The Legendre-Fenchel transform from a category theoretic perspective

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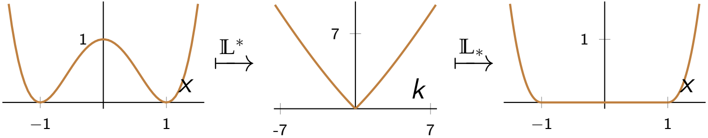
Legendre-Fenchel transform

V a real vector space, $V^{\#}$ is its linear dual, $\overline{\mathbb{R}} := [-\infty, +\infty]$.
There is a standard pair of transforms between function spaces:

$$\mathbb{L}^*\colon \operatorname{Fun}(V, \overline{\mathbb{R}}) \rightleftarrows \operatorname{Fun}(V^{\#}, \overline{\mathbb{R}})\colon \mathbb{L}_*,$$

$$\mathbb{L}^*(f)(k) := \sup_{x \in V} \left\{ \langle k, x \rangle - f(x) \right\}, \quad \mathbb{L}_*(g)(x) := \sup_{k \in V^{\#}} \left\{ \langle k, x \rangle - g(k) \right\}.$$



The image is always a (lower semicontinuous) convex function.
The composites $\mathbb{L}_* \circ \mathbb{L}^*$ and $\mathbb{L}^* \circ \mathbb{L}_*$ are **convex hull** operators.
We get an isomorphism between the sets of convex functions:

$$\operatorname{Cvx}(V, \overline{\mathbb{R}}) \cong \operatorname{Cvx}(V^{\#}, \overline{\mathbb{R}}).$$

$\overline{\mathbb{R}}$-metric structure

$\mathsf{Fun}(V, \overline{\mathbb{R}})$ has an "asymmetric metric with possibly negative distances":

$$\mathrm{d}\colon \mathsf{Fun}(V, \overline{\mathbb{R}}) \times \mathsf{Fun}(V, \overline{\mathbb{R}}) \to \overline{\mathbb{R}}; \quad \mathrm{d}(f_1, f_2) := \sup_{x \in V} \{ f_2(x) - f_1(x) \}.$$

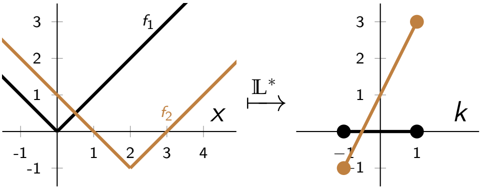
The Legendre-Fenchel transform is distance non-increasing:

$$\mathbb{L}^*\colon \mathsf{Fun}(V, \overline{\mathbb{R}}) \rightleftarrows \mathsf{Fun}(V^\#, \overline{\mathbb{R}})^{\mathrm{op}} \colon \mathbb{L}_* \,.$$

Theorem (Toland-Singer duality)

The Legendre-Fenchel transform gives an isomorphism of $\overline{\mathbb{R}}$-metric spaces:

$$\mathsf{Cvx}(V, \overline{\mathbb{R}}) \cong \mathsf{Cvx}(V^\#, \overline{\mathbb{R}})^{\mathrm{op}}.$$



$$\mathrm{d}(f_1, f_2) = 1 = \mathrm{d}(\mathbb{L}^*(f_2), \mathbb{L}^*(f_1))$$
$$\mathrm{d}(f_2, f_1) = 3 = \mathrm{d}(\mathbb{L}^*(f_1), \mathbb{L}^*(f_2))$$

Dualities and relations: Galois correspondences

Suppose that G and M are sets and $\mathcal{R}$ is a relation between them.
For example:

$$G = \text{some set of objects}, \quad M = \text{some set of attributes}$$
$$g \; \mathcal{R} \; m \text{ iff object } g \text{ has attribute } m$$

This gives rise to maps between the ordered sets of subsets

$$\mathcal{R}^* \colon \mathcal{P}(G) \leftrightarrows \mathcal{P}(M)^{\mathrm{op}} : \mathcal{R}_*$$

Both composites $\mathcal{R}_* \circ \mathcal{R}^*$ and $\mathcal{R}^* \circ \mathcal{R}_*$ are closure operators.
Restricts to an ordered isomorphism on the 'closed' subsets.

$$\mathcal{P}_{\mathrm{cl}}(G) \cong \mathcal{P}_{\mathrm{cl}}(M)^{\mathrm{op}}$$

Many classical dualities in mathematics arise in this way.

Monoidal categories

A monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ consists of a category $\mathcal{V}$ with a monoidal product $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and unit $\mathbb{1} \in \mathrm{Ob}(\mathcal{V})$, together with appropriate associativity and unit constraints.

category	objects	morphisms	$\otimes$	$\mathbb{1}$
Set	sets	functions	$\times$	$\{*\}$
Truth	$\{\mathrm{T}, \mathrm{F}\}$	$a \to b$ iff $a \vdash b$	$\&$	T
$\overline{\mathbb{R}_+}$	$[0, \infty]$	$a \to b$ iff $a \geq b$	$+$	0
$\overline{\mathbb{R}}$	$[-\infty, \infty]$	$a \to b$ iff $a \geq b$	$+$	0

Enriched categories

A category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathrm{Ob}(\mathcal{C})$ a specified set
$$\mathcal{C}(a, b)$$
- for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ a function
$$\circ_{a,b,c} \colon \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$$
- for each $a \in \mathrm{Ob}(\mathcal{C})$ an element
$$\mathrm{id}_a \in \mathcal{C}(a, a)$$

satisfying appropriate associativity and identity constraints.

