Monads, Partial Evaluations, and Rewriting

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SYCO 1, 2018

Idea

What do these things have in common?

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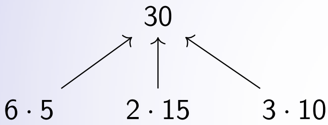

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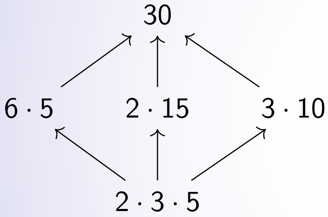

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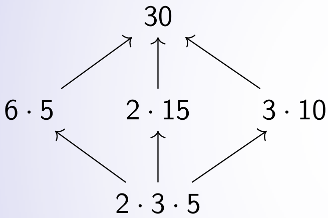

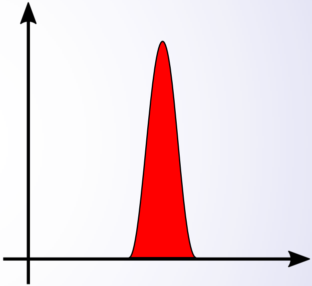

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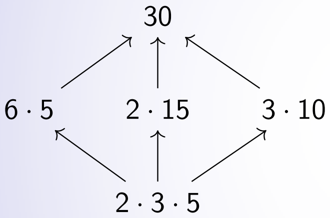

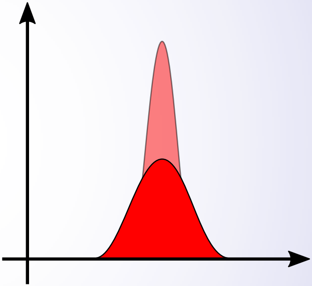

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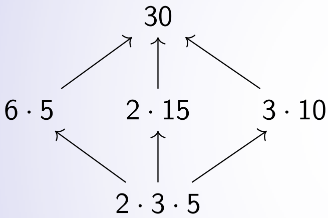

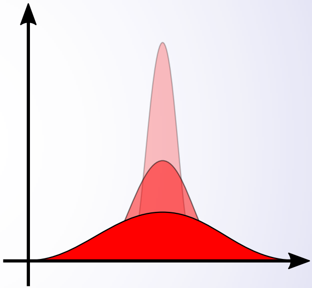

Monads and formal expressions

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Setting:

Let C be a *concrete category* and (T, μ, η) a monad with η monic.

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\begin{aligned}
X &= \{x\,,\ y\,,\ z\,,\ \dots\} \\
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TTX &= \{(x + y) + (x + z)\,,\ (x)\,,\ \dots\}
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$$f : X \to Y \quad \longmapsto \quad Tf\ :\ x + x' \mapsto f(x) + f(x')$$

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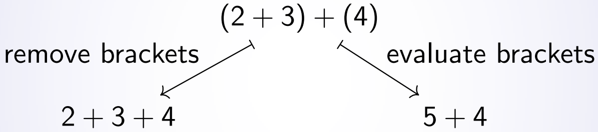
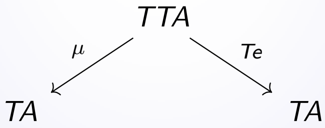
$$TA \xleftarrow{\;\mu\;} TTA \xrightarrow{\;Te\;} TA$$

Partial evaluations and partial decompositions

Definition:

Let $p, q \in TA$. If $\mu(m) = p$ and $(Te)(m) = q$ for some $m \in TTA$, we call q a *partial evaluation* of p and p a *partial decomposition* of q.





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$$\begin{array}{c} TTA \\ \uparrow T\eta \\ TA \end{array} \qquad\qquad \begin{array}{c} (2) + (3) \\ \upmapsto \\ 2 + 3 \end{array}$$

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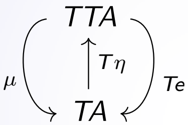

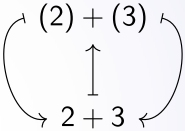

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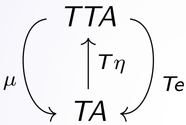

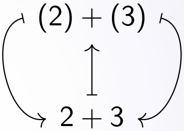

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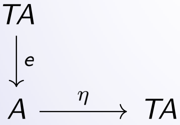

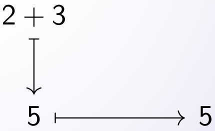

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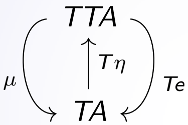

