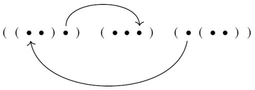
On Coherence for Stoic Conjunction

– a counterfactual look at ancient logic

Peter M. Hines — University of York

Symposium on Compositional Structures

Edinburgh, Dec. 19th 2022



Au fond de l'Inconnu pour trouver du nouveau!

A puzzle from antiquity . . .

The Logician/Philosopher vs. the Astronomer/Mathematician :

"Chrysippus says that the number of conjunctions[a] [constructible] from only ten assertibles exceeds one hundred myriads [i.e. 10^6]. However, Hipparchus refuted this, demonstrating that the affirmative encompasses 103049 conjoined assertibles and the negative 310952."

— *Plutarch,* **Quæstiones Convivales** *(2nd C. AD)*

[a]'combinations' in some documents . . .

This was reported as *common knowledge*,

"Chrysippus is refuted by *all the arithmeticians*, among them Hipparchus himself who proves that his error in calculation is enormous".

— *Plutarch,* **De Stoicorum Repugnantiis** *(2nd C. AD)*

but the precise meaning was lost.

"Since the exact terms of the problem are not stated, it is difficult to interpret the numerical answers . . . The Greeks took no interest in these matters".

— *N. L. Biggs* **The Roots of Combinatorics** *1979*

Interpretation and Composition

The significance of 103049 was realised in January 1994 by a graduate student (D. Hough) at George Washington University :

> Hipparchus, Plutarch, Schröder and Hough
> — *R. Stanley, American Mathematical Monthly (1997)*

It is simply the 10^{th} **little Schröder number**, counting (amongst other things[1]) the number of distinct **Rooted Planar Trees** with ten leaves.

A (too easy?) interpretation

It is tempting to interpret :

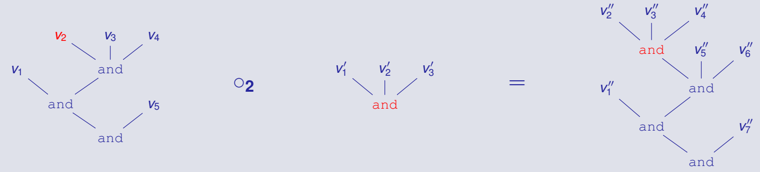
1. Each **branching** as a logical operation (conjunction?)
2. Each **leaf** as a simple assertible (variable?)

Building larger trees from smaller trees : **Substituting a tree for a given leaf.**

[1] e.g. the number of facets of the tenth associahedron

Replacing simple assertibles by non-simple composites

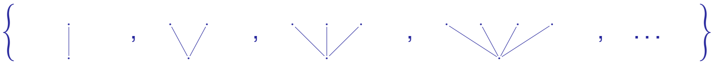
Operadic Composition :



Provided we

- Avoid clashes of free variable names, & identify α-equivalent trees,
- Identify up to (planar) topological equivalence,

we arrive at the non-symmetric operad $\mathbb{RPT}$ of **rooted planar trees**.
This is *freely generated* by one tree of each arity (number of leaves).



How did Hipparchus (and *"all the arithmeticians"*) calculate Schröder numbers??

> On the Shoulders of Hipparchus:
> A Reappraisal of Ancient Greek Combinatorics.
> — **F. Acerbi** *(2004)*

Why should we be interested?

> *Chrysippus' main achievement is the development of a propositional logic & deductive system*[2]. *He was innovative in topics central to contemporary formal and philosophical logic. The many close similarities with Gottlob Frege are especially striking.*
> — **Stanford Encyclopedia of Philosophy**

How & why did Chrysippus & Hipparchus come up with such different values?

> Combinatorics for Stoic Conjunction:
> Chrysippus Vindicated, Hipparchus Refuted.
> *Oxford Studies in Ancient Philosophy*, **S. Bobzien** *(2011)*

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A misunderstanding of logic?

