Quantum Sets

Andre Kornell

University of California, Davis

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two categories

Fun: sets and functions
qFun: quantum sets and quantum functions

$$\mathrm{Inc} \quad \dashv \quad \mathrm{Pts}$$

$$\mathbf{Fun} \xhookrightarrow{\mathrm{Inc}} \mathbf{qFun} \xrightarrow{\mathrm{Pts}} \mathbf{Fun}, \qquad \mathrm{Pts} \circ \mathrm{Inc} \underset{\mathrm{Id}}{\cong} \mathrm{Id}$$

Inc is full and faithful

quantum sets

definition

A quantum set $\mathcal{X}$ is a set of nonzero finite-dimensional Hilbert spaces.

$$\mathrm{Inc}(S) = {}`S = \left\{ \mathbb{C}^{\{s\}} \mid s \in S \right\}$$

$$\mathrm{Pts}(\mathcal{X}) = \{X \in \mathcal{X} \mid \dim(X) = 1\}$$

qFun has terminal object $\mathbf{1} = \{\mathbb{C}\} \cong {}`\{*\}$.
$\mathrm{Pts}(\mathcal{X}) \cong \mathrm{qFun}(\mathbf{1}, \mathcal{X})$

$+$ and $\times$

Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets.

definition

Cartesian product $\mathcal{X} \times \mathcal{Y} \; = \; \{X \otimes Y \,|\, X \in \mathcal{X},\; Y \in \mathcal{Y}\}$

$\mathcal{X} \times \mathcal{Y}$ is **not** the product of $\mathcal{X}$ and $\mathcal{Y}$

definition

disjoint union $\mathcal{X} + \mathcal{Y} \; = \; (\mathcal{X} \times `\{1\}) \cup (\mathcal{Y} \times `\{2\})$

$\mathcal{X} + \mathcal{Y}$ is the coproduct of $\mathcal{X}$ and $\mathcal{Y}$

$$\mathrm{Inc}(S + T) = \mathrm{Inc}(S) + \mathrm{Inc}(T) \qquad \mathrm{Pts}(S + T) = \mathrm{Pts}(S) + \mathrm{Pts}(T)$$

$$\mathrm{Inc}(S \times T) = \mathrm{Inc}(S) \times \mathrm{Inc}(T) \qquad \mathrm{Pts}(S \times T) = \mathrm{Pts}(S) \times \mathrm{Pts}(T)$$

quantum functions

expository definition

A quantum function F from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ assigns to each element X of $\mathcal{X}$ a unitary operator

$$X \quad \cong \quad (H^1 \otimes Y_1) \oplus (H^2 \otimes Y_2) \oplus \cdots \oplus (H^n \otimes Y_n),$$

up to unitary equivalence of the coefficients $H^1, \ldots, H^n$.

example: qubit measurement

$\mathcal{X} = \{\mathbb{C}^2\}$ $\qquad$ $\mathcal{X} \xrightarrow{F} {`S}$ $\qquad$ $S = \{\frac{1}{2}, -\frac{1}{2}\}$

$$\mathbb{C}^2 \quad \cong \quad (\mathbb{C} \otimes \mathbb{C}^{\{\frac{1}{2}\}}) \oplus (\mathbb{C} \otimes \mathbb{C}^{\{-\frac{1}{2}\}})$$

composing quantum functions

$$\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}$$

$$X \cong \bigoplus_i H^i \otimes Y_i$$

$$Y_i \cong \bigoplus_j K_i^j \otimes Z_j$$

$$\implies \quad X \cong \bigoplus_{i,j} H^i \otimes K_i^j \otimes Z_j \cong \bigoplus_j \left(\bigoplus_i H^i \otimes K_i^j \right) \otimes Z_j$$

qFun is like a topos

theorem (K)

The symmetric monoidal category $(\mathbf{qFun}, \times)$

1. has finite colimits,
2. has finite limits,
3. has a terminal monoidal unit,
4. is closed monoidal, and
5. classifies subobjects by "classical" quantum functions into $\mathbf{1} + \mathbf{1}$:

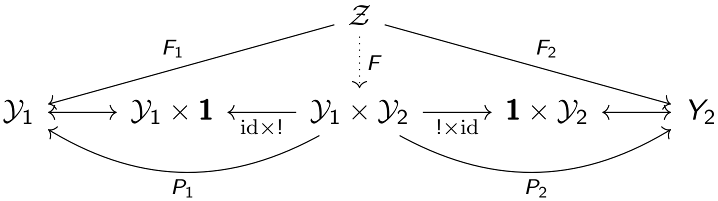
$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow T \\ \mathcal{X} & \overset{!}{\dashrightarrow} & \mathbf{1} + \mathbf{1} \end{array}$$

For a symmetric monoidal category $(\mathbf{C}, \times)$ satisfying (1) – (5):