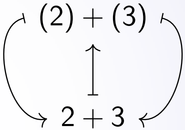

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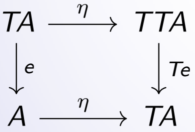

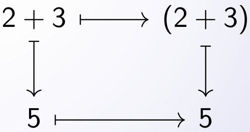

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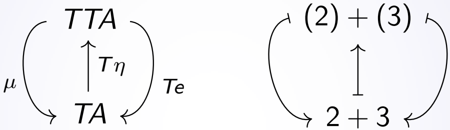

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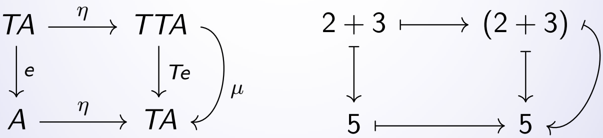

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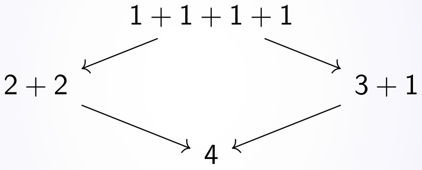

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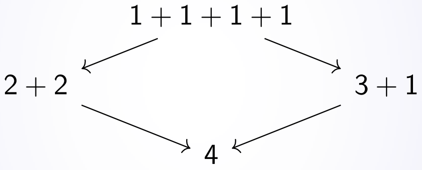

- The irreducible elements are the total evaluations.

Transitivity

Question:

Can partial evaluations be composed?

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$1 + 1 + 1 + 1$ $\qquad$ $2 + 2$ $\qquad$ 4

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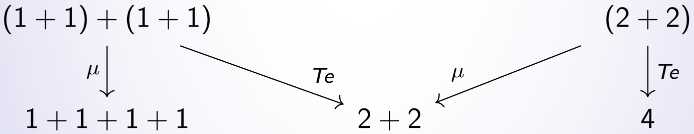

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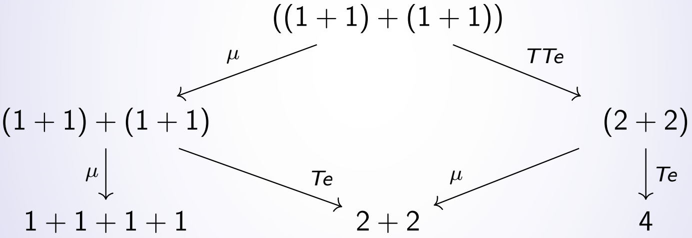

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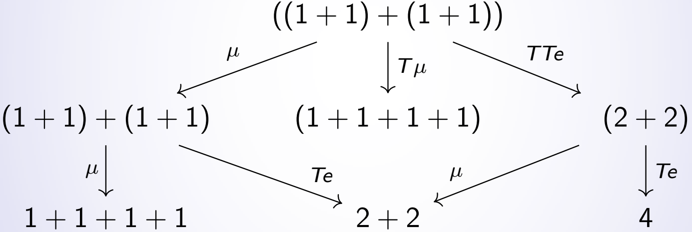

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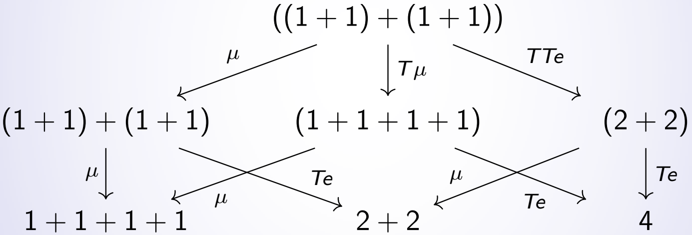

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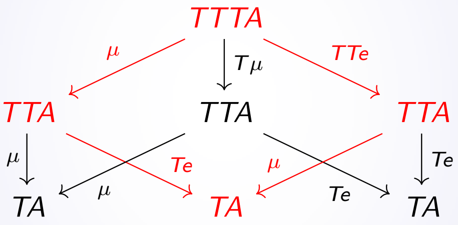

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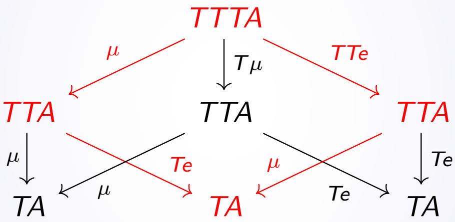

We are asking for the existence of a "rewriting of rewritings".