Bobzien's claim is that, "*Hipparchus, it seems, got his mathematics right. What I suggest in this paper is that he got his Stoic logic wrong.*"

Where it starts going wrong :

"He counts the *same sequence* of conjuncts but with *different bracketing* as different conjunctions . . . He counts

$$[P \wedge Q] \wedge R \quad \xrightarrow{\quad P \wedge Q \wedge R \quad} \quad P \wedge [Q \wedge R]$$

as different assertibles. Unlike modern propositional logic, Hipparchus assumes that a [elementary] conjunction can consist of <u>two or more</u> conjuncts.

In order to get to [the little Schröder number 103049], Hipparchus also had to take the order of the ten atomic assertibles as fixed.'" – S. B.

A synthesis via category theory

Between ‘equal’ and ‘not equal’ lies a compromise :

The Same $\overline{}$ Different

equal up to / *unique isomorphism*

This should be understood at the level of **semantic models**.

What might we need ?

- A family of k-ary **elementary conjunctions**

$$(\_ \star \_) \quad , \quad (\_ \star \_ \star \_) \quad , \quad (\_ \star \_ \star \_ \star \_) \quad , \quad \ldots$$

  (presumably, functors . . .)
- Under substitution / operadic composition these should freely generate an operad isomorphic to $\mathbb{RPT}$.
- A notion of ‘equivalence up to natural isomorphism’ that uniquely relates any two composites of the same arity.

An additional concern

We *cannot* assume idempotency of conjunction

"Non-simple [assertibles] are those that are, as it were, double $(\delta\iota\pi\lambda\alpha)$ – put together by means of a connecting particle from an assertible that is taken twice $(\delta\iota\zeta)$, or from two different assertibles." — S. B.

We find what we need by :

generalising a model of conjunction from Linear Logic.

(As a bonus!) Everything is based on **Euclidean division** & elementary **arithmetic**.

“Everything is *[in the endomorphism monoid of]* Numbers”

In J.-Y. Girard’s Geometry of Interaction system (Parts 0 – 2) :

Propositions are modelled by functions on $\mathbb{N}$.

- Bijections in the symmetric group $\mathcal{S}(\mathbb{N})$ for MLL
- Partial injections in the symmetric inverse monoid $\mathcal{I}(\mathbb{N})$ for MELL

Conjunction is modelled by the following operation :

$$(f \star g)(2n) = 2.f(n)$$

$$(f \star g)(2n+1) = 2.g(n) + 1$$

A simple description, based on Hilbert’s Grand Hotel

This “writes two functions as a single function”, by

replicating their behaviour on the **even** *and* **odd** *numbers respectively.*

This is an injective homomorphism $\mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$, and indeed a categorical (semi-monoidal) tensor

Potential vs. actual infinity

The Greeks feared infinity and tried to avoid it . . . According to tradition, they were frightened off by the paradoxes of Zeno. . . . Until the late C^{19th}, mathematicians were reluctant to accept infinity as more than "potential".
— *J. Stillwell,* **Mathematics and Its History** *2012*

Not Euclid There exists an infinite number of primes.

Euclid The prime numbers are more numerous than any proposed multitude of prime numbers.

Actual infinity was eventually forced by the requirements of medieval theology:

Duns Scotus on God (R. Cross, 2005)

John Duns Scotus (1266-1308) [ontological] argument may be summarised as,

"If God is composed of parts, then each part must be finite or infinite. . . . **If any given part is infinite, then it is equal to the whole**, *which is absurd"*

John Duns' **absurdity** was was (mostly!) stripped of theological interpretations, and taken as a **definition** by G. Cantor & company.

Not strictly the same . . .

In general : $(P \star Q) \star R \neq P \star (Q \star R)$

No faithful tensor on a non-abelian monoid can be strictly associative.

