Complete Algebraic Semantics for Second-Order Rewriting Systems based on Abstract Syntax with Variable Binding

Makoto Hamana

Faculty of Informatics, Gunma University, Japan

SYCO 10

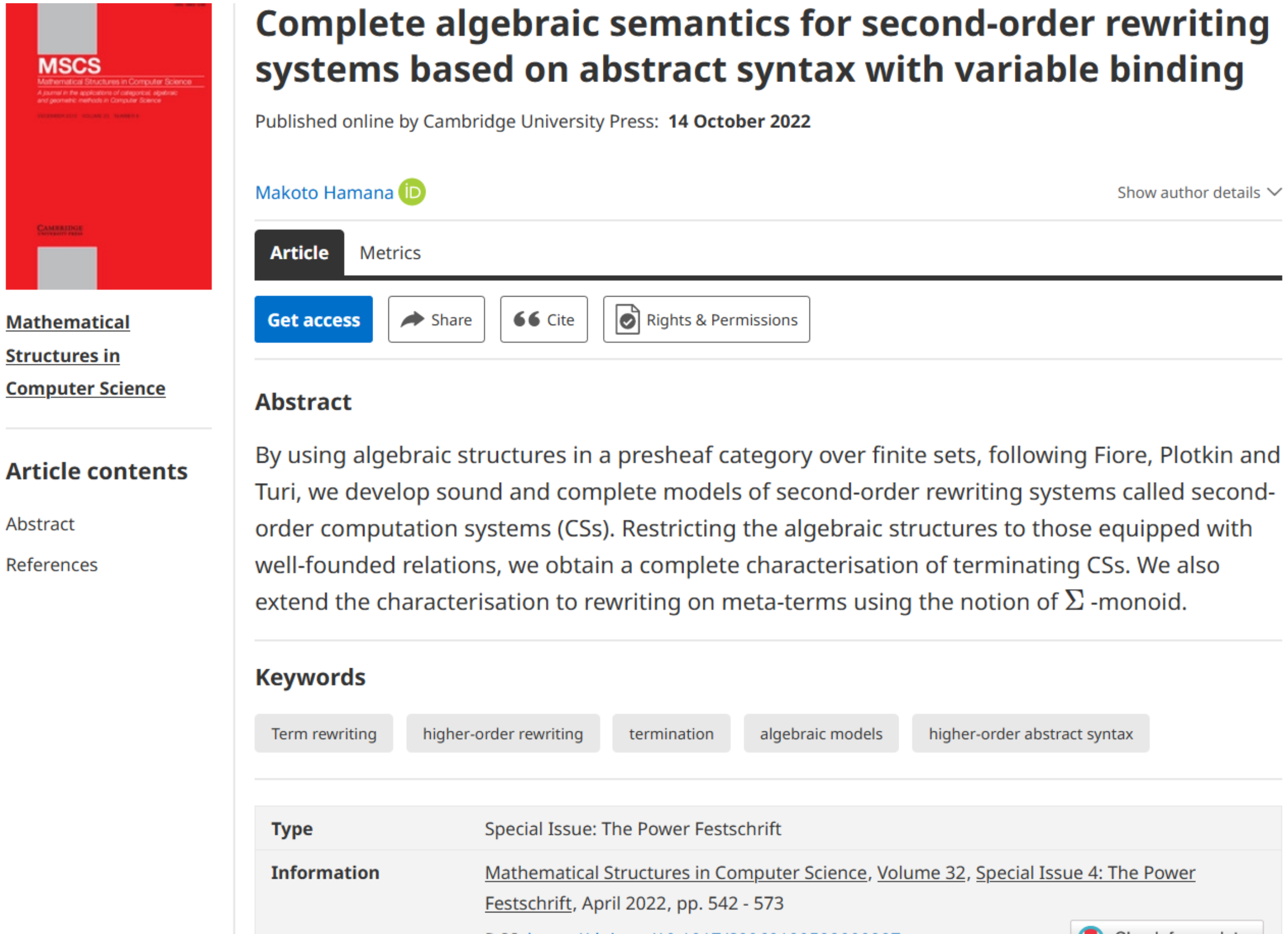
19th December, 2022, Edinburgh

This Talk

▷ Complete algebraic semantics of second-order rewriting

This Talk

- ▷ Complete algebraic semantics of second-order rewriting
- ▷ Based on my paper
  - – Complete Algebraic Semantics for Second-Order Rewriting Systems based on Abstract Syntax with Variable Binding
  - – MSCS, CUP, 2022,
    Special Issue of John Power Festschrift


MSCS
Mathematical Structures in Computer Science

Mathematical Structures in Computer Science

Article contents
Abstract
References

Complete algebraic semantics for second-order rewriting systems based on abstract syntax with variable binding

Published online by Cambridge University Press: 14 October 2022

Makoto Hamana

Show author details

Article | Metrics

Get access | Share | Cite | Rights & Permissions

Abstract

By using algebraic structures in a presheaf category over finite sets, following Fiore, Plotkin and Turi, we develop sound and complete models of second-order rewriting systems called second-order computation systems (CSs). Restricting the algebraic structures to those equipped with well-founded relations, we obtain a complete characterisation of terminating CSs. We also extend the characterisation to rewriting on meta-terms using the notion of Σ-monoid.

