Dagger linear logic for categorical quantum mechanics

Robin Cockett, Cole Comfort, and Priyaa Srinivasan

University of Calgary

Dagger compact closed categories

This work is available at **arXiv:1809.00275**

Dagger compact closed categories (†-KCC) provide a categorical framework to represent finite dimensional quantum processes.

What is a framework that supports infinite dimensional processes?

Dagger compact closed categories $\Rightarrow$ Finite-dimensionality on Hilbert Spaces. Because infinite dimensional Hilbert spaces are not compact closed.

One possibility is to drop the compact closure property and to consider † symmetric monoidal categories (†-SMC).

However, one loses the rich structure provided by the dualizing functor, $*$.

Dagger linear logic for quantum processes

Is there a way to generalize †-KCCs and still retain the goodness of the compact closed structure?

$*$-autonomous categories or more generally, linearly distributive categories (LDCs) generalize compact closed categories and allow for infinite dimensions.

? What is a dagger structure for LDCs?

? What are unitary isomorphisms in †-LDCs?

Linearly distributive categories

A **linearly distributive category (LDC)** has two monoidal structures $(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)$ and $(\oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$ linked by natural transformations called the linear distributors:

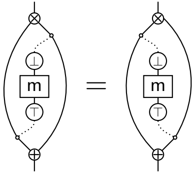
$$\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$
$$\partial_R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

LDCs are equipped with a graphical calculus.

LDCs provide a categorical semantics for multiplicative linear logic.

Mix categories

A **mix category** is a LDC with a mix map $\mathsf{m} : \perp \to \top$ in $\mathbb{X}$ such that

$$\mathsf{mx}_{A,B} : A \otimes B \longrightarrow A \oplus B :=$$ 

$$(1 \oplus (u_\oplus^L)^{-1})(1 \otimes (\mathsf{m} \oplus 1))\delta^L (u_\otimes^R \oplus 1) = ((u_\oplus^R)^{-1} \oplus 1)((1 \oplus \mathsf{m}) \otimes 1)\delta^R (1 \oplus u_\otimes^R)$$

mx is called a **mixor**. The mixor is a natural transformation.

It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the mixor an isomorphism.

The Core of mix category

The **core of a mix category**, $\mathrm{Core}(\mathbb{X}) \subseteq \mathbb{X}$, is the full subcategory determined by objects $U \in \mathbb{X}$ for which the natural transformations are isomorphisms:

$$U \otimes (\_) \xrightarrow{\mathsf{mx}_{U,(\_)}} U \oplus (\_) \qquad (\_) \otimes U \xrightarrow{\mathsf{mx}_{(\_),U}} (\_) \oplus U$$

The core of a mix category is closed to $\otimes$ and $\oplus$.

The core of an isomix category contains the monoidal units $\top$ and $\bot$.

A **compact LDC** is an LDC in which every mixor is an isomorphism i.e., in a compact LDC $\otimes \simeq \oplus$.

Compact LDCs $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ are linearly equivalent to underlying monoidal categories $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$.

Examples of mix categories

A monoidal category is trivially an isomix category: $\otimes = \oplus$

Finiteness spaces/matrices

Coherent spaces

$\mathrm{Chu}_I(X)$, The Chu construction over closed symmetric monoidal categories and the monoidal unit

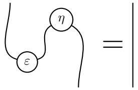
Linear duals

Suppose $\mathbb{X}$ is a LDC and $A, B \in \mathbb{X}$. Then, B is **left linear dual** $(\eta, \varepsilon) : B \dashv\!\!\vdash A$, if there exists

$$\eta : \top \to B \oplus A \qquad \qquad \varepsilon : A \otimes B \to \bot$$

such that the snake diagrams hold.

$$\begin{tikzcd}
A \arrow[r, "(u_\otimes^R)^{-1}"] \arrow[d, equal] & A \otimes \top \arrow[r, "1 \otimes \eta"] & A \otimes (B \oplus A) \arrow[d, "\partial_L"] \\
A & \bot \oplus A \arrow[l, "u_\oplus^L"] & (A \otimes B) \oplus A \arrow[l, "\varepsilon \oplus 1"]
\end{tikzcd}$$



A ***-autonomous category** is a category in which every object has a chosen left and right linear dual.