Coherence & Strictification for Self-Similarity
Journal Homotopy & related Structures *(P.M.H. 2016)*

There is a non-trivial natural isomorphism $(\_ \star (\_ \star \_)) \Rightarrow ((\_ \star \_) \star \_)$

$$\alpha(a \star (b \star c)) = ((a \star b) \star c)\alpha \quad \forall\ a, b, c \in \mathcal{S}(\mathbb{N})$$

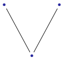
whose unique component (the associator) $\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$

is a congruential function, satisfying MacLane's pentagon condition

$$\alpha^2 = (\alpha \star Id)\ \alpha\ (Id \star \alpha)$$

A Hipparchus-style generalisation

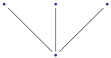
Girard gave a **binary** model of conjunction $(\_ \star \_) : \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$.
"$(a \star b)$ replicates a, b on the modulo classes $2\mathbb{N}$, $2\mathbb{N}+1$ respectively".

- We draw this as 

There is an obvious **ternary** analogue, $(\_ \star \_ \star \_) : \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$

$$(a \star b \star c)(3n+i) \;=\; \begin{cases} 3.a(n) & i=0 \\ 3.b(n)+1 & i=1 \\ 3.c(n)+2 & i=2 \end{cases}$$

"Replicate a, b, c on the modulo classes $3\mathbb{N}$, $3\mathbb{N}+1$, $3\mathbb{N}+2$ respectively".

- We draw this as 

The general case :

For any $k \geqslant 1$, we form the k^{th} **elementary conjunction** by :

$$(f_0 \star \dots f_{k-1})(kn+i) \;=\; k.f_i(n) + i \;\text{ where } i = 0, 1, 2, \dots, k-1$$

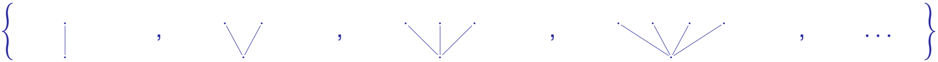
Alternatively & equivalently,

$$(f_0 \star \dots f_{k-1})(x) \;=\; k.f_i\left(\frac{x-i}{k}\right) + i \;\text{ where } x \equiv i \mod k$$

This gives, for any $k > 0$, an injective group homomorphism $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$ that :

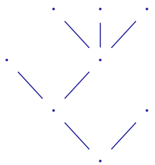
> *"replicates the action of f_0 , f_1 , ... , f_{k-1} on the modulo classes $k\mathbb{N}$, $k\mathbb{N}+1$, ... , $k\mathbb{N}+(k-1)$ respectively.*

For $k = 1, 2, 3, 4, \dots$, we draw these as



Composing elementary conjunctions

These ‘compose by substitution’ to give an operad $\mathcal{H}ipp$ of **generalised conjunctions**. Each k-leaf tree determines an injective hom. $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$.



$: \ \mathcal{S}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{S}(\mathbb{N})$

$$(f_0, f_1, f_2, f_3, f_4) \quad \mapsto \quad ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$$

More formally :

We have one operation of each arity > 0 in the (non-symmetric) endomorphism operad of $\mathcal{S}(\mathbb{N})$ within the category $(\mathbf{Grp}, \times)$ of groups / homomorphisms with Cartesian product.

These generate the sub-operad $\mathcal{H}ipp$.

An operad for Hipparchus

Claim :

The operad $\mathcal{H}ipp$ of generalised conjunctions extends Girard's operation from the Geometry of Interaction, to provide a semantic model for Hipparchus' (mis-)understanding of Chrysippus' Stoic Logic.

More concisely(!)

$\mathcal{H}ipp \cong \mathbb{RPT}$, so each tree determines a *distinct* homomorphism $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$.

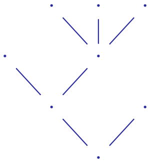
Proof?

Proving this requires a concept the Greeks (notoriously) did not have[3]:

> *The greatest calamity in the history of science was the failure of Archimedes to invent positional notation.* – C. F. Gauss

[3] . . . but may (occasionally) have borrowed from their neighbours

Let me see you counting like they do in Babylon



defines a homomorphism : $\mathcal{S}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{S}(\mathbb{N})$

In the operadic composite $(f_0, f_1, f_2, f_3, f_4) \mapsto ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$, the action of each f_j is mapped :

from The whole of the natural numbers $\mathbb{N}$

to Some modulo class $A_j\mathbb{N} + B_j$.