Keywords

Term rewriting | higher-order rewriting | termination | algebraic models | higher-order abstract syntax

Type	Special Issue: The Power Festschrift
Information	Mathematical Structures in Computer Science, Volume 32, Special Issue 4: The Power Festschrift, April 2022, pp. 542 - 573

First-order Rewriting: Review

First-Order Term Rewriting System (TRS) $\mathcal{R}$:

$$fact(0) \longrightarrow S(0)$$
$$fact(S(x)) \longrightarrow fact(x) * S(x)$$

Rewrite steps:

```
fact(S(S(0)))  =>  fact(S(0)) * S(S(0))  =>  (fact(0) * S(0)) * S(S(0))
 =>  (S(0) * S(0)) * S(S(0))  ==> S(S(0))   (normal form)
```

Fundametal problem

- ▷ Termination   (Strong Normalisation)
- ▷ How can we prove the termination of $\mathcal{R}$?

TRS: Sound and Complete Algebraic Characterisation

Thm. [Huet and Lankford'78]

A first-order term rewriting system $\mathcal{R}$ is terminating

$$\Leftrightarrow$$

there exists a well-founded monotone Σ-algebra $(A, >_A)$ that is compatible with $\mathcal{R}$.

Termination proof method

[$\Leftarrow$] Find a well-founded monotone Σ-algebra that is compatible with $\mathcal{R}$.

First-order Rewriting: Review

First-Order Term Rewriting System (TRS) $\mathcal{R}$:

$$\begin{aligned} fact(0) &\longrightarrow S(0) \\ fact(S(x)) &\longrightarrow fact(x) * S(x) \end{aligned}$$

Semantics: well-founded monotone Σ-algebra $(\mathbb{N}, >)$ given by

$$fact^{\mathbb{N}}(x) = 2x + 2 \qquad x *^{\mathbb{N}} y = x + y \qquad S^{\mathbb{N}}(x) = 2x + 1 \qquad 0^{\mathbb{N}} = 1$$

Then it is compatible with $\mathcal{R}$ as

$$\begin{array}{llll} fact^{\mathbb{N}}(0^{\mathbb{N}}) & = 2 + 2 & > \quad 2 + 1 & = S^{\mathbb{N}}(0^{\mathbb{N}}) \\ fact^{\mathbb{N}}(S^{\mathbb{N}}(x)) & = 2(2x + 1) + 2 & > \quad 2x + 2x + 1 & = fact^{\mathbb{N}}(x) * S^{\mathbb{N}}(x) \end{array}$$

Hence $\mathcal{R}$ is terminating.

Aim: Sound and Complete Algebraic Characterisation

Thm. [Huet and Lankford'78]

A first-order term rewriting system $\mathcal{R}$ is terminating

$$\Leftrightarrow$$

there exists a well-founded monotone Σ-algebra A that
is compatible with $\mathcal{R}$.

- Aim: Extend this to second-order rewriting
- Give: Complete algebraic semantics of second-order rewriting

Example of Second-Order Rewriting : Prenex normal forms

$$\mathrm{P}\wedge\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\mathrm{P}\wedge\mathrm{Q}[\boldsymbol{x}]) \qquad \neg\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \exists(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}]))$$

$$\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}])\wedge\mathrm{P} \longrightarrow \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]\wedge\mathrm{P}) \qquad \neg\exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}]))$$

Signature: $\neg, \wedge, \vee, \forall, \exists$

Example of Second-Order Rewriting : Prenex normal forms

$$\mathrm{P} \wedge \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\mathrm{P} \wedge \mathrm{Q}[\boldsymbol{x}]) \qquad \neg\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \exists(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}]))$$

$$\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \wedge \mathrm{P} \longrightarrow \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}] \wedge \mathrm{P}) \qquad \neg\exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}]))$$

Signature: $\neg, \wedge, \vee, \forall, \exists$

Second-Order Rewriting System is defined on
Second-Order Abstract Syntax [Hamana'04, Fiore LICS'06]

- ▷ Abstract syntax with variable binding [Fiore, Plotkin, Turi LICS'99]
- ▷ Metavariables with arities (e.g. P,Q)
- ▷ Substitutions (Metavars, object vars)

Example: the $\boldsymbol{\lambda}$-calculus as a Second-Order Rewriting System

$$\boldsymbol{\lambda}(\boldsymbol{x}.\mathrm{M}[\boldsymbol{x}]) \mathbin{@} \mathrm{N} \longrightarrow \mathrm{M}[\mathrm{N}]$$

$$\boldsymbol{\lambda}(\boldsymbol{x}.\mathrm{M} \mathbin{@} \boldsymbol{x}) \longrightarrow \mathrm{M}$$

▷ Signature: $\boldsymbol{\lambda}, @$

Abstract Syntax and Variable Binding [Fiore,Plotkin,Turi LICS'99]

▷ Aim: To model syntax with variable binding, e.g.

$$\frac{}{x_1, \ldots, x_n \vdash x_i} \qquad \frac{x_1, \ldots, x_n \vdash t \quad x_1, \ldots, x_n \vdash s}{x_1, \ldots, x_n \vdash t@s}$$

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash \lambda(x_{n+1}.t)}$$

▷ Syntax generated by 3 constructors

▷ λ is a special unary function symbol:
it decreases the context

Abstract Syntax and Variable Binding [Fiore,Plotkin,Turi LICS'99]

- Aim: model syntax with variable binding, e.g.

