Locally Cubical Gray Categories

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Overview

Gray defined a monoidal product for 2-categories that is left adjoint to the internal hom,

\[- \otimes - : \mathbf{2Cat} \times \mathbf{2Cat} \to \mathbf{2Cat}\]

and used this to give an algebraic presentation of the 3-dimensional structure of 2-categories and their morphisms [Gra74].
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Böhm recently defined a Gray-monoidal product functor for double categories [Böh19].

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Gray defined a monoidal product for 2-categories that is left adjoint to the internal hom,

$$- \otimes - : 2\text{CAT} \times 2\text{CAT} \to 2\text{CAT}$$

and used this to give an algebraic presentation of the 3-dimensional structure of 2-categories and their morphisms [Gra74].

Böhm recently defined a Gray-monoidal product functor for double categories [Böh19].

$$- \otimes - : \text{DblCAT} \times \text{DblCAT} \to \text{DblCAT}$$

We use this to give an algebraic presentation of the 3-dimensional structure of double categories and their morphisms, and consider the one-object case, which endows double categories with Gray-monoidal structure.
Double Categories

A double category [Ehr62] is a category internal to $\mathbb{C}AT$,

\[
\begin{array}{c}
\pi_0 \\
\downarrow \\
\pi_1
\end{array}
\]

$\mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \rightarrow \mathbb{C}_1 \rightarrow \mathbb{C}_0$

...or a 2-dimensional category of cubical shape, with:

- 0-cells or "objects", $A$
- vertical 1-cells or "arrows", $f : A \rightarrow A'$
- horizontal 1-cells or "proarrows", $M : A \leftrightarrow B$
- 2-cells or "squares", $\alpha : M f \triangleright g N$
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Double Categories

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$$\begin{array}{ccc}
\mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 & \xrightarrow{\pi_0} & \mathbb{C}_1 \\
\text{--} & \text{--} & \text{--} \\
\mathbb{C}_1 & \xleftarrow{\pi_1} & \mathbb{C}_0 \\
\end{array}$$

...or a 2-dimensional category of cubical shape, with:

- **0-cells** or “objects”, $A, A'$,
- **vertical 1-cells** or “arrows”, $f : A \to A'$,
- **horizontal 1-cells** or “proarrows”, $M : A \overset{\Rightarrow}{\to} B$,
- **2-cells** or “squares”, $\alpha : M \cdot f \circ g \cdot N$.

$$f \begin{array}{c}
A \\
\downarrow \\
A'
\end{array}$$
Double Categories

A double category [Ehr62] is a category internal to $\text{CAT}$,

$$\begin{array}{ccc}
\mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 & \xrightarrow{- \odot -} & \mathbb{C}_1 \\
\downarrow \pi_0 & & \downarrow \pi_1 \\
\mathbb{C}_1 & \xleftarrow{\text{R}} & \mathbb{C}_0 \xrightarrow{\text{L}} \mathbb{C}_1
\end{array}$$

...or a 2-dimensional category of cubical shape, with:

- **0-cells** or “objects”, $A, A', B$,
- **vertical 1-cells** or “arrows”, $f : A \rightarrow A'$,
- **horizontal 1-cells** or “proarrows”, $M : A \nrightarrow B$,
Double Categories

A double category [Ehr62] is a category internal to $\mathbf{Cat}$,

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\pi_0} & C_1 \\
\xleftarrow{\pi_1} & \circ & \xleftarrow{\pi_1} \\
C_0 & \xrightarrow{\pi_0} & C_1
\end{array}
\]

...or a 2-dimensional category of cubical shape, with:

- **0-cells** or “objects”, $A, A', B, B'$,
- **vertical 1-cells** or “arrows”, $f : A \to A', g : B \to B'$,
- **horizontal 1-cells** or “proarrows”, $M : A \to B, N : A' \to B'$,
- **2-cells** or “squares”, $\alpha : M_f \bowtie g_N$. 

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & B \\
\downarrow{f} & & \downarrow{g} \\
A' & & B'
\end{array}
\]
Composition in Double Categories
Squares compose by pasting in both dimensions [Mye16]:

\[
\begin{align*}
\begin{tikzpicture}
  \node (M) at (0,0) {M};
  \node (P) at (0,-1) {P};
  \node (R) at (-1,-1) {R};
  \node (f) at (0,-0.5) {f};
  \node (g) at (0,-1.5) {g};
  \node (i) at (-1,-0.5) {i};
  \node (j) at (-1,-2) {j};
  \node (α) at (-0.5,-0.75) {α};
  \node (γ) at (-0.5,-1.75) {γ};
  \draw [-stealth] (f) -- (M);
  \draw [-stealth] (g) -- (P);
  \draw [-stealth] (i) -- (R);
  \draw [-stealth] (j) -- (R);
\end{tikzpicture}
\end{align*}
\]

