Quantum quirks, classical contexts
Towards a Bohrification of effect algebras

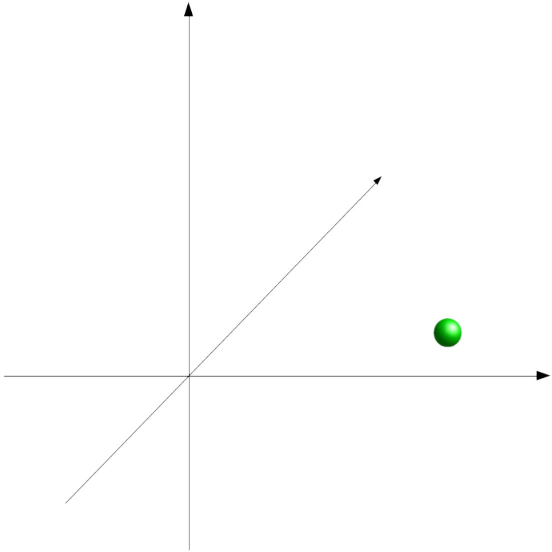
Leo Lobski
University College London
leo.lobski.21@ucl.ac.uk

20 December 2022
Tenth Symposium on Compositional Structures

The structure of a measurement in quantum mechanics

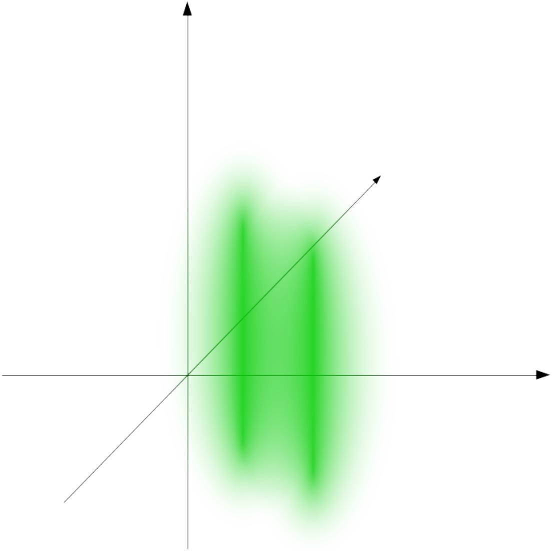
(theorist's perspective)

The structure of a measurement in quantum mechanics
(theorist's perspective)



The structure of a measurement in quantum mechanics

(theorist's perspective)



The structure of a measurement in quantum mechanics
(theorist's perspective)

- A system is modelled by a Hilbert space

The structure of a measurement in quantum mechanics
(theorist's perspective)

- A system is modelled by a Hilbert space
- We have access to the system via measurements only

The structure of a measurement in quantum mechanics
(theorist's perspective)

- A system is modelled by a Hilbert space
- We have access to the system via measurements only
  - A finite *positive operator valued measure* is a finite set $\{A_i\}_{i \in I}$ of positive semi-definite self-adjoint operators such that
$$\sum_{i \in I} A_i = I.$$

The structure of a measurement in quantum mechanics
(theorist's perspective)

- A system is modelled by a Hilbert space
- We have access to the system via measurements only
  - A finite *positive operator valued measure* is a finite set $\{A_i\}_{i\in I}$ of positive semi-definite self-adjoint operators such that
$$\sum_{i\in I} A_i = I.$$
  - Philosophical problem: is the information contained in the measurements sufficient to know the system?

The structure of a measurement in quantum mechanics
(theorist's perspective)

- A system is modelled by a Hilbert space
- We have access to the system via measurements only
  - A finite *positive operator valued measure* is a finite set $\{A_i\}_{i \in I}$ of positive semi-definite self-adjoint operators such that
    $$\sum_{i \in I} A_i = I.$$
  - Philosophical problem: is the information contained in the measurements sufficient to know the system?
- Operational quantum mechanics: replace the Hilbert space with the set of *effects*: physical outcomes which may actually occur

The structure of a measurement in quantum mechanics

A formalisation program

- Heunen, Landsman and Spitters showed that for any C^*-algebra A, the presheaf topos over the poset of commutative subalgebras has an internal *commutative* C^*-algebra, called the *Bohrification* of A:

The structure of a measurement in quantum mechanics

A formalisation program

- Heunen, Landsman and Spitters showed that for any C^*-algebra A, the presheaf topos over the poset of commutative subalgebras has an internal *commutative* C^*-algebra, called the *Bohrification* of A:

$$[\mathcal{C}(A), \mathbf{Set}]$$
$$S \mapsto S$$
$$(S \subseteq T) \mapsto (S \hookrightarrow T)$$

The structure of a measurement in quantum mechanics

A formalisation program

- Heunen, Landsman and Spitters showed that for any C^*-algebra A, the presheaf topos over the poset of commutative subalgebras has an internal *commutative* C^*-algebra, called the *Bohrification* of A:

$$[\mathcal{C}(A), \mathbf{Set}]$$
$$S \mapsto S$$
$$(S \subseteq T) \mapsto (S \hookrightarrow T)$$

- Moreover, $\mathcal{C}(A) \simeq \mathcal{C}(B)$ implies $A \simeq B$ (Jordan isomorphism)

The structure of a measurement in quantum mechanics

A formalisation program

- Heunen, Landsman and Spitters showed that for any C^*-algebra A, the presheaf topos over the poset of commutative subalgebras has an internal *commutative* C^*-algebra, called the *Bohrification* of A:

$$[\mathcal{C}(A), \mathbf{Set}]$$
$$S \mapsto S$$
$$(S \subseteq T) \mapsto (S \hookrightarrow T)$$

- Moreover, $\mathcal{C}(A) \simeq \mathcal{C}(B)$ implies $A \simeq B$ (Jordan isomorphism)
- C^*-algebras come with a lot of analytic structure, and are difficult to motivate from foundational/operational perspective

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1, ', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1, ', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1,', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as

$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1,', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as

$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$

(E3) $a \perp a'$ and $a \oplus a' = 1$, and if $a \perp b$ such that $a \oplus b = 1$, then $b = a'$,

Effect algebras

Definition

An *effect algebra* is a partial algebra $(E, 0, 1, ', \perp, \oplus)$ such that the following hold for all $a, b, c \in E$:

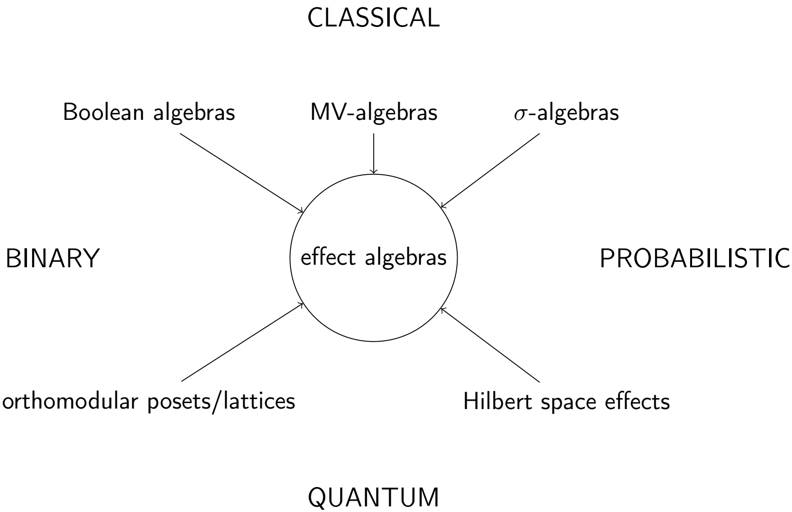
(E1) if $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$,

(E2) if $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$ and $a \perp (b \oplus c)$ as well as
$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$

(E3) $a \perp a'$ and $a \oplus a' = 1$, and if $a \perp b$ such that $a \oplus b = 1$, then $b = a'$,

(E4) if $a \perp 1$, then $a = 0$.

