher category theory we can instead require that equivalences exist between certain arrows.

Coherence

For a 1-cell \( f: x \to y \), there are unitors \( \lambda_f: \text{id}_x \circ f \to f \) and \( \rho_f: f \circ \text{id}_y \to f \).

\( \lambda_{\text{id}_x} \) and \( \rho_{\text{id}_x} \) are both arrows \( \text{id}_x \circ \text{id}_x \to \text{id}_x \).

These should be equivalent.
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However this is no longer possible at dimensions 3 and higher.
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Therefore, we look for \textit{semistrict} definitions of infinity categories.
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- Associators
- Unitors
- Interchangers
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We can strictify:

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Semistrictness

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CaTT is a type theory for weak infinity categories\textsuperscript{3}.

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There are 4 pieces of syntax, all defined by mutual induction:

- Contexts: Generating data of an infinity category.
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- Types: Source and Target for a term.
- Substitutions: A mapping from variables of one context to terms of another.

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Types in CaTT have 2 constructors.

- The \( \star \) constructor takes no arguments. A term of type \( \star \) represents a 0-cell.
- The \( \text{arrow} \) constructor takes 2 terms and a type as arguments. A term of type \( s \to A \) has source \( s \), target \( t \) and lower dimensional sources and targets given by \( A \).
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\begin{align*}
  x & \xrightarrow{\alpha} y \\
  & \quad \Downarrow \alpha \\
  & \quad \Downarrow f \\
  & \quad \Downarrow g
\end{align*}
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\[
\begin{align*}
  &\xymatrix{
    x 
    \ar[r]_{f} 
    \ar@/^/[rr]^g 
    
    
    & y 
    
    \ar@/^/[rr]^g 
    
    \ar[u]_{\alpha} 
    }

  \alpha : f \rightarrow_{x \rightarrow \star y} g
\end{align*}
\]
Contexts consist of a list of pairs of variable names and types.
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Disc contexts

For each natural number we can define the disc context $D_n$.

\[ D_0 \]

\[ D_1 \]

\[ D_2 \]

\[ D_3 \]

\[ D_2 := x : *, y : *, f : x \to_\ast y, g : x \to_\ast y, \alpha : f \to_{x \to_\ast y} g \]
Composition can be done with the coh constructor.

**coh constructor**

Given:

- A context $\Gamma$ - the shape of the composition,
- A type $A$ in $\Gamma$ - the boundary of the composition,
- A substitution $\sigma : \Gamma \rightarrow \Delta$ - the terms to be composed,

we get a term in $\Delta$:

$$\text{coh} \ (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*.
Suppose we have:

\[ f \rightarrow g \rightarrow h \rightarrow \]

Let \( \Gamma = a \rightarrow b \rightarrow \). \( \Gamma \) is a pasting context. Then:

\[ f \cdot g = \text{coh} (\Gamma : x \rightarrow z)[a \mapsto f, b \mapsto g] \]

\[ (f \cdot g) \cdot h = \text{coh} (\Gamma : x \rightarrow z)[a \mapsto f \cdot g, b \mapsto h] \]
Example composition

Suppose we have:

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \]

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(f \cdot g) \cdot h := \text{coh} (\Gamma : x \to z)[a \mapsto f \cdot g, \quad b \mapsto h]
\]
CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories.
CaTT has a definitional equality based on an operation we call insertion.

\[
\begin{align*}
\text{1-associator} & \\
\xrightarrow{f} x & \xrightarrow{g} y & \xrightarrow{z} z' & \xrightarrow{f'} x' & \xrightarrow{g'} y' & \xrightarrow{z'}
\end{align*}
\]

is sent to:

\[
\begin{align*}
\xrightarrow{f} x & \xrightarrow{f'} x' & \xrightarrow{g'} y' & \xrightarrow{z'}
\end{align*}
\]
Components of insertion

\[ \Delta = x \xrightarrow[\beta \uparrow]{g} y \xrightarrow[k]{z} \]

\[ \Theta = x' \xrightarrow[\beta' \uparrow]{g'} y' \]

Given \( \sigma: \Delta \rightarrow \Gamma \) and \( \tau: \Theta \rightarrow \Gamma \) we get:

\( \sigma \ll \alpha \tau: \Delta \ll \alpha \Theta \rightarrow \Gamma \)
Given $\sigma : \Delta \rightarrow \Gamma$ and $\tau : \Theta \rightarrow \Gamma$ we get:

$$\sigma \ll \alpha \tau : \Delta \ll \alpha \Theta \rightarrow \Gamma$$
Components of insertion

\[ \Delta = x \xrightarrow{\beta \uparrow} g \xrightarrow{\alpha \uparrow} y \xrightarrow{k} z \]

\[ \Theta = x' \xrightarrow{\beta' \uparrow} g' \xrightarrow{\alpha' \uparrow} y' \]

\[ \Delta \ll \alpha \Theta = x' \xrightarrow{\beta' \uparrow} h' \xrightarrow{\alpha' \uparrow} g' \xrightarrow{f'} y' \xrightarrow{k} z \]

\[ \iota : \Theta \rightarrow \Delta \ll \alpha \Theta \]
Components of insertion

\[ \Delta = x \xrightarrow{\alpha} y \xrightarrow{k} z \]

\[ \Theta = x' \xrightarrow{\alpha'} y' \xrightarrow{k} z \]

\[ \Delta \ll \alpha \Theta = x' \xrightarrow{\alpha'} y' \xrightarrow{k} z \]

\[ \iota : \Theta \to \Delta \ll \alpha \Theta \]

\[ \kappa : \Delta \to \Delta \ll \alpha \Theta \]
Components of insertion

\[ \Delta = x \xrightarrow{\alpha} g \xrightarrow{\beta} y \xrightarrow{k} z \]

\[ \Theta = x' \xrightarrow{\alpha'} g' \xrightarrow{\beta'} y' \]

\[ \Delta \ll_{\alpha} \Theta = x' \xrightarrow{\alpha'} g' \xrightarrow{\beta'} y' \xrightarrow{k} z \]

\[ \iota : \Theta \rightarrow \Delta \ll_{\alpha} \Theta \]

\[ \kappa : \Delta \rightarrow \Delta \ll_{\alpha} \Theta \]

Given \( \sigma : \Delta \rightarrow \Gamma \) and \( \tau : \Theta \rightarrow \Gamma \) we get:

\[ \sigma \ll_{\alpha} \tau : \Delta \ll_{\alpha} \Theta \rightarrow \Gamma \]
Insertion also satisfies a *universal property*. Suppose we have \( \text{coh} (\Delta : A)[\sigma] \) where \( \sigma(\alpha) = \text{coh} (\Theta : B)[\tau] \).
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\[
\begin{array}{c}
D_n \xrightarrow{\alpha} \Delta \\
\Theta \xrightarrow{\iota} \Delta \triangleleft \alpha \Theta \\
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\end{array}
\]
Properties of Insertion

Insertion generates a reduction relation for $\text{Catt}_{sa}$:

$$\text{coh} \left( \Delta : A \right)[\sigma] \rightsquigarrow \text{coh} \left( \Delta \ll_{\alpha} \Theta : A[\kappa] \right)[\sigma \ll_{\alpha} \tau]$$

where $\sigma(\alpha) = \text{coh} \left( \Delta : B \right)[\tau]$. 
Insertion generates a reduction relation for $\text{Catt}_{sa}$:

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$$

where $\sigma(\alpha) = \text{coh } (\Delta : B)[\tau]$.

This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence
Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak $\omega$-Categories.* 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