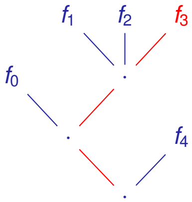
For example : f_3 is translated onto $12\mathbb{N} + 10$.

Question:

How do we derive these coefficients *from the tree*?

A root and branch approach

Deriving $12\mathbb{N} + 10$, from the leaf-to-root path :



Branch number 2 of 3

Branch number 1 of 2

Branch number 0 of 2

Multiplicative coefficient : $12 = 3 \times 2 \times 2$

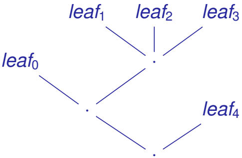
Additive coefficient :	(Decimal)		<u>Base 3</u>	<u>Base 2</u>	<u>Base 2</u>
	10	=	2	1	0

Positional mixed-radix number systems

First formal study by G. Cantor, *Über einfache Zahlensysteme* (1869)

Covering the Numbers with Trees

Rooted Planar Trees are uniquely determined by the addresses of their leaves



$leaf_0$		$(0,2)$	$(0,2)$	$4\mathbb{N}$
$leaf_1$	$(0,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+2$
$leaf_2$	$(1,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+6$
$leaf_3$	$(2,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+10$
$leaf_4$			$(1,2)$	$2\mathbb{N}+1$

which uniquely determine ordered **exact covering systems**, such as

$$4\mathbb{N} \ , \ 12\mathbb{N}+2 \ , \ 12\mathbb{N}+6 \ , \ 12\mathbb{N}+10 \ , \ 2\mathbb{N}+1$$

Heavily studied by P. Erdös (1950s)

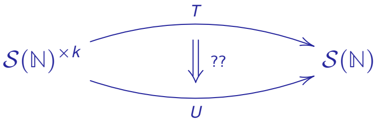
Ordered sets of pairwise-disjoint modulo classes, whose union is the whole of $\mathbb{N}$

— also (finite) open covers of $(\mathbb{N},+)$ w.r.t. the profinite topology.

(Corol: Distinct trees determine distinct homomorphisms).

Mappings between between gen. conjunctions

Given generalised conjunctions $T, U : \mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$, can we find a (well-behaved) natural isomorphism between them?



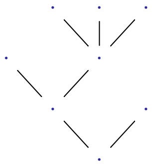
Convention : As generalised conjunctions are **group** homomorphisms, natural transformations have a **single** component.

We identify nat. iso.s with their unique component in $\mathcal{S}(\mathbb{N})$.

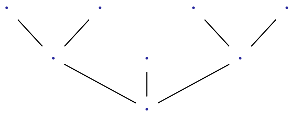
Congruential functions as natural isomorphisms

Consider the generalised conjunctions[4] $T, U : \mathcal{S}(\mathbb{N})^{\times 6} \hookrightarrow \mathcal{S}(\mathbb{N})$

$T =$



$U =$



We build a natural isomorphism $\eta_{T,U} : T \Rightarrow U$ by monotonically mapping between their respective ordered covering systems :

leaf 0	$4\mathbb{N}$	$\mapsto$	$6\mathbb{N}$
leaf 1	$12\mathbb{N}+2$	$\mapsto$	$6\mathbb{N}+3$
leaf 2	$12\mathbb{N}+6$	$\mapsto$	$3\mathbb{N}+1$
leaf 3	$12\mathbb{N}+10$	$\mapsto$	$6\mathbb{N}+2$
leaf 4	$2\mathbb{N}+1$	$\mapsto$	$6\mathbb{N}+4$

This gives, as desired,

$$\eta_{T,U}.((a \star (b \star c \star d)) \star e) \;=\; ((a \star b) \star c \star (d \star e)).\eta_{T,U}$$

[4] edges of the fifth associahedron $\mathcal{K}_5$

A category for Chrysippus

Observe that :