Examples



CLASSICAL
Boolean algebras
MV-algebras
σ-algebras
BINARY
effect algebras
PROBABILISTIC
orthomodular posets/lattices
Hilbert space effects
QUANTUM

Partitions of unity

Let E be an effect algebra. A finite multiset (A, η) such that $A \subseteq E$ is *summable* if the sum

Partitions of unity

Let E be an effect algebra. A finite multiset (A, η) such that $A \subseteq E$ is *summable* if the sum

$$\bigoplus_{a \in A} \eta(a) \cdot a$$

exists (if it exists it is well-defined).

Partitions of unity

Let E be an effect algebra. A finite multiset (A, η) such that $A \subseteq E$ is *summable* if the sum

$$\bigoplus_{a \in A} \eta(a) \cdot a$$

exists (if it exists it is well-defined).

Definition

Let E be an effect algebra. A finite multiset (A, η) such that $A \subseteq E$ is a *partition of unity* if it is summable, $0 \notin A$, and

$$\bigoplus_{a \in A} \eta(a) \cdot a = 1.$$

Partitions of unity

- Part(E) is partially ordered "by refinement":

Partitions of unity

- Part(E) is partially ordered "by refinement":
  - $\mathcal{P} \leq \mathcal{Q}$ if $\mathcal{P}$ can be partitioned into $|\mathcal{Q}|$ parts such that the sum of each such part is a unique (up to the multiplicity) element of $\mathcal{Q}$.

Partitions of unity

- Part(E) is partially ordered "by refinement":
  - $\mathcal{P} \leq \mathcal{Q}$ if $\mathcal{P}$ can be partitioned into $|\mathcal{Q}|$ parts such that the sum of each such part is a unique (up to the multiplicity) element of $\mathcal{Q}$.
- Partitions of unity are in one-to-one correspondence with finite positive operator valued measures (POVMs).
  - The refinement order corresponds to coarse-graining.

The partitions of unity functor

- Partitions of unity extend to a functor $\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$.

The partitions of unity functor

- Partitions of unity extend to a functor $\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$.

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathfrak{C}$ be an isomorphism-closed subclass of objects of $\mathcal{C}$. We say that F is *essentially injective on* $\mathfrak{C}$*-objects* if for any objects $C, B \in \mathfrak{C}$, having $F(C) \simeq F(B)$ implies $C \simeq B$.

The partitions of unity functor

- Partitions of unity extend to a functor $\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$.

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $\mathfrak{C}$ be an isomorphism-closed subclass of objects of $\mathcal{C}$. We say that F is *essentially injective on $\mathfrak{C}$-objects* if for any objects $C, B \in \mathfrak{C}$, having $F(C) \simeq F(B)$ implies $C \simeq B$.

Conjecture

The functor

$$\mathrm{Part} : \mathbf{EAlg} \to \mathbf{Pos}$$

is essentially injective on effect algebras which do not have minimal partitions of unity of cardinality 2 *or less.*

Boolean algebras and orthoalgebras

Theorem
The functor

$$\mathrm{FinSub} : \mathbf{BAlg} \to \mathbf{Pos}$$

is essentially injective on Boolean algebras with more than four elements.

Boolean algebras and orthoalgebras

Theorem

The functor

$$\mathrm{FinSub} : \mathbf{BAlg} \to \mathbf{Pos}$$

is essentially injective on Boolean algebras with more than four elements.

- Finite subalgebra poset of a Boolean algebra is dually isomorphic to its partitions of unity poset.

Boolean algebras and orthoalgebras

Theorem
The functor

$$\mathrm{FinSub} : \mathbf{BAlg} \to \mathbf{Pos}$$

is essentially injective on Boolean algebras with more than four elements.

- Finite subalgebra poset of a Boolean algebra is dually isomorphic to its partitions of unity poset.
- Proved for the poset of all Boolean algebras by Sachs (1961) and independently by Filippov (1965). Grätzer, Koh and Makkai (1972) gave an alternative proof.

Boolean algebras and orthoalgebras

Theorem
The functor

$$\mathrm{FinSub} : \mathbf{BAlg} \to \mathbf{Pos}$$

is essentially injective on Boolean algebras with more than four elements.

- Finite subalgebra poset of a Boolean algebra is dually isomorphic to its partitions of unity poset.
- Proved for the poset of all Boolean algebras by Sachs (1961) and independently by Filippov (1965). Grätzer, Koh and Makkai (1972) gave an alternative proof.