$$\frac{}{\boldsymbol{n} \vdash \boldsymbol{i}} \qquad \frac{\boldsymbol{n} \vdash \boldsymbol{t} \quad \boldsymbol{n} \vdash \boldsymbol{s}}{\boldsymbol{n} \vdash \boldsymbol{t@s}}$$

$$\frac{\boldsymbol{n+1} \vdash \boldsymbol{t}}{\boldsymbol{n} \vdash \boldsymbol{\lambda}(\boldsymbol{n+1.t})}$$

- Category $\mathbb{F}$ for variable contexts

  objects: $\boldsymbol{n} = \{\mathbf{1}, \ldots, \boldsymbol{n}\}$ (variable contexts)

  arrows: all functions $\boldsymbol{n} \longrightarrow \boldsymbol{n}'$ (renamings)

- Presheaf category $\mathbf{Set}^{\mathbb{F}}$

Models of Syntax with Binding: $\boldsymbol{\Sigma}$-Algebras in $\mathbf{Set}^{\mathbb{F}}$

Def. A binding signature $\boldsymbol{\Sigma}$ is a set of function symbols with binding arities:

$$f : \langle n_1, \ldots, n_l \rangle$$

which has l arguments and binds n_i variables in the i-th argument .

Def. A $\boldsymbol{\Sigma}$-algebra $A = (A, [f^A]_{f \in \Sigma})$ in $\mathbf{Set}^{\mathbb{F}}$ consists of

▷ carrier: a presheaf $A \in \mathbf{Set}^{\mathbb{F}}$

▷ operations: arrows of $\mathbf{Set}^{\mathbb{F}}$

$$f^A : \delta^{n_1} A \times \ldots \times \delta^{n_l} A \longrightarrow A$$

corresponding to function symbols $f : \langle n_1, \ldots, n_l \rangle \in \Sigma$.

▷ Context extension: $\delta A \in \mathbf{Set}^{\mathbb{F}}$; $(\delta A)(n) = A(n+1)$

Example: λ-terms

▷ Binding signature Σ_λ for λ-terms

$$\lambda \;:\; \langle 1 \rangle, \qquad\qquad @ \;:\; \langle 0, 0 \rangle$$

▷ Carrier: the presheaf Λ of all λ-terms

$$\Lambda(n) = \{ t \mid n \vdash t \}$$

$$\Lambda(\rho) : \Lambda(m) \to \Lambda(n) \qquad \text{renaming on } \lambda\text{-terms for } \rho : m \to n \text{ in } \mathbb{F}.$$

▷ Forms a $\mathrm{V}+\Sigma_\lambda$-algebra

$$\begin{array}{llll} \mathrm{var}^\Lambda : \mathrm{V} \to \Lambda & @^\Lambda : \Lambda \times \Lambda \to \Lambda & \lambda^\Lambda \quad : \delta\Lambda & \to \Lambda \\ \quad i \mapsto i & \quad s \; , \; t \mapsto s@t & \lambda^\Lambda(n) : \Lambda(n+1) & \to \Lambda(n) \\ & & \qquad\quad t & \mapsto \lambda n{+}1.t \end{array}$$

▷ Presheaf of variables: $\mathrm{V} \in \mathbf{Set}^{\mathbb{F}}; \mathrm{V}(n) = \{1, \ldots, n\}$

▷ Thm. $\Lambda \; (= \mathrm{T}_\Sigma \mathrm{V})$ is an initial $\mathrm{V} + \Sigma_\lambda$-algebra.

Second-Order Abstract Syntax

- Abstract syntax with variable binding
- Metavariables with arities
- Substitutions (Metavars, object vars)

Models of Secound-Order Abstract Syntax: $\boldsymbol{\Sigma}$-monoids

- A $\boldsymbol{\Sigma}$-monoid [Fiore, Plotkin, Turi'99] is
  - a $\boldsymbol{\Sigma}$-algebra $\boldsymbol{A}$ with
  - a monoid structure

$$\mathbf{V} \xrightarrow{\nu} \boldsymbol{A} \xleftarrow{\mu} \boldsymbol{A} \bullet \boldsymbol{A}$$

    in the monoidal category $(\mathbf{Set}^{\mathbb{F}}, \bullet, \mathbf{V})$,
  - both are compatible.

- Idea
  - Unit $\boldsymbol{\nu}$ models the embedding of variables
  - Multiplication $\boldsymbol{\mu}$ models substitution for object variables

Algebraic Characterisation of Syntax with Binding

Given a binding signature $\mathbf{\Sigma}$

- The presheaf of all $\mathbf{\Sigma}$-terms

$$\mathbf{T_\Sigma V}(n) = \{t \mid n \vdash t\}$$

- Multiplication $\mu : \mathbf{T_\Sigma V} \bullet \mathbf{T_\Sigma V} \longrightarrow \mathbf{T_\Sigma V}$

$$\mu_n^{(m)}(t;\ s_1, \ldots, s_m) \triangleq t[1 := s_1, \ldots, n := s_m]$$

(the substitution of $\mathbf{\Sigma}$-terms for de Bruijn variables)

Algebraic Characterisation of Syntax with Binding

Given a binding signature $\mathbf{\Sigma}$

▷ The presheaf of all $\mathbf{\Sigma}$-terms

$$\mathbf{T_\Sigma V}(n) = \{t \mid n \vdash t\}$$

▷ Multiplication $\mu : \mathbf{T_\Sigma V} \bullet \mathbf{T_\Sigma V} \longrightarrow \mathbf{T_\Sigma V}$

$$\mu_n^{(m)}(t;\ s_1, \ldots, s_m) \triangleq t[1 := s_1, \ldots, n := s_m]$$

(the substitution of $\mathbf{\Sigma}$-terms for de Bruijn variables)

▷ Thm. [Fiore, Plotkin, Turi'99]

– $(\mathbf{T_\Sigma V}, \nu, \mu)$ is an initial $\mathbf{\Sigma}$-monoid.