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Definition:

The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[n]{} & D \end{array}$$

is called a *weak* or *meek* pullback if for every $b \in B$ and $c \in C$ such that $m(b) = n(c)$ there exists an $a \in A$ such that $f(a) = b$ and $g(a) = c$.

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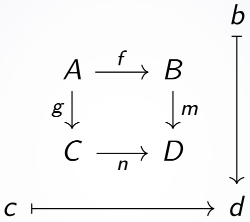

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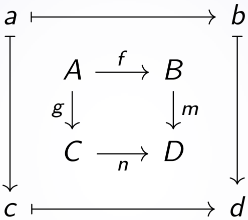

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A *cartesian monad* is a monad (T, η, μ) such that:

- T preserves pullbacks;
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Proposition:

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- For cartesian monads, composition is always *uniquely* defined.

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- Most monads of classical algebras are *not* weakly cartesian;
- As we prove, the Kantorovich probability monad is weakly cartesian (more on that later).

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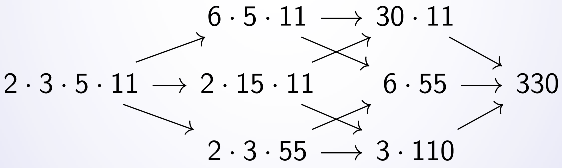

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Let $(g, x), (h, y) \in G \times A$.

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Let G be an internal monoid (or group) in a (cartesian) monoidal category.

- $X \mapsto G \times X$ is a monad;
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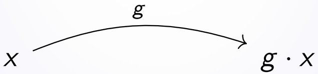

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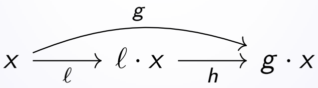

Probability monads

Idea [Giry, 1982]:

Spaces of random elements as formal convex combinations.

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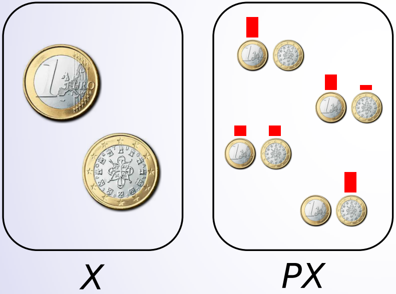

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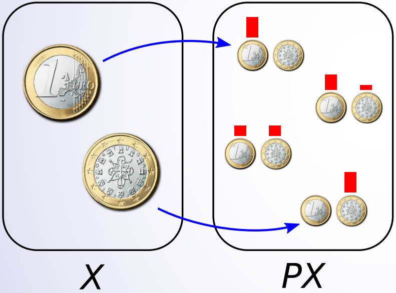

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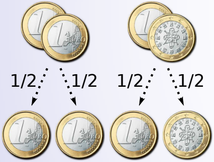

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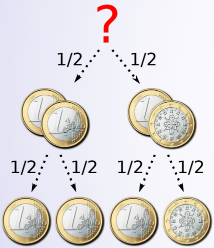

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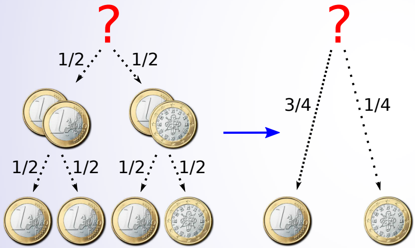

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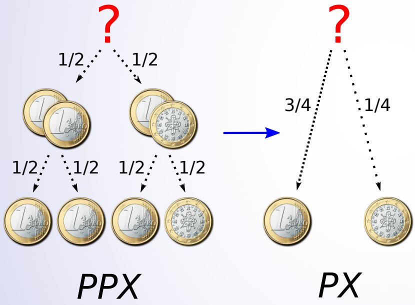

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- Composition $E : PPX \to PX$

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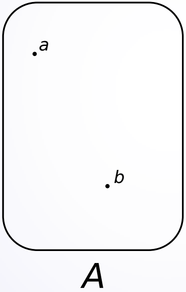

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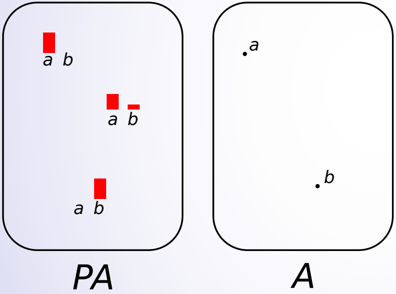