Forging the $\dagger$

The definition of $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ cannot be directly imported to LDCs because the dagger has to flip the tensor products:

$$(A \otimes B)^\dagger = A^\dagger \oplus B^\dagger$$

Why? If the dagger is identity-on-objects, then the linear distributor degenerates to an associator:

$$\frac{\delta_R : (A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)}{(\delta^R)^\dagger : A \oplus (B \otimes C) \longrightarrow (A \oplus B) \otimes C}$$

†-LDCs

A †-**LDC** is an LDC $\mathbb{X}$ with a dagger functor $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ and the natural isomorphisms:

$$
\begin{aligned}
\text{tensor laxors: } & \lambda_\oplus : A^\dagger \oplus B^\dagger \to (A \otimes B)^\dagger \\
& \lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger \\
\text{unit laxors: } & \lambda_\top : \top \to \bot^\dagger \\
& \lambda_\bot : \bot \to \top^\dagger \\
\text{involutor: } & \iota : A \to A^{\dagger\dagger}
\end{aligned}
$$

such that certain coherence conditions hold.

Coherences for †-LDCs

Coherences for the interaction between the tensor laxors and the basic natural isomorphisms (6 coherences):

$$\begin{tikzcd}
A^\dagger \otimes (B^\dagger \otimes C^\dagger) \arrow[r, "a_\otimes"] \arrow[d, "1\otimes\lambda_\otimes"'] & (A^\dagger \otimes B^\dagger) \otimes C^\dagger \arrow[d, "\lambda_\otimes\otimes 1"] \\
(A^\dagger \otimes (B \oplus C)^\dagger) \arrow[d, "\lambda_\otimes"'] & (A \oplus B)^\dagger \otimes C^\dagger \arrow[d, "\lambda_\otimes"] \\
(A \oplus (B \oplus C))^\dagger \arrow[r, "(a_\oplus^{-1})^\dagger"'] & ((A \oplus B) \oplus C)^\dagger
\end{tikzcd}$$

Coherences for †-LDCs (cont.)

Interaction between the unit laxors and the unitors (4 coherences):

$$\begin{tikzcd} \top \otimes A^\dagger \arrow[r, "\lambda_\top \otimes 1"] \arrow[d, "u^l_\otimes"'] & \bot^\dagger \otimes A^\dagger \arrow[d, "\lambda_\otimes"] \\ A^\dagger \arrow[r, "(u^l_\oplus)^\dagger"'] & (\bot \oplus A)^\dagger \end{tikzcd} \qquad \begin{tikzcd} \bot \oplus A^\dagger \arrow[r, "\lambda_\bot \oplus 1"] \arrow[d, "u^l_\oplus"'] & \top^\dagger \oplus A^\dagger \arrow[d, "\lambda_\oplus"] \\ A^\dagger \arrow[r, "(u^l_\otimes)^\dagger"'] & (\top \otimes A)^\dagger \end{tikzcd}$$

Interaction between the involutor and the laxors (4 coherences):

$$\begin{tikzcd} A \oplus B \arrow[r, "\iota"] \arrow[d, "i \oplus i"'] & ((A \oplus B)^\dagger)^\dagger \arrow[d, "\lambda_\otimes^\dagger"] \\ (A^\dagger)^\dagger \oplus (B^\dagger)^\dagger \arrow[r, "\lambda_\oplus"'] & (A^\dagger \otimes B^\dagger)^\dagger \end{tikzcd} \qquad \begin{tikzcd} \bot \arrow[r, "\iota"] \arrow[dr, "\lambda_\bot"'] & (\bot^\dagger)^\dagger \arrow[d, "\lambda_\top^\dagger"] \\ & \top^\dagger \end{tikzcd}$$

$$\iota_{A^\dagger} = (\iota_A^{-1})^\dagger : A^\dagger \rightarrow A^{\dagger\dagger\dagger}$$

†-mix categories

A **†-mix category** is a †-LDC with $m : \bot \to \top$ such that:

$$\begin{tikzcd} \bot \arrow[r, "m"] \arrow[d, "\lambda_\bot"'] & \top \arrow[d, "\lambda_\top"] \\ \top^\dagger \arrow[r, "m^\dagger"'] & \bot^\dagger \end{tikzcd}$$

If m is an isomorphism, then $\mathbb{X}$ is a **†-isomix category**.

Lemma 1: The following diagram commutes in a mix †-LDC:

$$\begin{tikzcd} A^\dagger \otimes B^\dagger \arrow[r, "\mathsf{mx}"] \arrow[d, "\lambda_\otimes"'] & A^\dagger \oplus B^\dagger \arrow[d, "\lambda_\oplus"] \\ (A \oplus B)^\dagger \arrow[r, "\mathsf{mx}^\dagger"'] & (A \otimes B)^\dagger \end{tikzcd}$$

$\dagger$-mix categories

Lemma 2: Suppose $\mathbb{X}$ is a $\dagger$-mix category and $A \in \mathrm{Core}(\mathbb{X})$ then $A^\dagger \in \mathrm{Core}(\mathbb{X})$.