– $(\mathbf{T_\Sigma V}, \nu)$ is an initial $\mathbf{V} + \mathbf{\Sigma}$-algebra.

▶ How to model metavariables and substitutions for metavariables?

Algebraic Characterisation of Syntax with Binding

Given a binding signature $\mathbf{\Sigma}$

▷ The presheaf of all $\mathbf{\Sigma}$-terms

$$\mathbf{T}_{\Sigma}\mathbf{V}(n) = \{t \mid n \vdash t\}$$

▷ Multiplication $\mu : \mathbf{T}_{\Sigma}\mathbf{V} \bullet \mathbf{T}_{\Sigma}\mathbf{V} \longrightarrow \mathbf{T}_{\Sigma}\mathbf{V}$

$$\mu_n^{(m)}(t;\ s_1, \ldots, s_m) \triangleq t[1 := s_1, \ldots, n := s_m]$$

(the substitution of $\mathbf{\Sigma}$-terms for de Bruijn variables)

▷ Thm. [Fiore, Plotkin, Turi'99]

- $(\mathbf{T}_{\Sigma}\mathbf{V}, \nu, \mu)$ is an initial $\mathbf{\Sigma}$-monoid.
- $(\mathbf{T}_{\Sigma}\mathbf{V}, \nu)$ is an initial $\mathbf{V} + \mathbf{\Sigma}$-algebra.

▶ How to model metavariables and substitutions for metavariables?

▶ Free $\mathbf{\Sigma}$-monoids [Hamana, APLAS'04]

Meta-terms: Terms with Metavariables [Aczel '78]

- A binding signature Σ
- Z is an $\mathbb{N}$-indexed set of metavariables parameterised by arities:

$$Z(l) \triangleq \{\mathrm{M} \mid \mathrm{M}^{l}, \text{ where } l \in \mathbb{N}\}.$$

- Raw meta-terms generated by Z:

$$t ::= x \mid f(x_1 \cdots x_{i_1}.t_1, \ldots, x_1 \cdots x_{i_l}.t_l) \mid \mathrm{M}[t_1, \ldots, t_l]$$

- A meta-term t is a raw meta-term derived from:

$$\frac{x \in n}{n \vdash x} \qquad \frac{f : \langle i_1, \ldots, i_l \rangle \in \Sigma \quad n{+}i_1 \vdash t_1 \ \cdots \ n{+}i_l \vdash t_l}{n \vdash f(\, n{+}1 \ldots n{+}i_1.t_1, \ \ldots, \ n{+}1 \ldots n{+}i_l.t_l \,)}$$

$$\frac{\mathrm{M} \in Z(l) \quad n \vdash t_1 \ \cdots \ n \vdash t_l}{n \vdash \mathrm{M}[t_1, \ldots, t_l]}$$

Meta-terms: Terms with Metavariables

- Presheaf $M_\Sigma Z \in \mathbf{Set}^{\mathbb{F}}$

$$M_\Sigma Z(n) = \{t \mid n \vdash t\}$$

- $\mathbf{V}+\mathbf{\Sigma}$-algebra $(M_\Sigma Z, [\nu, f_T]_{f\in\Sigma})$

$$\begin{aligned} \nu(n) : \mathrm{V}(n) &\longrightarrow M_\Sigma Z(n), \\ x &\longmapsto x \end{aligned}$$

$$\begin{aligned} f^T : \delta^{i_1} M_\Sigma Z \times \cdots \times \delta^{i_l} M_\Sigma Z &\longrightarrow M_\Sigma Z \\ (t_1, \ldots, t_l) &\longmapsto f(n+\overline{i_1}.t_1, \ldots, n+\overline{i_l}.t_l). \end{aligned}$$

