A compositional approach to quantum functions

David Reutter

University of Oxford

First Symposium on Compositional Structures
University of Birmingham

20 September, 2018

Overview

This talk is based on joint work with Ben Musto and Dominic Verdon:

A compositional approach to quantum functions

The Morita theory of quantum graph isomorphisms

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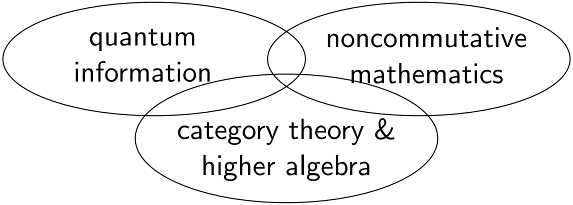

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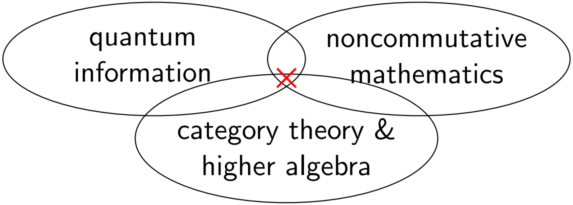

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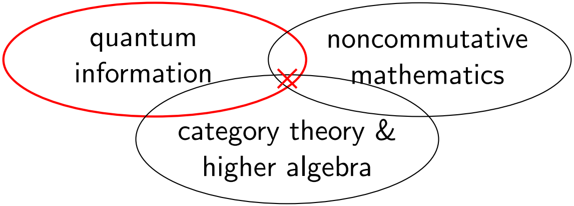

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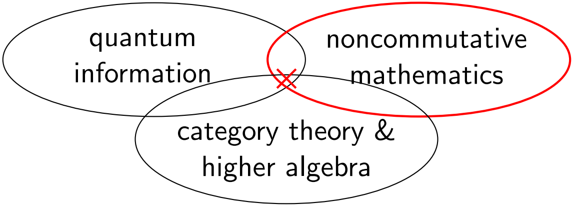

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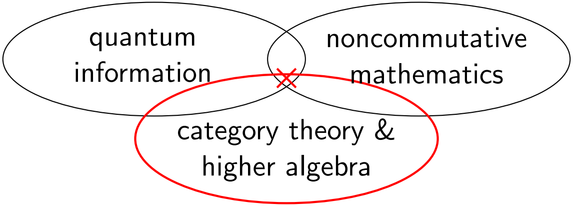

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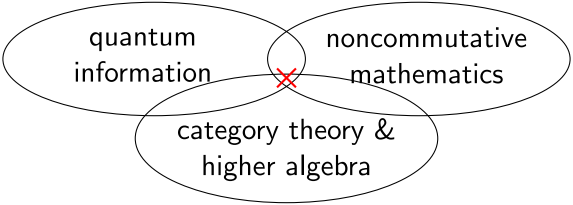

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Part 1: Getting started

Part 2: Quantum functions

Part 3: Classifying quantum isomorphic graphs

Part 1
Getting started

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A perfect winning strategy is a graph isomorphisms.

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Example (Quantum graph isomorphism game [1])

Let G and H be graphs.
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There are graphs that are quantum but not classically isomorphic!

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Quantum graph isomorphisms — the algebra

A quantum graph isomorphism between graphs G and H is:
A matrix of projectors $\{P_{xy}\}_{x\in V(G),y\in V(H)}$ on a Hilbert space $\mathcal{H}$ such that:

$$P_{xy}P_{xy'} = \delta_{y,y'}P_{xy} \qquad \sum_{y\in V(H)} P_{xy} = \mathrm{id}_{\mathcal{H}}$$

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Are there also notions of quantum bijections? Quantum functions?

What is quantum set and quantum graph theory?

Setting the stage

The stage:
Hilb — the category of finite-dimensional Hilbert spaces and linear maps.