Theorem (Harding, Heunen, Lindenhovius and Navara, 2019)
If A is a proper orthoalgebra, then $\mathrm{BSub}(A)$ *has enough directions and* $\mathrm{Dir}(\mathrm{BSub}(A))$ *is an orthoalgebra isomorphic to A.*

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1, ', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1, ', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

(MV7) $a + a' = 1$,

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1, ', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

(MV7) $a + a' = 1$,

(MV8) $(a' + b)' + b = (a + b')' + a$.

MV-algebras

Definition

An *MV-algebra* is an algebra $(M, 0, 1,', +)$ such that the following axioms hold:

(MV1) $a + (b + c) = (a + b) + c$, *(associativity)*

(MV2) $a + b = b + a$, *(commutativity)*

(MV3) $a + 0 = a$,

(MV4) $a + 1 = 1$,

(MV5) $a'' = a$,

(MV6) $0' = 1$,

(MV7) $a + a' = 1$,

(MV8) $(a' + b)' + b = (a + b')' + a$.

- Models of Łukasiewicz many-valued logics

Examples

- Boolean algebras

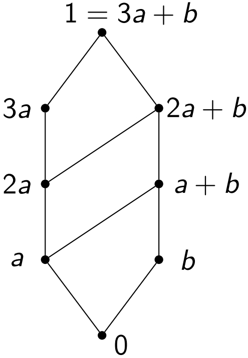
Examples

- Boolean algebras
- The unit interval

Examples

- Boolean algebras
- The unit interval

Example



Partitions determine finite MV-algebras

Theorem (L. 2020)

The functor

$$\mathrm{Part} : \mathbf{FinMV} \to \mathbf{Pos}$$

is essentially injective on algebras with more than four elements.

Conclusion

- The result for Boolean algebras shows that classical measurements are informationally complete.

Conclusion

- The result for Boolean algebras shows that classical measurements are informationally complete.
- The result for MV-algebras can have significance for the theory of MV-algebras, setoids and multisets.

Conclusion

- The result for Boolean algebras shows that classical measurements are informationally complete.
- The result for MV-algebras can have significance for the theory of MV-algebras, setoids and multisets.
- Open problem 1: Show that partitions of unity have enough information to reconstruct an effect algebra.

Conclusion

- The result for Boolean algebras shows that classical measurements are informationally complete.
- The result for MV-algebras can have significance for the theory of MV-algebras, setoids and multisets.
- Open problem 1: Show that partitions of unity have enough information to reconstruct an effect algebra.
- Open problem 2: Can effect algebras be Bohrified?

References

- Paul Busch, Marian Grabowski, and Pekka J. Lahti. *Operational Quantum Physics*. Lecture Notes in Physics. Springer-Verlag, 1995.
- Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. *Bohrification*. Deep Beauty. Ed. by H. Halvorson. Cambridge University Press, 2010.
- Anatolij Dvurečenskij and Sylvia Pulmannová. *New Trends in Quantum Structures*. Mathematics and Its Applications. Kluwer Academic Publishers, 2000.
- N. D. Filippov. *Projections of lattices*. (in Russian). Mat. Sb. (N.S.) 70(112).1 (1966).
- G. Grätzer, K. M. Koh, and M. Makkai. *On the lattice of subalgebras of a Boolean algebra*. Proceedings of the American mathematical society 36(1):87-92 (1972).
- John Harding, Chris Heunen, Bert Lindenhovius, and Mirko Navara. *Boolean Subalgebras of Orthoalgebras*. Order 36:563-609 (2019).
- Leo Lobski. *Quantum quirks, classical contexts: Towards a Bohrification of effect algebras*. Master of Logic Thesis (MoL) Series, MoL-2020-09.
- Sam Staton and Sander Uijlen. *Effect algebras, presheaves, non-locality and contextuality*. Information and Computation, volume 261, part 2. Elsevier 2018.
- David Sachs. *The lattice of subalgebras of a Boolean algebra*. Canadian Journal of Mathematics 14:451-460 (1962).

Thank you for your attention!