\[
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\begin{tikzpicture}
  \node (M) at (0,0) {M};
  \node (R) at (-1,0) {R};
  \node (P) at (0,-1) {P};
  \node (Q) at (1,-1) {Q};
  \node (M⊙N) at (2,0) {M ⊙ N};
  \node (f) at (0,-0.5) {f};
  \node (g) at (0,-1.5) {g};
  \node (h) at (2,-0.5) {h};
  \node (α) at (0,-0.75) {α};
  \node (β) at (0,-1.75) {β};
  \node (γ) at (1,-0.75) {γ};
  \node (δ) at (1,-1.75) {δ};
  \draw [-stealth] (f) -- (M);
  \draw [-stealth] (g) -- (P);
  \draw [-stealth] (h) -- (M⊙N);
  \draw [-stealth] (f) -- (i);
  \draw [-stealth] (g) -- (j);
\end{tikzpicture}
\end{align*}
\]

By functoriality of \( U \) and \( ⊙ \), these squares are well-defined:

Technically, our double categories are weak, but coherence lets us pretend they are strict. [GP99]
Composition in Double Categories

Squares compose by pasting in both dimensions [Mye16]:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & f \cdot i \\
\downarrow_{\alpha} & & \downarrow_{\alpha \cdot \gamma}
\end{array}
\]

By functoriality of \(U\) and \(\odot\), these squares are well-defined:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & f' \\
\downarrow_{\alpha} & & \downarrow_{\alpha \odot \beta}
\end{array}
\]
Composition in Double Categories

Squares compose by pasting in both dimensions [Mye16]:

\[
\begin{align*}
M & \quad P & \quad f \alpha g \\
& \quad i \gamma j & \quad R \\
& = & & \\
& \quad f \cdot i \alpha \gamma g \cdot j & \quad R \\
& \quad f \cdot i \alpha \gamma g \cdot j & \quad R \\
\end{align*}
\]

By functoriality of \(U\) and \(\circ\), these squares are well-defined:

\[
\begin{align*}
A & \quad M \quad N & \quad f \\
& \quad g & \quad M \quad N \\
& = & & \\
& \quad f \circ \alpha \beta h & \quad P \quad Q \\
\end{align*}
\]

Technically, our double categories are weak, but coherence lets us pretend they are strict. [GP99]
Morphisms of Double Categories

There is a hierarchy of morphisms of double categories:

- functors, $F : \mathcal{C} \to \mathcal{D}$,
- vertical transformations, $\alpha : F \to F'$,
- horizontal transformations, $\gamma : F \swarrow G$,
- cubical modifications, $\mu : \gamma \triangleright\triangleleft \alpha \beta \delta$.

Classically, these are defined either internally or by components \cite{GP99}, but we would like to understand them algebraically.

Gray worked out the algebraic structure for the 3-dimensional category comprising 2-dimensional globular categories and their morphisms \cite{Gra74}.

Using Böhm's Gray tensor product of double categories \cite{Böh19}, we do the same thing in the cubical setting.
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There is a hierarchy of morphisms of double categories:

- Functors, \( F : C \to D \),
- Vertical transformations, \( \alpha : F \to F' \),
- Horizontal transformations, \( \gamma : F \leftrightarrow G \),
- Cubical modifications, \( \mu : \alpha \diamond \beta \).

Classically, these are defined either internally or by components [GP99], but we would like to understand them *algebraically*.
Morphisms of Double Categories

There is a hierarchy of morphisms of double categories:

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- vertical transformations, \( \alpha : F \to F' \),
- horizontal transformations, \( \gamma : F \leftrightarrow G \),
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Morphisms of Double Categories

There is a hierarchy of morphisms of double categories:

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- horizontal transformations, $\gamma : F \leftrightarrow G$,
- cubical modifications, $\mu : \gamma \alpha \Diamond \beta$.

Classically, these are defined either internally or by components [GP99], but we would like to understand them \textit{algebraically}.

Gray worked out the algebraic structure for the 3-dimensional category comprising 2-dimensional globular categories and their morphisms [Gra74].