- Multiplication $\mu : M_\Sigma Z \bullet M_\Sigma Z \to M_\Sigma Z$

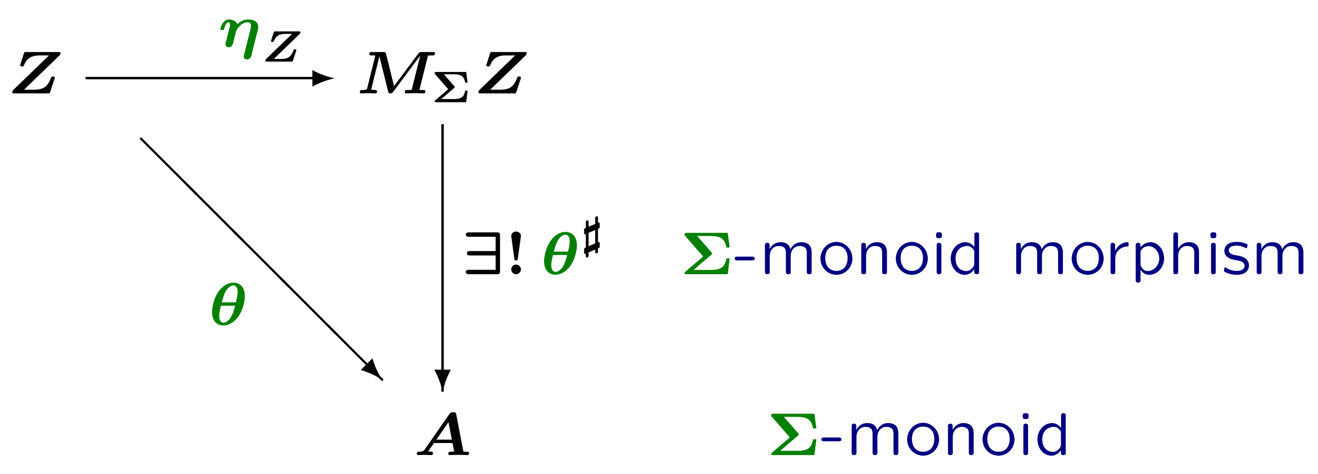
$$t, \quad \overline{s} \longmapsto t[1 := s_1, \ldots, n := s_n]$$

$\cdots$ substitution of meta-terms for object variables

Free Σ-monoids: Syntax with Metavariables [Hamana, APLAS'04]

Thm. $(M_\Sigma Z, \nu, \mu)$ forms a free Σ-monoid over Z.

▷ Freeness of $M_\Sigma Z$: in $\mathbf{Set}^{\mathbb{F}}$, given assignment θ



▷ The unique Σ-monoid morphism $\theta^\sharp$ that extends θ.

Instance: Substitution for Metavariables

Case $\boldsymbol{A} = \mathbf{T_\Sigma V}$ $\cdots$ a $\boldsymbol{\Sigma}$-monoid of terms,

$$\begin{array}{ccc} \boldsymbol{Z} & \xrightarrow{\eta_Z} & \boldsymbol{M_\Sigma Z} \\ & \searrow^{\theta} & \downarrow \exists!\, \theta^\sharp \quad \boldsymbol{\Sigma}\text{-monoid morphism} \\ & & \mathbf{T_\Sigma V} \end{array}$$

- ▷ $\theta^\sharp$ is a substitution of terms for metavariables $\boldsymbol{Z}$
- ▷ E.g. $\boldsymbol{\Sigma}$: signature for $\boldsymbol{\lambda}$-terms, for $\theta(\mathrm{M}^{(1)}) = a@a$

$$\theta^\sharp(\ \lambda(x.\mathrm{M}[x]@y)\ ) = \lambda(\ x.(x@x)@y\ )$$

- ▷ Other examples of $\boldsymbol{\Sigma}$-monoid $\boldsymbol{A}$:
  - $\boldsymbol{M_\Sigma Z}$: meta-substitution: substitution of meta-terms for metavars
  - Any $\boldsymbol{\Sigma}$-monoid as a model – $\theta^\sharp$ is compositional interpretation

Second-Order Rewriting System

Eg. A transformation to prenex normal forms

$$\mathrm{P}\wedge\forall(x.\mathrm{Q}[x]) \;\to\; \forall(x.\mathrm{P}\wedge\mathrm{Q}[x]) \qquad\qquad \neg\forall(x.\mathrm{Q}[x]) \;\to\; \exists(x.\neg(\mathrm{Q}[x]))$$

Def.

Rewrite rules $\mathcal{R}$ $\quad l \to r \quad$ on meta-terms $M_\Sigma Z$

(with some syntactic conditions)

Rewrite relation $\quad \to_{\mathcal{R}} \quad$ on terms $\mathbf{T}_\Sigma \mathbf{V}$

$$\frac{l \to r \in \mathcal{R}}{\theta^\sharp(l) \;\to_{\mathcal{R}}\; \theta^\sharp(r)} \qquad \frac{s \;\to_{\mathcal{R}}\; t}{f(\ldots,\overline{x}.s,\ldots) \;\to_{\mathcal{R}}\; f(\ldots,\overline{x}.t,\ldots)}$$

▷ Substitution $\theta : Z \to \mathbf{T}_\Sigma \mathbf{V}$ maps metavariables to terms

▷ NB. rewriting is defined on terms (without metavars)

Presheaf with relation $(A, >_A)$

Def. A presheaf $A \in \mathbf{Set}^{\mathbb{F}}$ is equipped with a binary relation $>_A$, if

1. $>_A$ is a family $\{>_{A(n)}\}_{n \in \mathbb{F}}$,
2. which is compatible with presheaf action.

(for all $a, b \in A(m)$ and $\rho : m \longrightarrow n$ in $\mathbb{F}$,
if $a >_{A(m)} b$, then $A(\rho)(a) >_{A(n)} A(\rho)(b)$.)

Monotone Algebra

Def. A monotone $\mathbf{V}+\mathbf{\Sigma}$-algebra $(\boldsymbol{A}, >_{\boldsymbol{A}})$ is a $\mathbf{V}+\mathbf{\Sigma}$-algebra $(\boldsymbol{A}, [\boldsymbol{\nu}, \boldsymbol{f}^{\boldsymbol{A}}]_{\boldsymbol{f}\in\mathbf{\Sigma}})$

- ▷ equipped with a relation $>_{\boldsymbol{A}}$ such that
- ▷ every operation $\boldsymbol{f}^{\boldsymbol{A}}$ is monotone.