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- for each $a \in \mathrm{Ob}(\mathcal{C})$ a morphism in Set
$$\mathrm{id}_a \colon \{*\} \to \mathcal{C}(a, a)$$

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Enriched categories

A $\mathcal{V}$-category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathrm{Ob}(\mathcal{C})$ a specified object
$$\mathcal{C}(a, b) \in \mathrm{Ob}(\mathcal{V})$$
- for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ a morphism in $\mathcal{V}$
$$\circ_{a,b,c} \colon \mathcal{C}(a, b) \otimes \mathcal{C}(b, c) \to \mathcal{C}(a, c)$$
- for each $a \in \mathrm{Ob}(\mathcal{C})$ a morphism in $\mathcal{V}$
$$\mathrm{id}_a \colon \mathbb{1} \to \mathcal{C}(a, a)$$

satisfying appropriate associativity and identity constraints.

Enriched categories

A Truth-category $\mathcal{C}$ consists of a set $\mathsf{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathsf{Ob}(\mathcal{C})$ a specified truth value
$$\mathcal{C}(a, b) \in \{\mathrm{T}, \mathrm{F}\}$$
- for each $a, b, c \in \mathsf{Ob}(\mathcal{C})$ an entailment
$$\mathcal{C}(a, b) \,\&\, \mathcal{C}(b, c) \vdash \mathcal{C}(a, c)$$
- for each $a \in \mathsf{Ob}(\mathcal{C})$ an entailment
$$\mathrm{T} \vdash \mathcal{C}(a, a)$$

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A Truth-category is a preorder: write $a \leq b$ iff $\mathcal{C}(a, b) = \mathrm{T}$.
[Fails to be a poset as $(a \leq b) \,\&\, (b \leq a) \not\vdash a = b$.]

Enriched categories

A $\overline{\mathbb{R}}$-category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathrm{Ob}(\mathcal{C})$ a specified number
$$\mathcal{C}(a, b) \in [-\infty, \infty]$$
- for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ an inequality
$$\mathcal{C}(a, b) + \mathcal{C}(b, c) \geq \mathcal{C}(a, c)$$
- for each $a \in \mathrm{Ob}(\mathcal{C})$ an inequality
$$0 \geq \mathcal{C}(a, a)$$

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An $\overline{\mathbb{R}}$-category is a $\overline{\mathbb{R}}$-metric space: write $d(a, b) := \mathcal{C}(a, b)$.

More structure

Suppose $\mathcal{V}$ is particularly nice (braided, closed, complete and cocomplete). We can define a $\mathcal{V}$-category structure $[\mathcal{C}, \mathcal{V}]$ on the collection of $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{V}$.

$\mathcal{V}$	$\mathcal{V}$-functor	$\mathcal{C} \to \mathcal{V}$	$[\mathcal{C}, \mathcal{V}]$
Set	functor	copresheaf	category of copresheaves and natural transformations
Truth	order-preserving function	upper closed subset	poset of upper closed subsets ordered by inclusion
$\overline{\mathbb{R}}$	distance non-increasing map	$X \to [-\infty, \infty]$	$\mathrm{Fun}(X, \overline{\mathbb{R}})$ with sup-metric $\mathrm{d}(f_1, f_2) := \sup_x (f_2(x) - f_1(x))$

Generalizing the relation-to-duality idea

- $\mathcal{V}$, suitable category to enrich over,
- $\mathcal{C}$, a $\mathcal{V}$-category,
- $\mathcal{D}$, a $\mathcal{V}$-category,
- $P\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$, a $\mathcal{V}$-functor (i.e. profunctor from $\mathcal{C}$ to $\mathcal{D}$).

Get an adjunction of $\mathcal{V}$-categories

$$P^*\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{D}, \mathcal{V}]^{\mathrm{op}} \colon P_*$$

which restricts to an equivalence of $\mathcal{V}$-categories

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_{\mathrm{cl}} \cong [\mathcal{D}, \mathcal{V}]^{\mathrm{op}}_{\mathrm{cl}}.$$

This is Pavlovic's profunctor nucleus.

The examples of interest

- $\mathcal{V} = \mathrm{Truth}$
- $\mathcal{C} = G$ a set, i.e. a discrete preorder,
- $\mathcal{D} = M$ a set, i.e. a discrete preorder,
- $P = \mathcal{R}$ a relation $G \times M \to \{\mathrm{T}, \mathrm{F}\}$

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$$\mathcal{P}_{\mathrm{cl}}(G) \cong \mathcal{P}_{\mathrm{cl}}(M)^{\mathrm{op}}.$$

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- $\mathcal{C} = V$ a vector space, as a discrete $\overline{\mathbb{R}}$-space,
- $\mathcal{D} = V^{\#}$ a vector space, as a discrete $\overline{\mathbb{R}}$-space,
- P the canonical pairing $V \otimes V^{\#} \to \mathbb{R} \subset \overline{\mathbb{R}}$.

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We get all of the Legendre-Fenchel transform machinery.
In particular we get Toland-Singer duality, an isomorphism of $\overline{\mathbb{R}}$-spaces:

$$\mathsf{Cvx}(V, \overline{\mathbb{R}}) \cong \mathsf{Cvx}(V^{\#}, \overline{\mathbb{R}})^{\mathrm{op}}.$$

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