Proof: The natural transformation $A^\dagger \otimes X \xrightarrow{\mathsf{mx}} A^\dagger \oplus X$ is an isomorphism:

$$\begin{tikzcd}
A^\dagger \otimes X \arrow[r, "1 \otimes \iota"] \arrow[d, "\mathsf{mx}"'] \arrow[dr, phantom, "\text{nat. mx}"] & A^\dagger \otimes X^{\dagger\dagger} \arrow[r, "\lambda_\otimes"] \arrow[d, "\mathsf{mx}"'] \arrow[dr, phantom, "\text{Lemma 1}"] & (A \oplus X^\dagger)^\dagger \arrow[d, "\mathsf{mx}^\dagger"] \\
A^\dagger \oplus X \arrow[r, "1 \oplus \iota"'] & A^\dagger \oplus A^{\dagger\dagger} \arrow[r, "\lambda_\oplus"'] & (A \otimes X^\dagger)^\dagger
\end{tikzcd}$$

commutes.

Example of a †-isomix category

Category of finite-dimensional framed vector spaces, FFVec_K

Objects: The objects are pairs $(V, \mathcal{V})$ where V is a finite dimensional K-vector space and $\mathcal{V} = \{v_1, ..., v_n\}$ is a basis;

Maps: These are vector space homomorphisms which ignore the basis information;

Tensor product:
$$(V, \mathcal{V}) \otimes (W, \mathcal{W}) = (V \otimes W, \{v \otimes w | v \in \mathcal{V}, w \in \mathcal{W}\})$$

Tensor unit: $(K, \{e\})$ where e is the unit of the field K.

Example (cont.)

To define the "dagger" we assume that the field has an involution $\overline{(\_)} : K \to K$, that is a field homomorphism with $k = \overline{(\overline{k})}$.

This involution then can be extended to a (covariant) functor:

$$\overline{(\_)} : \mathsf{FFVec}_K \longrightarrow \mathsf{FFVec}_K; \quad \begin{array}{c}(V,\mathcal{V})\\ \big\downarrow f\\ (W,\mathcal{W})\end{array} \quad \mapsto \quad \begin{array}{c}\overline{(V,\mathcal{V})}\\ \big\downarrow \overline{f}\\ \overline{(W,\mathcal{W})}\end{array}$$

where $\overline{(V,\mathcal{V})}$ is the vector space with the same basis but the conjugate action $c \,\bar{\cdot}\, v = \overline{c} \cdot v$. $\overline{f}$ is the same underlying map.

Example (cont.)

FFVec_K is also a compact closed category with $(V, \mathcal{B})^* = (V^*, \{\widetilde{b_i} | b_i \in \mathcal{B}\})$ where

$$V^* = V \multimap K \quad \text{and} \quad \widetilde{b_i} : V \longrightarrow K; \left(\sum_j \beta_j \cdot b_j \right) \mapsto \beta_i$$

Hence, we have a contravariant functor $(\_)^* : \mathrm{FFVec}_K^{\mathrm{op}} \longrightarrow \mathrm{FFVec}_K$.

$$(V, \mathcal{B})^\dagger = \overline{(V, \mathcal{B})^*}$$

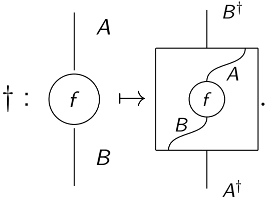
$$\iota : (V, \mathcal{V}) \longrightarrow ((V, \mathcal{V})^\dagger)^\dagger; v \mapsto \lambda f.f(v)$$

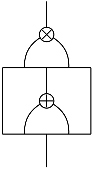
FFVec_K is a compact LDC: $\otimes$ and $\oplus$ coincides.

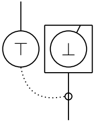
Diagrammatic calculus for †-LDC

Extends the diagrammatic calculus of LDCs

The action of dagger is represented diagrammatically using dagger boxes:



$\lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger =$ 

$\lambda_\top : \top \to \bot^\dagger =$ 

Next step: Unitary structure

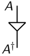
✓ Define †-LDC

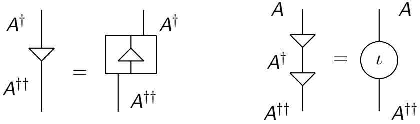
? Define unitary isomorphisms

The usual definition of unitary maps $(f^\dagger : B^\dagger \to A^\dagger = f^{-1} : B \longrightarrow A)$ is applicable only when the † functor is stationary on objects.