Thm. $(\mathbf{T}_{\mathbf{\Sigma}}\mathbf{V}, \longrightarrow_{\mathcal{R}}\ )$ is a monotone $\mathbf{V}+\mathbf{\Sigma}$-algebra.

Models of Rewrite System $\mathcal{R}$: $(\mathbf{V}+\mathbf{\Sigma}, \mathcal{R})$-algebras

A $(\mathbf{V}+\mathbf{\Sigma}, \mathcal{R})$-algebra $(A, >_A)$ is a monotone $\mathbf{V}+\mathbf{\Sigma}$-algebra satisfying all rules in $\mathcal{R}$ as:

$$Z \xrightarrow{\eta_Z} M_\Sigma Z \qquad\qquad l \rightarrow r \in \mathcal{R} \qquad \cdots \text{ a rule}$$

θ : $Z \to \mathbf{T}_\Sigma \mathbf{V}$

$\theta^\sharp$: $M_\Sigma Z \to \mathbf{T}_\Sigma \mathbf{V}$ — a unique $\mathbf{\Sigma}$-monoid mor. extends θ

$$\text{initial } \mathbf{V}+\mathbf{\Sigma}\text{-algebra} \quad \mathbf{T}_\Sigma \mathbf{V} \qquad\qquad \theta^\sharp_n(l) \rightarrow_{\mathcal{R}} \theta^\sharp_n(r) \qquad \cdots \text{ a rewrite}$$

$!_A$: $\mathbf{T}_\Sigma \mathbf{V} \to A$ — a unique $\mathbf{V}+\mathbf{\Sigma}$-algebra homomor.

$$\mathbf{V}+\mathbf{\Sigma}\text{-algebra} \quad A \qquad\qquad !_A\theta^\sharp_n(l) >_{A(n)} !_A\theta^\sharp_n(l) \qquad \cdots \text{ an interpretation}$$

Soundness and Completeness of Models

Prop. $\quad s \rightarrow_{\mathcal{R}} t$

$$\Leftrightarrow$$

$!_A(s) \;>_A\; !_A(t)$ for all $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebras A, assignments θ.

Proof. [$\Rightarrow$]: By induction of the proof of rewrite.

[$\Leftarrow$]: Take $(A, >_A) = (\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}, \rightarrow_{\mathcal{R}}\ )$.

Complete Characterisation of Terminating Second-Order Rewriting

Thm. A second-order rewriting system $\mathcal{R}$ is terminating iff there is a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra $(A, >_A)$.

Proof. ($\Leftarrow$): Suppose a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra $(A, >_A)$.

Assume $\mathcal{R}$ is non-terminating:

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots .$$

By soundness,

$$!_A(t_1) \;>_{A(n)}\; !_A(t_2) \;>_A \cdots .$$

Contradiction.

($\Rightarrow$): When $\mathcal{R}$ is terminating, the $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra $(\mathbf{T}_\Sigma\mathbf{V}, \to_{\mathcal{R}})$ is a well-founded algebra.

▶ Because of the algebraic chatersiations of abstract sytanx with binding [FPT'99] and meta-terms [H.04]

Application: Termination by Interpretation

$$\mathrm{P} \wedge \forall(x.\mathrm{Q}[x]) \longrightarrow \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) \qquad \neg\forall(x.\mathrm{Q}[x]) \longrightarrow \exists(x.\neg(\mathrm{Q}[x]))$$

$$\forall(x.\mathrm{Q}[x]) \wedge \mathrm{P} \longrightarrow \forall(x.\mathrm{Q}[x] \wedge \mathrm{P}) \qquad \neg\exists(x.\mathrm{Q}[x]) \longrightarrow \forall(x.\neg(\mathrm{Q}[x]))$$

Take a well-founded monotone $\mathbf{V}+\mathbf{\Sigma}$-algebra $(K, >_K)$ where $K(n) = \mathbb{N}$ with $>_{K(n)} = >$ on $\mathbb{N}$.