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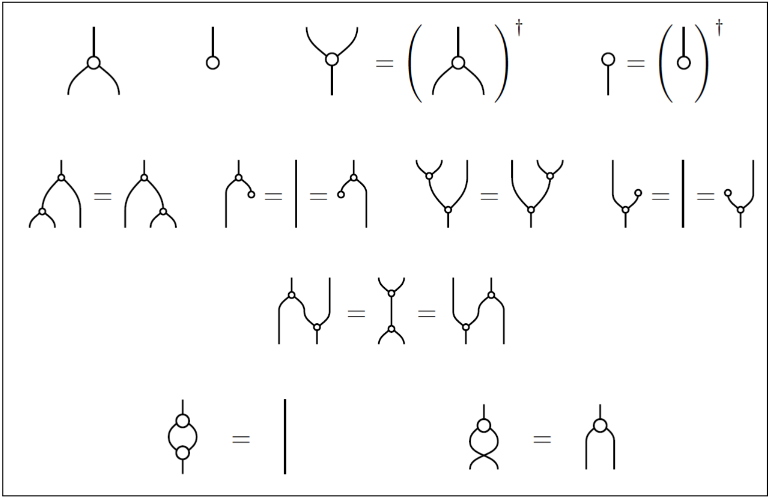

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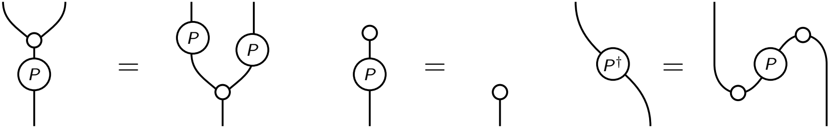
Philosophy: Do finite set theory with string diagrams in Hilb.

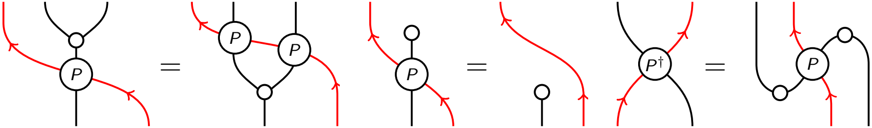
Part 2
Quantum functions

Function P between finite sets:



Quantizing functions

Quantum function $(\mathcal{H}, P)$ between finite sets:



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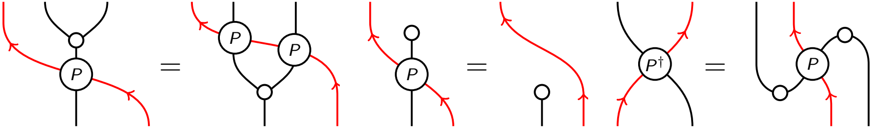

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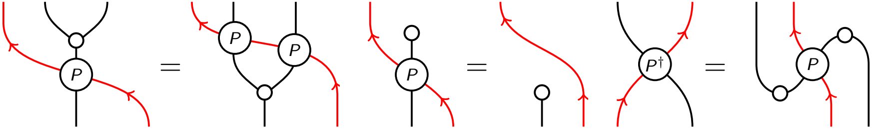

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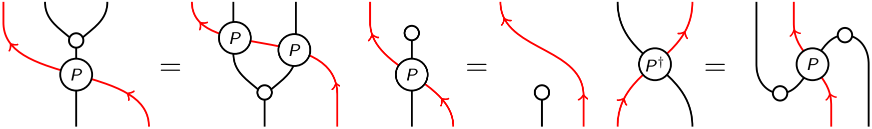

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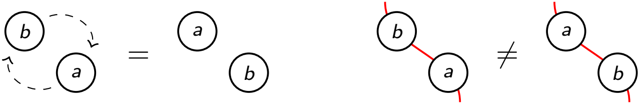

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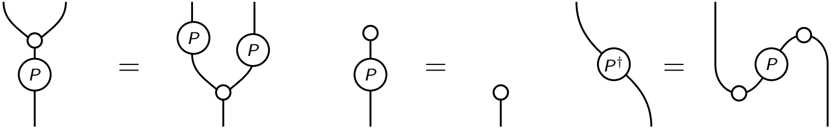

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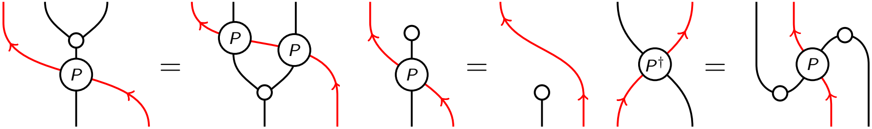

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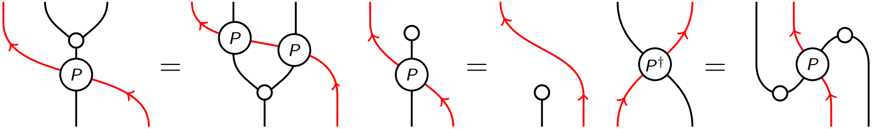

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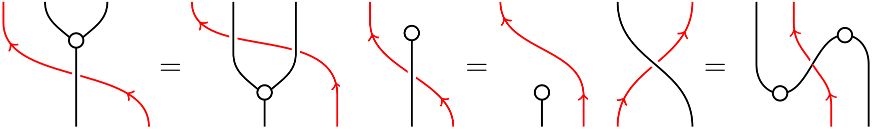

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These look like the equations satisfied by a braiding.