Unitary structure

A $\dagger$-isomix category has **unitary structure** in case there is an essentially small class of objects called **unitary objects** such that:

- Every unitary object, $A \in \mathcal{U}$, is in the core;
- Each unitary object $A \in \mathcal{U}$ comes equipped with an isomorphism, called the **unitary strucure** of A,  $: A \xrightarrow{\varphi_A} A^\dagger$ such that



$$\varphi_{A^\dagger} = ((\varphi_A)^{-1})^\dagger \qquad (\varphi_A \varphi_{A^\dagger}) = \iota$$

Unitary structure (cont.)

- $\top, \bot$ are unitary objects with:

$$\varphi_\bot = \mathsf{m}\lambda_\top \qquad \varphi_\top = \mathsf{m}^{-1}\lambda_\bot$$

- If A and B are unitary objects then $A \otimes B$ and $A \oplus B$ are unitary objects such that:

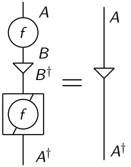
$$A \otimes B \xrightarrow[\simeq]{\varphi_{A\otimes B}} (A \otimes B)^\dagger \xrightarrow[\simeq]{\lambda_\oplus^{-1}} A^\dagger \oplus B^\dagger \xrightarrow[\simeq]{\varphi_A^{-1} \oplus \varphi_B^{-1}} A \oplus B = \mathsf{mx}$$

$$A \otimes B \xrightarrow[\simeq]{\varphi_A \otimes \varphi_B} (A^\dagger \otimes B^\dagger \xrightarrow[\simeq]{\lambda_\otimes} (A \oplus B)^\dagger \xrightarrow[\simeq]{\varphi_{(A\oplus B)}^{-1}} A \oplus B = \mathsf{mx}$$

Unitary isomorphisms

Suppose A and B are unitary objects. An isomorphism $A \xrightarrow{f} B$ is said to be a **unitary isomorphism** if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & A^\dagger \\ B \arrow[r, "\varphi_B"] & B^\dagger \arrow[u, "f^\dagger"'] \end{tikzcd}$$



Lemma: In a $\dagger$-isomix category with unitary structure,

- $f^\dagger$ is a unitary map iff f is;
- $f \otimes g$ and $f \oplus g$ are unitary maps whenever f and g are.
- $a_\otimes, a_\oplus, c_\otimes, c_\oplus, \delta^L$, m, and mx are unitary isomorphisms.
- $\lambda_\otimes, \lambda_\oplus, \lambda_\top, \lambda_\perp$, and ι are unitary isomorphisms.
- φ_A is a unitary isomorphisms for for all unitary objects A.

Unitary structure of FFVec_K

Unitary structure for FFVec_K is $\varphi_{(V,\mathcal{V})} : (V,\mathcal{V}) \longrightarrow (V,\mathcal{V})^\dagger; v_i \mapsto \widetilde{v_i}$

Define a functor $U : \mathrm{FFVec}_K \longrightarrow \mathrm{Mat}(K)$

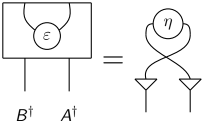
- for each object in FFVec_K we choose a total order on the elements of the basis
- any map is given by a matrix acting on the bases

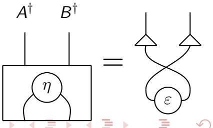
Lemma: An isomorphism $u : (A,\mathcal{A}) \longrightarrow (B,\mathcal{B})$ in FFVec_K is unitary if and only if $U(f)$ is unitary in $\mathrm{Mat}(K)$.

Unitary duals

A linear dual $(\eta, \varepsilon) : A \dashv\!\!\vdash_u B$ is a **unitary linear dual** if A and B are unitary objects satisfying in addition:

$$
\begin{tikzcd}
\top \arrow[rr, "\eta_A"] \arrow[d, "\lambda_\top"'] & & A \oplus B \arrow[d, "\varphi_A \oplus \varphi_B"] \\
\bot^\dagger \arrow[d, "\varepsilon^\dagger"'] & & A^\dagger \oplus B^\dagger \arrow[d, "c_\oplus"] \\
(B \otimes A)^\dagger \arrow[rr, "\lambda_\oplus^{-1}"'] & & B^\dagger \oplus A^\dagger
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[rr, "\varphi_A \otimes \varphi_B"] \arrow[d, "c_\otimes"'] & & A^\dagger \otimes B^\dagger \arrow[d, "\lambda_\otimes"] \\
B \otimes A \arrow[d, "\varepsilon_A"'] & & (A \oplus B)^\dagger \arrow[d, "\eta_A^\dagger"] \\
\bot \arrow[rr, "\lambda_\bot"'] & & \top^\dagger
\end{tikzcd}
$$





Unitary categories

A **unitary category** is a compact †-LDC in which every object is unitary.