Operations

$$\nu_n^K(i) = 0 \qquad \wedge_n^K(x, y) = \vee_n^K(x, y) = 2x + 2y$$

$$\neg_n^K(x) = 2x \qquad \forall_n^K(x) = \exists_n^K(x) = x + 1.$$

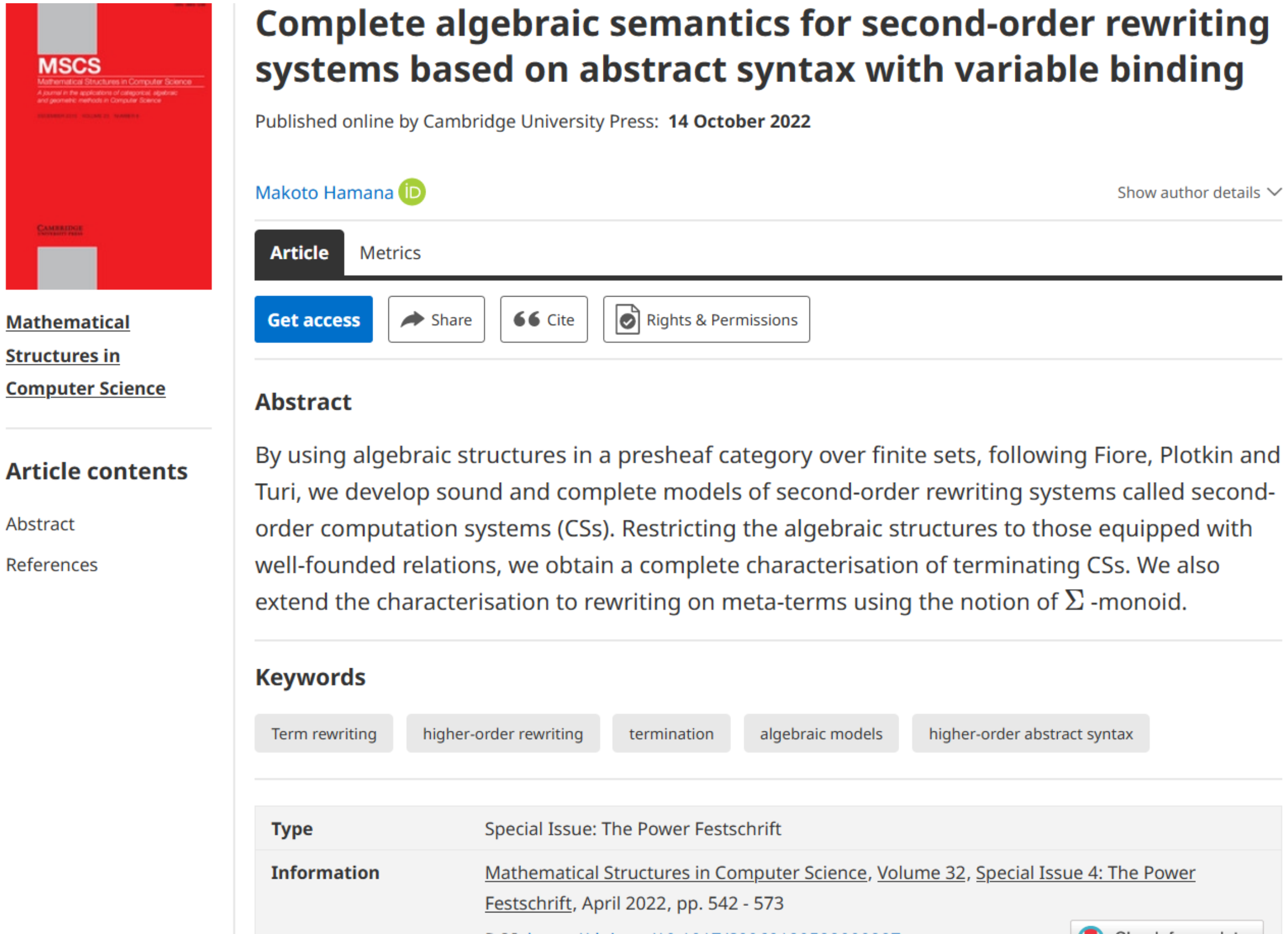
(V+Σ,$\mathcal{R}$)-algebra

$$!\theta_0^\sharp(\mathrm{P} \wedge \forall(1.\mathrm{Q}[1])) = 2x + 2(y+1) >_{K(0)} (2x + 2y) + 1 = !\theta_0^\sharp(\forall(1.\mathrm{P} \wedge \mathrm{Q}[1]))$$

$$!\theta_0^\sharp(\neg\exists(1.\mathrm{Q}[1])) = 2(y+1) >_{K(0)} 2y + 1 = !\theta_0^\sharp(\forall(1.\neg(\mathrm{Q}[1]))).$$

Summary

- ▷ Complete algebraic semantics of second-order rewriting systems
- ▷ Based on my paper
  - – Complete Algebraic Semantics for Second-Order Rewriting Systems based on Abstract Syntax with Variable Binding
  - – MSCS, CUP, 2022,
    Special Issue of John Power Festschrift


MSCS
Mathematical Structures in Computer Science
Article contents
Abstract
References

Complete algebraic semantics for second-order rewriting systems based on abstract syntax with variable binding

Published online by Cambridge University Press: **14 October 2022**

Makoto Hamana
Show author details

Article | Metrics

Get access | Share | Cite | Rights & Permissions

Abstract

By using algebraic structures in a presheaf category over finite sets, following Fiore, Plotkin and Turi, we develop sound and complete models of second-order rewriting systems called second-order computation systems (CSs). Restricting the algebraic structures to those equipped with well-founded relations, we obtain a complete characterisation of terminating CSs. We also extend the characterisation to rewriting on meta-terms using the notion of Σ-monoid.