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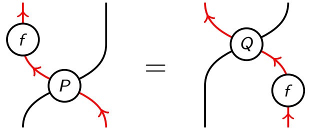
a linear map $f : \mathcal{H} \to \mathcal{H}'$ such that 

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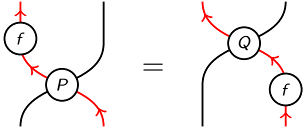

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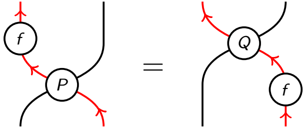

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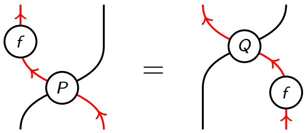

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$\mathrm{Set}(A, B)$: Set of functions between finite sets A and B

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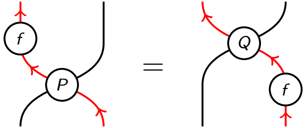

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$\mathrm{QSet}(A, B)$: Category of quantum functions between finite sets A and B

The 2-category QSet

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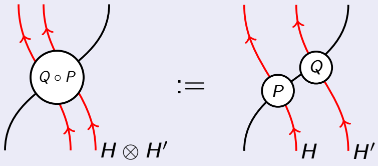
Definition

The 2-category QSet is built from the following structures:

- **objects** are finite sets $A, B, \ldots$;
- **1-morphisms** $A \to B$ are quantum functions $(H, P) : A \to B$;
- **2-morphisms** $(H, P) \to (H', P')$ are intertwiners

The composition of two quantum functions $(H, P) : A \to B$ and $(H', Q) : B \to C$ is a quantum function $(H \otimes H', Q \circ P)$ defined as follows:



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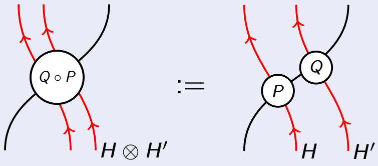

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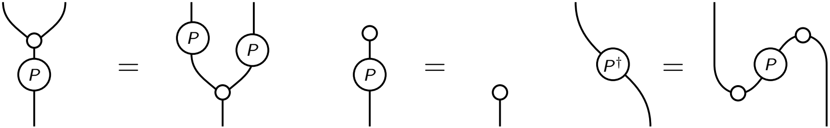
Can be extended to also include 'non-commutative sets' as objects.

Quantum bijections

Function P between finite sets:



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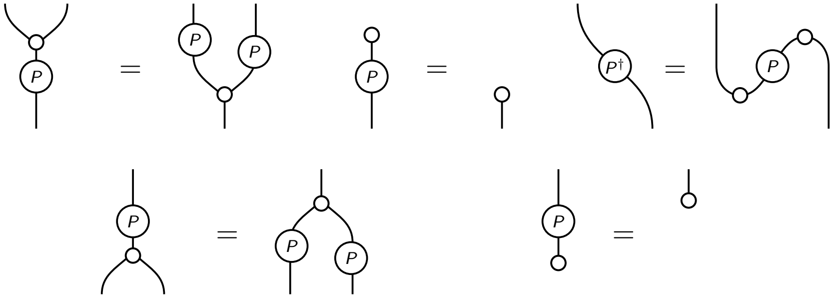

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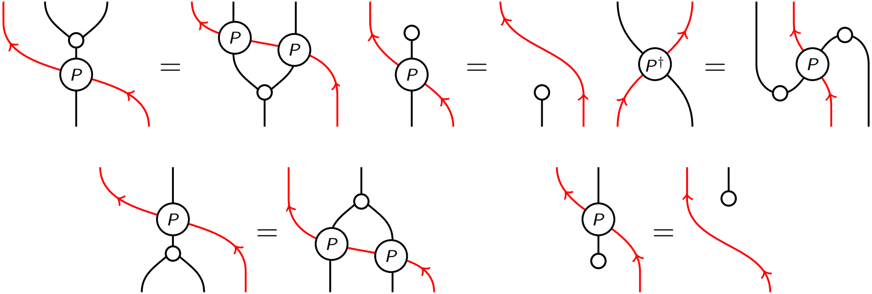