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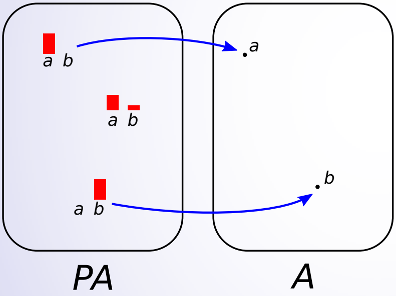

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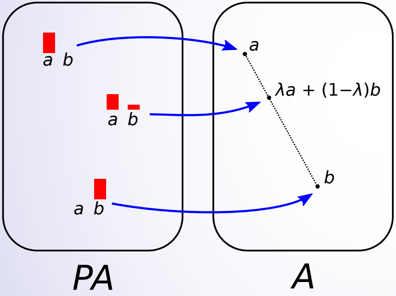

- Algebras $e : PA \to A$ are "convex spaces"
- Formal averages are mapped to actual averages

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Kantorovich monad [van Breugel, 2005, Fritz and Perrone, 2017]:

- Given a complete metric space X, PX is the set of Radon probability measures of finite first moment, equipped with the *Wasserstein distance*, or *Kantorovich-Rubinstein distance*, or *earth mover's distance*:

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- Algebras of P are closed convex subsets of Banach spaces.

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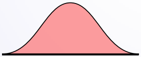

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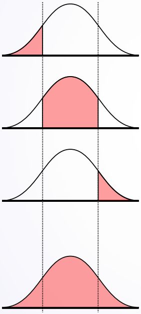

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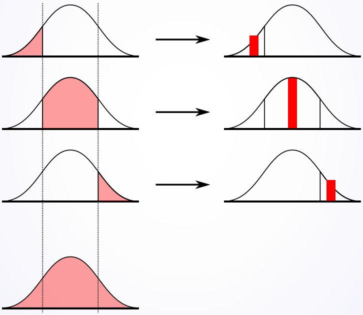

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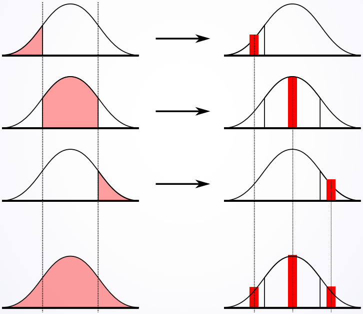

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2. *Partial expectations can always be composed (not uniquely)*;
3. The relation on PA induced by partial evaluations is a closed partial order, which is known in the literature as the *Choquet* or *convex order*, used in statistics and finance [Winkler, 1985], [Rothschild and Stiglitz, 1970].

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Theorem, extending [Winkler, 1985, Theorem 1.3.6]

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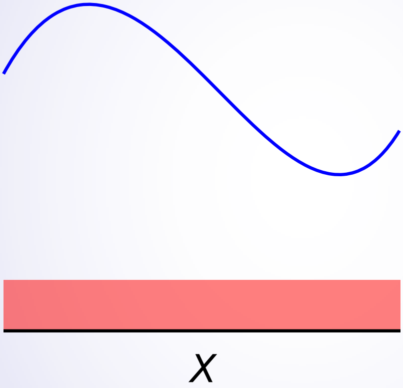

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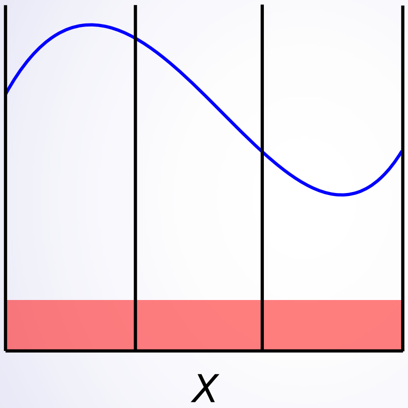

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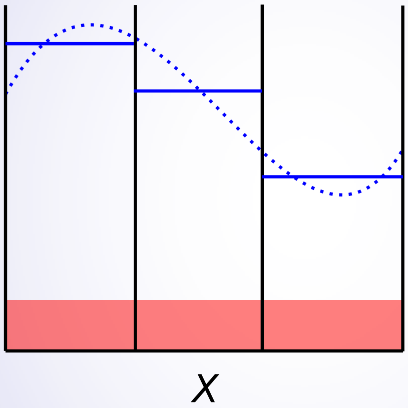