$(\mathbf{C}, \times)$ is a topos $\iff$ $\times$ is a category-theoretic product

compatible quantum functions

we say that F_1 and F_2 are compatible just in case



definition

A quantum function out of $\mathcal{X}$ is classical iff it is compatible with every quantum function out of $\mathcal{X}$. A quantum set is classical iff $I_{\mathcal{X}}$ is classical.

classical quantum sets and classical quantum functions

proposition (K)

A quantum set $\mathcal{X}$ is classical iff there is a set S such that $\mathcal{X} \cong {`S}$.

proposition (K)

A quantum function $F\colon \mathcal{X} \to \mathcal{Y}$ iff there is a function $f\colon \mathcal{X} \to \mathrm{Pts}(\mathcal{Y})$ with

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;F\;} & \mathcal{Y} \\ {\scriptstyle Q}\downarrow\!\!\downarrow & & \uparrow{\scriptstyle J} \\ `\mathcal{X} & \xrightarrow[`f]{} & `\mathrm{Pts}(\mathcal{Y}) \end{array}$$

$X \quad \cong_Q \quad X \otimes \mathbb{C}^{\{X\}}$ $\qquad\qquad$ $\mathbb{C}^{\{Y\}} \quad \cong_J \quad \mathbb{C} \otimes Y$

von Neumann algebras

proposition (K)

There is a full and faithful contravariant functor $\ell^\infty_q \colon \mathbf{qFun} \to \mathbf{vNalg}$.

$$\ell_q(\mathcal{X}) = \prod_{X \in \mathcal{X}} L(X) \qquad \ell^\infty_q(\mathcal{X}) = \left\{ a \in \ell_q(\mathcal{X}) \,\middle|\, \sup_{X \in \mathcal{X}} \|a(X)\| < \infty \right\}$$

theorem (K)

Let A be a von Neumann algebra. The following are equivalent:

1. $A \cong \ell^\infty_q(\mathcal{X})$ for some quantum set $\mathcal{X}$
2. every von Neumann subalgebra of A is atomic
3. if $a^\dagger = a$, then there is an orthogonal family of projections $(p_\alpha | \alpha \in \mathbb{R})$

$$a = \sum_{\alpha \in \mathbb{R}} \alpha \cdot p_\alpha$$

internal ring of quantum complex numbers

Write $\mathcal{X} * \mathcal{Y}$ for category theoretic product of $\mathcal{X}$ and $\mathcal{Y}$. Write $\mathcal{C} = {}^{`}\mathbb{R} * {}^{`}\mathbb{R}$.

There are quantum functions

$$\mathcal{C} * \mathcal{C} \xrightarrow{+} \mathcal{C} \qquad \mathcal{C} * \mathcal{C} \xrightarrow{\cdot} \mathcal{C} \qquad \mathcal{C} \xrightarrow{\dagger} \mathcal{C} \qquad {}^{`}\mathbb{C} \hookrightarrow \mathcal{C}$$

such that the set $\mathrm{qFun}(\mathcal{X}, \mathcal{C})$ has the structure of a $\dagger$-algebra over $\mathbb{C}$.

proposition (K)

We have a natural isomorphism $\mathrm{qFun}(\mathcal{X}, \mathcal{C}) \cong \ell_q(\mathcal{X})$.

interlude: quantum relations

definition (essentially, Weaver)

A quantum relation R from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ assigns to each element X of $\mathcal{X}$ and each element Y of $\mathcal{Y}$ a subspace

$$R(X,Y) \leq L(X,Y)$$

Quantum relations correspond to quantum functions $\mathcal{X} \times \mathcal{Y}^* \to \mathbf{1} + \mathbf{1}$.

The category **qRel** of quantum sets and quantum relations is a dagger compact category enriched over ortholattices.

Definition

A quantum function from $\mathcal{X}$ to $\mathcal{Y}$ is a quantum relation such that

$$R^\dagger \circ R \geq I_\mathcal{X} \qquad R \circ R^\dagger \leq I_\mathcal{Y}$$

the graph coloring game

parameters: a finite simple graph G and a finite set S
players: Alice and Bob, cooperating blindly against a Referee

round 1: Referee plays a pair $(g_A, g_B) \in G \times G$
(Alice sees only g_A, and Bob sees only g_B)

round 2: Alice plays a color s_A and Bob plays a color s_B
(Alice sees only s_A and Bob sees only s_B)

scoring: Alice and Bob lose iff

$$(g_A = g_B \text{ and } s_A \neq s_B) \text{ or } (g_A \sim g_B \text{ and } s_A = s_B)$$

Alice and Bob have a winning strategy $\Leftrightarrow$ G can be properly colored by S

true if Alice and Bob share classical randomness
false if Alice and Bob share quantum randomness

from the graph coloring game to quantum functions

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quantum families of graph colorings

$$`G \times \mathcal{Z} \xrightarrow{F} `S$$

$$F \circ (E_G \otimes I_{\mathcal{Z}}) \quad \leq \quad (\neg I_S) \circ F$$

proposition

Alice and Bob have a winning strategy using quantum entanglement iff there is a quantum family of graph colorings of G by S.