Unitary categories are '†- linearly equivalent' to dagger monoidal categories.

A unitary category is **closed** if every object has a unitary dual.

Closed unitary categories are '†- linearly equivalent' to dagger compact closed categories

Mixed Unitary Categories

A **Mixed Unitary Category** is a

$$\boxed{\dagger\text{-isomix functor: Unitary category} \longrightarrow \dagger\text{-isomix category}}$$

The functor factors through the Core of the †-isomix category.

Examples: Finiteness matrices, Coherent spaces, Chu Spaces over involutive closed monoidal categories and the monoidal unit

Unitary construction

In a †-mix category a **pre-unitary object** is an object U, which in the core, together with an isomorphism:

$$\alpha : U \longrightarrow U^\dagger \text{ such that } \alpha(\alpha^{-1})^\dagger = \iota$$

Suppose $\mathbb{X}$ is a †-isomix category, then define Unitary($\mathbb{X}$):

Objects: Pre-unitary objects (U, α)

Maps: $(U, \alpha) \xrightarrow{f} (V, \beta)$ where $U \xrightarrow{f} V$ is in $\mathbb{X}$

Unitary construction (cont.)

$\otimes$ on objects: $(A, \alpha) \otimes (B, \beta) :=$

$$(A \otimes B,\ A \otimes B \xrightarrow{\mathsf{mx}} A \oplus B \xrightarrow{\alpha \oplus \beta} A^\dagger \oplus B^\dagger \xrightarrow{\lambda_\oplus} (A \otimes B)^\dagger)$$

Unit of $\otimes$: $(\top,\ \mathsf{m}^{-1}\lambda_\perp : \top \rightarrow \top^\dagger)$

$\oplus$ on objects: $(A, \alpha) \oplus (B, \beta) :=$

$$(A \oplus B,\ A \oplus B \xrightarrow{\mathsf{mx}^{-1}} A \otimes B \xrightarrow{\alpha \otimes \beta} A^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (A \oplus B)^\dagger$$

Unit of $\oplus$: $(\perp,\ \mathsf{m}\lambda_\top : \perp \rightarrow \perp^\dagger)$

Unitary construction (cont.)

Lemma: Unitary($\mathbb{X}$) is a unitary category.
Proof:

$$(U, \alpha)^\dagger := (U^\dagger, (\alpha^{-1})^\dagger)$$

$(U, \alpha)^\dagger \in \text{Unitary}(X)$:

$$(\alpha^{-1})^\dagger(((\alpha^{-1})^\dagger)^{-1})^\dagger = (\alpha^{-1})^\dagger(\alpha^\dagger)^\dagger = (\alpha^\dagger\alpha^{-1})^\dagger = (\iota^{-1})^\dagger = \iota$$

Every object (U, α) has an obvious unitary structure:

$$(U, \alpha) \xrightarrow{\alpha} (U^\dagger, (\alpha^{-1})^\dagger)$$

Proposition: If $\mathbb{X}$ is any $\dagger$-isomix category, then Unitary($\mathbb{X}$) is a unitary category with a full and faithful underlying $\dagger$-isomix functor to $U : \text{Unitary}(\mathbb{X}) \to \mathbb{X}$.

Summary

A Mixed Unitary Category (a MUC) is:

$$\boxed{\dagger\text{-isomix functor: Unitary category} \longrightarrow \dagger\text{-isomix category}}$$

Unitary categories are compact †-LDCs in which every object is unitary.

Unitary categories are '†-linearly equivalent' to † monoidal categories.

The dagger functor is non-stationary on objects in †-isomix categories.

Bibliography

LDC: Robin Cockett, and Robert Seely. **Weakly distributive categories.** Journal of Pure and Applied Algebra 114.2 (1997): 133-173.

The core of a mix category: Richard Blute, Robin Cockett, and Robert Seely. **Feedback for linearly distributive categories: traces and fixpoints.** Journal of Pure and Applied Algebra 154.1-3 (2000): 27-69.

Graphical calculus for LDCs: Richard Blute, Robin Cockett , Robert Seely, and Tood Trimble. **Natural deduction and coherence for weakly distributive categories.** Journal of Pure and Applied Algebra 113.3 (1996): 229-296.

Linear logic: Girard, Jean-Yves. **Linear logic.** Theoretical computer science 50.1 (1987): 1-101.