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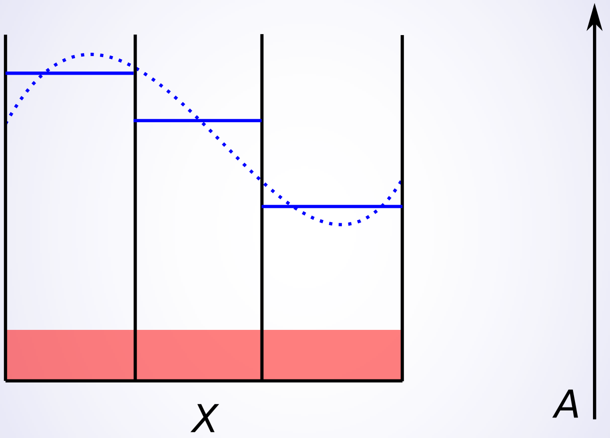

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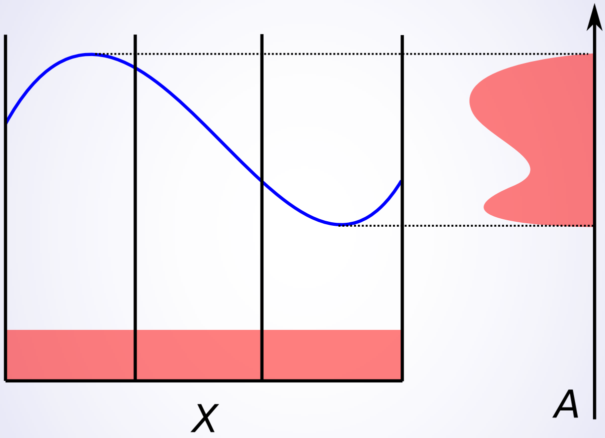

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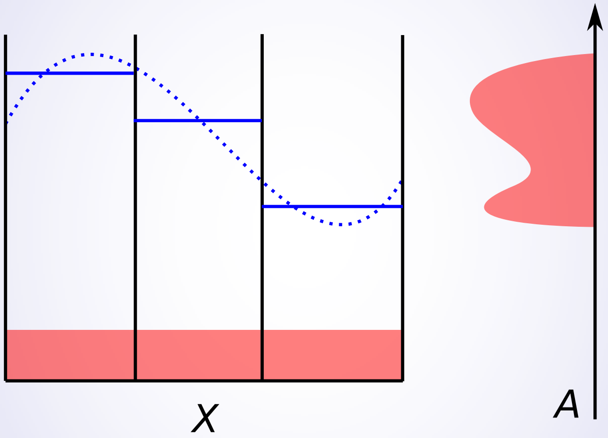

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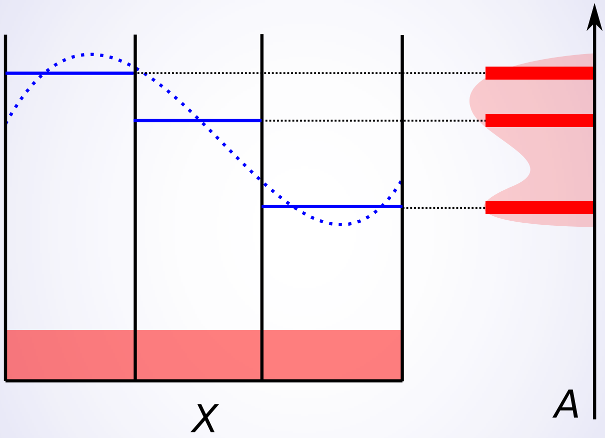

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Corollary

A chain of composable partial decompositions in PA is (basically) the same as a *martingale* on A.

Towards higher rewritings (work in progress)

A

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$$TA \xrightarrow{e} A$$

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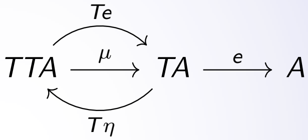

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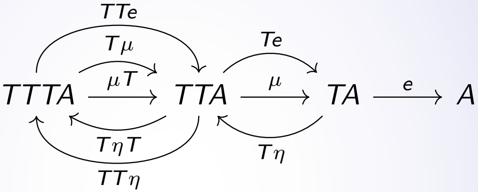