Using Böhm’s \textit{Gray tensor product of double categories} [Böh19] we do the same thing in the cubical setting.
A locally cubical Gray category $\mathbb{C}$ has

0-cells, $A$,
A locally cubical Gray category $\mathbb{C}$ has

- 0-cells, $A, B$,
- 1-cells, $f : A \to B$, 

A $\mathbb{C}$ $\mathbb{C}$
A locally cubical Gray category $\mathbb{C}$ has

0-cells, $A, B$,

1-cells, $f, f' : A \to B$,

vertical 2-cells, $\alpha : f \to f'$,
A locally cubical Gray category $\mathcal{C}$ has

0-cells, $A, B$,
1-cells, $f, f', g : A \to B$,
vertical 2-cells, $\alpha : f \to f'$,
horizontal 2-cells, $\gamma : f \Rightarrow g$. 
A locally cubical Gray category $\mathcal{C}$ has

0-cells, $A, B,$
1-cells, $f, f', g, g' : A \to B,$
vertical 2-cells, $\alpha : f \to f', \beta : g \to g',$
horizontal 2-cells, $\gamma : f \leftrightarrow g, \delta : f' \leftrightarrow g',$
3-cells, $\varphi : \gamma \Diamond \delta.$
For each pair of 0-cells we have a hom double category $\mathbb{C}(A \to B)$.
For \( m, n \in \mathbb{N} \) with \( m + n \leq 2 \), composing an \((m + 1)\)-cell with 0-cell boundary \( A \to B \) with an \((n + 1)\)-cell with 0-cell boundary \( B \to C \) yields an \((m + n + 1)\)-cell with 0-cell boundary \( A \to C \).
Locally Cubical Gray Categories – principal composition

For $m, n \in \mathbb{N}$ with $m + n \leq 2$, composing an $(m + 1)$-cell with 0-cell boundary $A \to B$ with an $(n + 1)$-cell with 0-cell boundary $B \to C$ yields an $(m + n + 1)$-cell with 0-cell boundary $A \to C$.

We read off the boundaries of composite cells from the projection string diagram of a surface diagram.
Whiskerings

When $m = 0$ or $n = 0$ the principal composition is called *whiskering* $(- \otimes -)$. 
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When \( m = n = 1 \) the principal composition is called \textit{interchange} \( (\chi_{(-,-)}) \).
Interchangers

When $m = n = 1$ the principal composition is called *interchange* ($\chi(-,-)$).

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This variance for homogeneous interchangers is called "oplax", and its opposite "lax".

\[
\begin{array}{c}
\begin{array}{c}
A \\
\gamma \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\beta \quad g \\
g' \quad \beta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\gamma \quad \gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma \otimes g \\
\gamma \otimes g' \\
\gamma \otimes g'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f \otimes \beta \\
f \otimes g \\
f \otimes g' \\
f' \otimes \beta \\
f' \otimes g \\
f' \otimes g'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma \otimes \beta \\
\gamma \otimes g \\
\gamma \otimes g'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\chi(\gamma,\beta)
\end{array}
\end{array}
\]
Interchangers

When \( m = n = 1 \) the principal composition is called *interchange* \( (\chi_{(-,-)}) \).

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Principal Composition Laws

Principal composition is strictly unital and associative, and compatible with local composition in hom double categories. We can “read off” laws from diagrams without critical points. E.g.

\[ \chi_{(\alpha \otimes b, \gamma)} = \chi_{(\alpha, b \otimes \gamma)} \]
Principal Composition Laws

Principal composition is strictly unital and associative, and compatible with local composition in hom double categories.

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E.g.

\[(\varphi \otimes \psi) \otimes b = (\varphi \otimes b) \circ (\psi \otimes b)\]
Principal Composition Laws

Principal composition is strictly unital and associative, and compatible with local composition in hom double categories.

We can “read off” laws from diagrams without critical points.

E.g.

$$\chi_{(\alpha, \beta \cdot \beta')} = (\chi_{(\alpha, \beta)} \cdot U(f' \otimes \beta')) \odot (U(f \otimes \beta) \cdot \chi_{(\alpha, \beta')})$$
Principal Composition Laws

We don’t get a *structure* by composing a 2-cell with a 3-cell because there are no 4-cells.
Principal Composition Laws

We don’t get a *structure* by composing a 2-cell with a 3-cell because there are no 4-cells.

Instead we get the *property* of a naturality equation. We can read these off of diagrams by perturbing them away from critical points [Mor22].
Proposition

There is a locally cubical Gray category where

0-cells are double categories,
1-cells are strict functors,
vertical 2-cells are (lax and/or oplax) vertical transformations,
horizontal 2-cells are (lax and/or oplax) horizontal transformations,
3-cells are cubical modifications.
Locally Cubical Gray Categories of Interest

Proposition
There is a locally cubical Gray category where

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1-cells are strict functors,
vertical 2-cells are (lax and/or oplax) vertical transformations,
horizontal 2-cells are (lax and/or oplax) horizontal transformations,
3-cells are cubical modifications.

Proposition
A (locally globular) Gray category is a locally cubical Gray category with trivial horizontal 2-cells.
(The loop space of) a one-object locally cubical Gray category is a Gray-monoidal double category.
Gray-Monoidal Double Categories

(The loop space of) a one-object locally cubical Gray category is a \textit{Gray-monoidal double category}.