Keywords

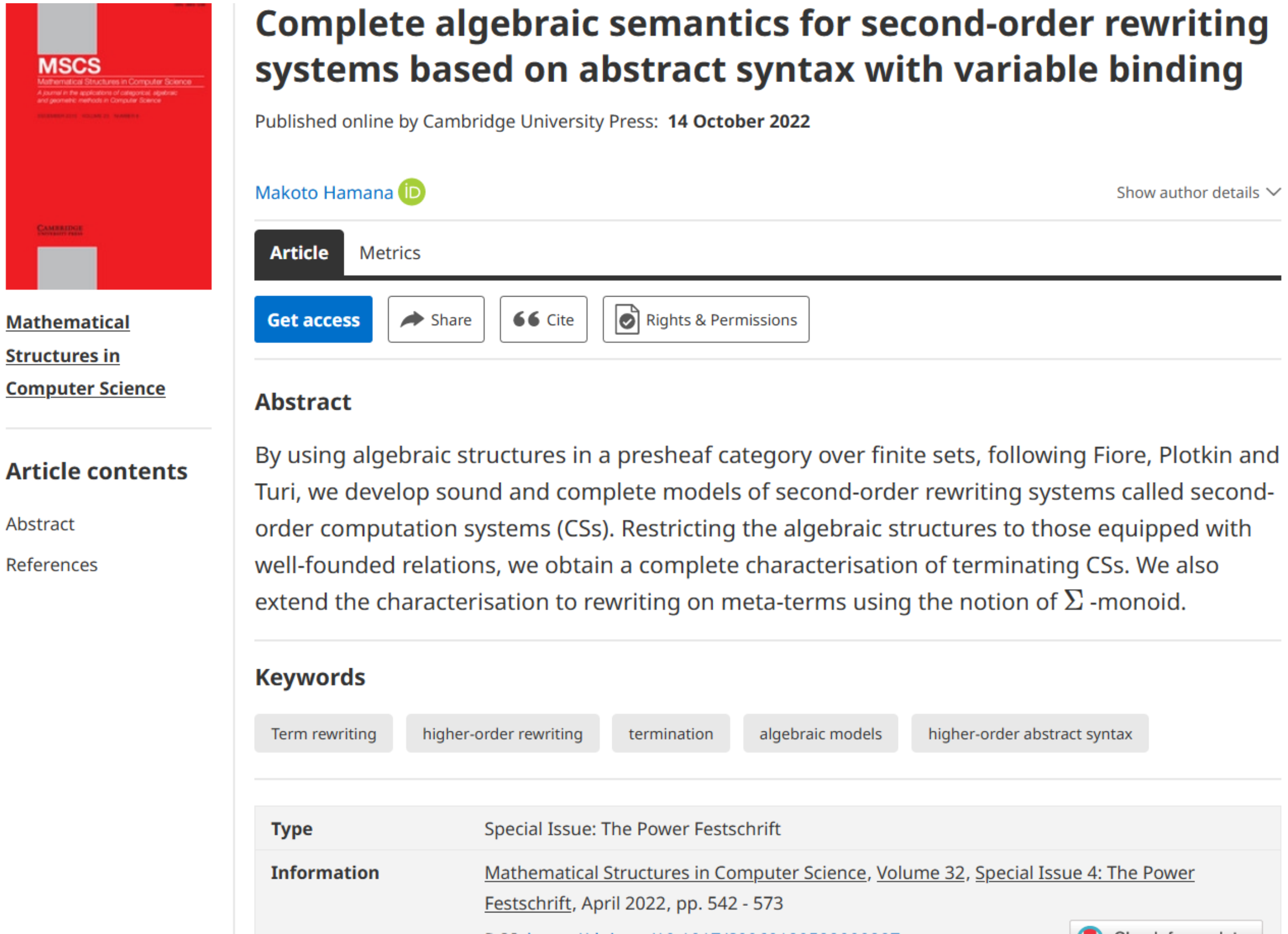
Term rewriting | higher-order rewriting | termination | algebraic models | higher-order abstract syntax

Type Special Issue: The Power Festschrift

Information Mathematical Structures in Computer Science, Volume 32, Special Issue 4: The Power Festschrift, April 2022, pp. 542 - 573

Summary

- Complete algebraic semantics of second-order rewriting systems
- Based on my paper
  - Complete Algebraic Semantics for Second-Order Rewriting Systems based on Abstract Syntax with Variable Binding
  - MSCS, CUP, 2022,
    Special Issue of John Power Festschrift
- Short history: I visted LFCS, Edinburgh in 1999-2000 as a JSPS postdoc.
- Thanks to John Power, Gordon Plotkin



Mathematical Structures in Computer Science

Article contents

Abstract

References

Complete algebraic semantics for second-order rewriting systems based on abstract syntax with variable binding

Published online by Cambridge University Press: **14 October 2022**

Makoto Hamana

Show author details

Article | Metrics

Get access | Share | Cite | Rights & Permissions

Abstract

By using algebraic structures in a presheaf category over finite sets, following Fiore, Plotkin and Turi, we develop sound and complete models of second-order rewriting systems called second-order computation systems (CSs). Restricting the algebraic structures to those equipped with well-founded relations, we obtain a complete characterisation of terminating CSs. We also extend the characterisation to rewriting on meta-terms using the notion of Σ-monoid.

Keywords

Term rewriting | higher-order rewriting | termination | algebraic models | higher-order abstract syntax

Type	Special Issue: The Power Festschrift
Information	Mathematical Structures in Computer Science, Volume 32, Special Issue 4: The Power Festschrift, April 2022, pp. 542 - 573

Summary

- Complete algebraic characterisation of second-order rewriting systems
- using algebraic models of second-order abstrax syntax

Further Topics and Applications

- Meta-rewriting: rewriting on meta-terms using monotone $\mathbf{\Sigma}$-monoids
- **Modularity** of Termination for Second-Order rewriting [H. LMCS'21]
  $\mathbf{A}$: terminating & $\mathbf{B}$ terminating $\quad\Rightarrow\quad$ $\mathbf{A} \uplus \mathbf{B}$: terminating
  with several conditions
- **Tool** SOL for termination and confluence checking
  1st places in the Higher-order Category of
  - International Confluence Competition 2020
  - Termination Competition 2022

  `http://solweb.mydns.jp/webcui/sol/`