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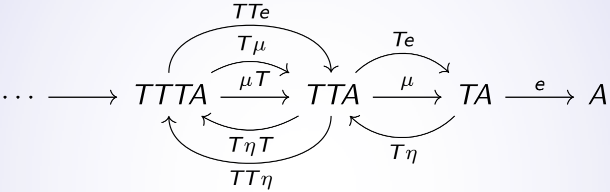

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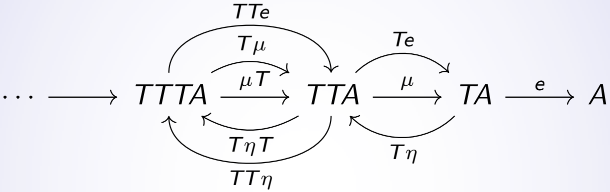

- The partial evaluation rewriting system is the 1-dimensional truncation of a simplicial set.

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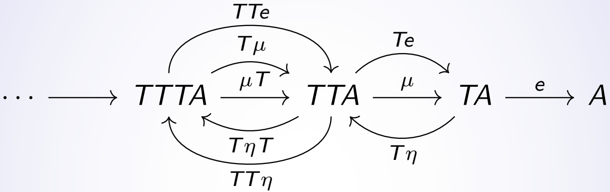

- The partial evaluation rewriting system is the 1-dimensional truncation of a simplicial set.
- Composition is a 2-simplex of $TTTA$, which can be seen as a Kan filler condition for inner 2-horns.

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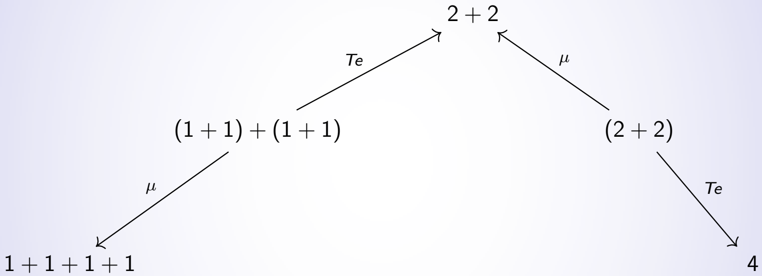

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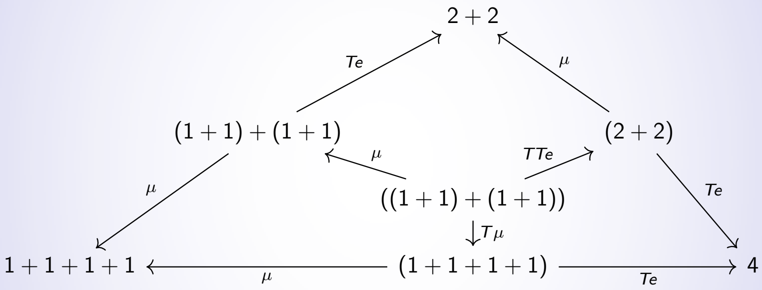

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Joint work with Tobias Fritz

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and Sharwin Rezagholi
(MPI MIS Leipzig)

References

Book, R. V. and Otto, F. (1993).
String Rewriting Systems.
Springer.

Clementino, M. M., Hofmann, D., and Janelidze, G. (2014).
The monads of classical algebra are seldom weakly cartesian.
J. Homotopy Relat. Struct., 9:175–197.

Fritz, T. and Perrone, P. (2017).
A Probability Monad as the Colimit of Finite Powers.
Submitted. arXiv:1712.05363.

Giry, M. (1982).
A Categorical Approach to Probability Theory.
In *Categorical aspects of topology and analysis*, volume 915 of *Lecture Notes in Mathematics*.

Hyland, M. and Power, J. (2007).
The category-theoretic understanding of universal algebra: Lawvere theories and monads.
ENTCS.

Perrone, P. (2018).
Categorical Probability and Stochastic Dominance in Metric Spaces.
PhD thesis.
Submitted. www.paoloperrone.org/phdthesis.pdf.

Rothschild, M. and Stiglitz, J. E. (1970).
Increasing risk: I. A definition.
Journal of Economic Theory, 2:225–243.

van Breugel, F. (2005).
The Metric Monad for Probabilistic Nondeterminism.

Weber, M. (2004).
Generic morphisms, parametric representations and weakly cartesian monads.
Theory and Applications of Categories, 13(14):191–234.

Winkler, G. (1985).
Choquet order and simplices with applications in probabilistic models.
Lecture Notes in Mathematics. Springer.

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