This is essentially Böhm’s “double category analogue of Gray monoids” obtained from the \textit{Gray monoidal product functor for double categories} 

\[- \otimes - : \mathbf{DblCat} \times \mathbf{DblCat} \to \mathbf{DblCat}. \] [Böh19]
(The loop space of) a one-object locally cubical Gray category is a *Gray-monoidal double category*.

This is essentially Böhm’s “double category analogue of Gray monoids” obtained from the *Gray monoidal product functor for double categories* $- \otimes - : \text{DblCat} \times \text{DblCat} \to \text{DblCat}$. [Böh19]

Generating $(m + n)$-cells of $\mathcal{C} \otimes \mathcal{D}$ are ordered pairs of an $m$-cell of $\mathcal{C}$ and an $n$-cell of $\mathcal{D}$. 
Gray-Monoidal Double Categories

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This is essentially Böhm’s “double category analogue of Gray monoids” obtained from the \textit{Gray monoidal product functor for double categories} 

\[- \otimes - : \text{DblCat} \times \text{DblCat} \to \text{DblCat}. \] \cite{Boehm19}

Generating \((m + n)\)-cells of \(\mathbb{C} \otimes \mathbb{D}\) are ordered pairs of an \(m\)-cell of \(\mathbb{C}\) and an \(n\)-cell of \(\mathbb{D}\).

Double category \(\mathbb{C}\) is Gray-monoidal if functors \(\otimes_{\mathbb{C}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}\) and \(I_{\mathbb{C}} : 1 \to \mathbb{C}\) form a monoid.
Braiding

The swap functor \( S_{(\mathbb{C}, \mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C} \) reverses ordered pairs.

\[
\begin{array}{c}
\begin{array}{c}
M \\
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
V \\
M
\end{array}
\end{array}
\quad \overset{S}{\mapsto}
\quad \begin{array}{c}
\begin{array}{c}
V \\
M
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
M \\
V
\end{array}
\end{array}
\end{array}
\]
Braiding

The swap functor $S_{(\mathbb{C},\mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C}$ reverses ordered pairs.

A braiding for a Gray-monoidal double category with invertible interchangers $\mathbb{C}$ is a vertical pseudo transformation $\sigma : (\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}) (\otimes \mathbb{C} \to S_{(\mathbb{C},\mathbb{C})} \cdot \otimes \mathbb{C})$
Braiding

The swap functor $S_{(\mathbb{C}, \mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C}$ reverses ordered pairs.

A braiding for a Gray-monoidal double category with invertible interchangers $\mathbb{C}$ is a vertical pseudo transformation $\sigma : (\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}) \cdot (\otimes \mathbb{C} \to S_{(\mathbb{C}, \mathbb{C})} \cdot \otimes \mathbb{C})$ that is coherent for monoidal composition [KV94; Cra98]
Braiding

The swap functor $S_{(\mathbb{C}, \mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C}$ reverses ordered pairs.

\[ M \quad V \quad \mapsto \quad \Rightarrow \quad \quad V \quad M \]

A braiding for a Gray-monoidal double category with invertible interchangers $\mathbb{C}$ is a vertical pseudo transformation $\sigma : (\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}) (\otimes_{\mathbb{C}} \to S_{(\mathbb{C}, \mathbb{C})} \cdot \otimes_{\mathbb{C}})$ that is coherent for monoidal composition [KV94; Cra98] and Yang-Baxterators [BN96].
A *syllepsis* for a braided Gray-monoidal double category $\mathbb{C}$ is an invertible globular modification $\nu : (\otimes \mathbb{C} \to \otimes \mathbb{C})(\text{id}(\otimes \mathbb{C}) \leftrightarrow \sigma \cdot (S \cdot \sigma))$ relating the unbraiding to a pair of consecutive braidings.
A syllepsis for a braided Gray-monoidal double category $\mathbb{C}$ is an invertible globular modification $\nu : (\otimes_\mathbb{C} \rightarrow \otimes_\mathbb{C}) (\text{id}(\otimes_\mathbb{C}) \leftrightarrow \sigma \cdot (S \cdot \sigma))$ relating the unbraiding to a pair of consecutive braidings that is coherent for monoidal composition [DS97].
Syllepsis

A syllepsis for a braided Gray-monoidal double category \( \mathbb{C} \) is an invertible globular modification \( \nu : (\otimes \mathbb{C} \to \otimes \mathbb{C}) (\text{id}(\otimes \mathbb{C}) \leftrightarrow \sigma \cdot (S \cdots \sigma)) \) relating the unbraiding to a pair of consecutive braidings that is coherent for monoidal composition [DS97]. A syllepsis is a symmetry if it is the unit of an adjoint equivalence \( \sigma \dashv S \cdot \sigma \).
Wrap-Up

The algebra of 3-dimensional Gray categories can be cumbersome, but the geometry is helpful in understanding what is going on.
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Preprint available: Cartesian Gray-Monoidal Double Categories
References