- $\eta_{T,T} = Id \in \mathcal{S}(\mathbb{N})$
- $\eta_{T,U}\eta_{S,T} = \eta_{S,U}$
- $\eta^{-1}_{T,U} = \eta_{U,T}$

We have a **posetal groupoid**[5] $\mathcal{C}hrys$ of functors / natural iso.s, given by :

Objects Generalised conjunctions (operations of $\mathcal{H}ipp$)

Arrows $\mathcal{C}hrys(T,U) = \begin{cases} \{\eta_{T,U}\} & T, U \text{ have the same arity,} \\ \emptyset & \text{otherwise.} \end{cases}$

We can take generalised conjunctions of *arrows*

What about objects – are generalised conjunctions functors ??

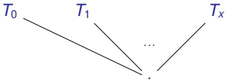
[5] within which, 'all diagrams commute'

Generalised conjunctions within a posetal category

We define generalised conjunctions of **objects** of $\mathcal{C}hrys$ in the obvious manner :

Given $T_0, \ldots, T_x \in Ob(\mathcal{C}hrys)$, define

$$(T_0 \star \ldots \star T_x) \stackrel{def.}{=}$$



Rather neatly (but entirely expectedly) :

The unique arrow $T_0 \star \ldots \star T_x \Rightarrow U_0 \star \ldots \star U_x$ is given by

$$\eta_{(T_0 \star \ldots \star T_x),(U_0 \star \ldots U_x)} = (\eta_{T_0,U_0} \star \ldots \star \eta_{T_x,U_x})$$

Generalised conjunction defines $\mathbb{N}^+$-indexed family of functors on a posetal groupoid :

$$\left\{ (\_ \star \ldots \star \_) : \prod_k Chrys \rightarrow \mathcal{C}hrys \right\}_{k \in \mathbb{N}^+}$$

Very special case: $\mathcal{C}hrys$ contains a copy of (a unitless version of) MacLane's posetal monoidal groupoid $(\mathcal{W}, \_\square\_)$.

Concrete formulæ for arrows of $\mathcal{C}hrys$

Given two ordered exact covering systems, determined by k-ary generalised conjunctions T, U

leaf 0	$A_0\mathbb{N} + B_0$	$\mapsto$	$C_0\mathbb{N} + D_0$
leaf 1	$A_1\mathbb{N} + B_1$	$\mapsto$	$C_1\mathbb{N} + D_1$
$\vdots$	$\vdots$		$\vdots$
leaf $k-1$	$A_{k-1}\mathbb{N} + B_{k-1}$	$\mapsto$	$C_{k-1}\mathbb{N} + D_{k-1}$

The natural isomorphism $\eta_{T,U}$ is the bijection

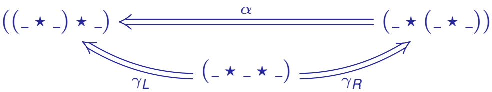
$$\eta_{T,U}(x) = \frac{1}{A_j}\left(C_j x + \begin{vmatrix} A_j & B_j \\ C_j & D_j \end{vmatrix}\right) \text{ where } x \equiv B_j \mod A_j$$

We arrive at John Conway's **congruential functions**

"Unpredictable Iterations" — J. Conway (1972)

as used to demonstrate undecidability in elementary arithmetic.

Bobzien's 'three simple assertibles' example



$$\gamma_L(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \mod 3, \\ \frac{4n+2}{3} & n \equiv 1 \mod 3, \\ \frac{2n-1}{3} & n \equiv 2 \mod 3. \end{cases} \qquad \gamma_R(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3. \end{cases}$$

$$\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$$

These are familiar from *other areas* :

- α : the canonical associativity isomorphism for Girard's conjunction.
- γ_R : the core arithmetic operator from the Original Collatz conjecture.
- γ_L : a 'slightly shifted' version of γ_R, given by $1 + \gamma_L(n) = \gamma_R(n+1)$.

From the third to the fourth associahedron