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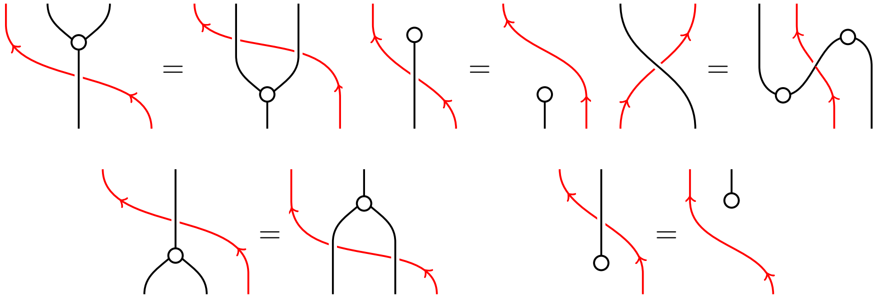

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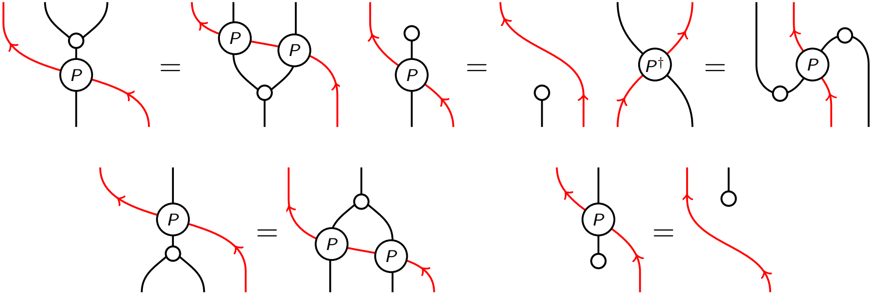

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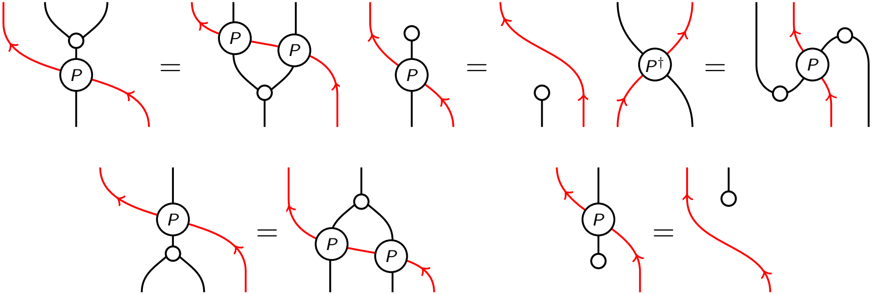

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Quantum bijections are not invertible but only dualizable quantum functions.

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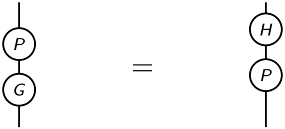

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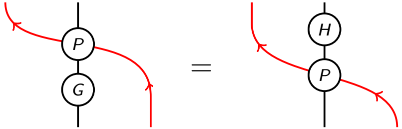

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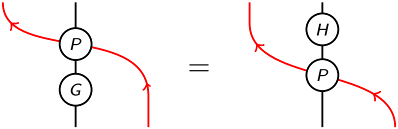

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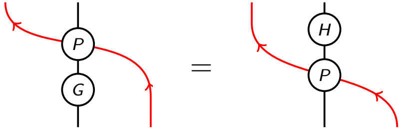

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The 2-category QGraph is built from the following structures:

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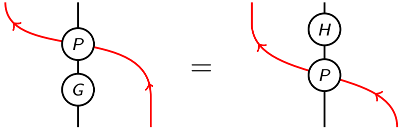

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Can we understand quantum isomorphisms in terms of the quantum automorphism categories $\mathrm{QAut}(G)$?

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Part 3
Classifying quantum isomorphic graphs

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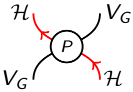

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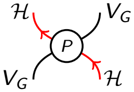

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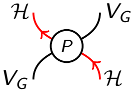

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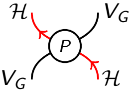

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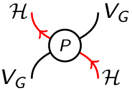

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drop commutativity condition $\rightsquigarrow$ classify quantum graphs [1,2]

[1] Weaver — Quantum graphs as quantum relations. 2015

[2] Duan, Severini, Winter — Zero error communication [...] theta functions. 2010

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Thanks for listening!

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If G has no quantum symmetries: get all quantum isomorphic graphs G'

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- Reconnect the disjoint components of $\sqcup_O G_O$ as follows:

$$v \in O \sim_{G'} w \in O' \quad \Leftrightarrow \quad h_{O,O}v \sim_G w$$