An Allegorical Semantics of Modal Logic

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e.g. bisimulation theorems, correspondence theory, duality theory.

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- In effect, Kripke semantics will be extended to regular categories.

Outline

1. Recast Kripke semantics and its model theory using **Rel**.
2. Briefly review allegories.
3. Give allegorical semantics of modal logic, and model theory.

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- A Kripke frame, a set X plus $R_i : X \nrightarrow X$.
  Each R_i interprets $\Box_i$, $\Diamond_i$.
- A Kripke model, a frame (X, R_i) plus $[\![p]\!] \subseteq X$.
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$x \vDash \varphi$ "φ is true at x", for a world / state $x \in X$ and a formula φ.

$$
\begin{aligned}
x \vDash p &\iff x \in [\![p]\!] && \text{(via the model)},\\
x \vDash \varphi \wedge \psi &\iff x \vDash \varphi \text{ and } x \vDash \psi, &&\\
x \vDash \Box_i \varphi &\iff y \vDash \varphi \text{ for all } y \text{ s.th. } x R_i y && \text{(via the frame)},\\
x \vDash \Diamond_i \varphi &\iff y \vDash \varphi \text{ for some } y \text{ s.th. } x R_i y && \text{(via the frame)}.
\end{aligned}
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"Standard translation": "$x \models \varphi$" $\overset{\text{tr}}{\rightsquigarrow}$ $\varphi(x)$

$$\begin{aligned}
\text{tr}(p) &= Px, \\
\text{tr}(\varphi \wedge \psi) &= \text{tr}(\varphi) \wedge \text{tr}(\psi), \\
\text{tr}(\Box_i \varphi) &= \forall y.\, R_i x y \Rightarrow \text{tr}(\varphi)[y/x], \\
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Two layers of semantic structures $\Longrightarrow$ two (split) perspectives:

- **Bisimulation theorems**:
  "modal logic is about LTSs (Kripke models)."
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Rel gives a more unifying approach to these perspectives.

Also, **some variants of modal logic**:

- Temporal logic has modalities about the future and about the past, i.e. modalities of opposite relations.
- Dynamic logic has composition and union of transitions.
- “Dynamic epistemic logic” has modalities of transitions across different models.
- Different $\vdash_\sigma$ for different stages σ of computation (e.g. quote and unquote as modalities).

Thus we need involution, union, etc., and categorification—hence **Rel**.

Semantics Using Rel (take 1)

Every relation $R : X \nrightarrow Y$ induces two adjoint pairs:

$$\mathcal{P}X \underset{\forall_{R^\dagger}}{\overset{\exists_R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}Y \qquad\qquad \mathcal{P}X \underset{\forall_{R}}{\overset{\exists_{R^\dagger}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{P}Y$$

$$\exists_R(S) = \{\, v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \,\},$$
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Complete atomic Boolean algebras ("caBas", $\simeq$ powerset algebras):

- $\mathbf{caBa}_\vee$ with all-$\vee$-preserving maps,
- $\mathbf{caBa}_\wedge$ with all-$\wedge$-preserving maps.

Then $\exists_{-} : \mathbf{Rel} \to \mathbf{caBa}_\vee$ and $\forall_{-} : \mathbf{Rel} \to \mathbf{caBa}_\wedge$, and moreover . . . .

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Thm (Thomason 1975)**.**

$$\text{Kripke frames} \simeq (\text{caBas with } \vee\text{-preserving operators})^{\mathrm{op}}.$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ R \,\big\uparrow\!\!\!\!+ & \leq & +\!\!\!\!\big\uparrow\, S \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P}X & \xleftarrow{f^{-1}} & \mathcal{P}Y \\ \exists_R \big\downarrow & \leq & \big\downarrow \exists_S \\ \mathcal{P}X & \xleftarrow[f^{-1}]{} & \mathcal{P}Y \end{array}$$

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Thm. Bisimulations preserve satisfaction.

Pf. Because they are spans of homomorphisms.

$$\begin{array}{ccccc} X & \xleftarrow{f} & Z & \xrightarrow{g} & Y \\ R\uparrow\!\!\!\!+ & \geq & +\!\!\!\!\uparrow U & \leq & +\!\!\!\!\uparrow S \\ X & \xleftarrow[f]{} & Z & \xrightarrow[g]{} & Y \end{array}$$

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- $\exists_{-} : \mathbf{Rel} \to \mathbf{caBa}_{\vee}$ is a 2-equivalence.
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Thm (Lemmon-Scott 1977). $(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger}$ corresponds to

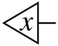
$$\Diamond^m \Box^k \varphi \vdash \Box^n \Diamond^{\ell} \varphi, \qquad \Diamond^n \Box^{\ell} \varphi \vdash \Box^m \Diamond^k \varphi.$$

Pf.

$$\dfrac{\dfrac{\dfrac{(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger}}{\blacklozenge^n \circ \Diamond^m \leqslant \Diamond^{\ell} \circ \blacklozenge^k}}{\Diamond^m \leqslant \Box^n \circ \Diamond^{\ell} \circ \blacklozenge^k}}{\Diamond^m \circ \Box^k \leqslant \Box^n \circ \Diamond^{\ell}} \qquad \dfrac{\dfrac{\dfrac{(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger}}{\Box^{\ell} \circ \blacksquare^k \leqslant \blacksquare^n \circ \Box^m}}{\Diamond^n \circ \Box^{\ell} \circ \blacksquare^k \leqslant \Box^m}}{\Diamond^n \circ \Box^{\ell} \leqslant \Box^m \circ \Diamond^k}$$

E.g.
- $\varphi \vdash \Diamond\varphi, \Box\varphi \vdash \varphi \iff 1 \subseteq R$ (reflexivity);
- $\Diamond\Diamond\varphi \vdash \Diamond\varphi, \Box\varphi \vdash \Box\Box\varphi \iff R;R \subseteq R$ (transitivity);
- $\varphi \vdash \Box\Diamond\varphi, \Diamond\Box\varphi \vdash \varphi \iff R^{\dagger} \subseteq R$ (symmetry).

Semantics in Rel (take 2)

Worlds $x \in X$ are functions $x : 1 \to X$, or 
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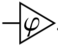
Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or 
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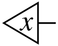
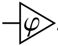
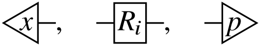
Worlds $x \in X$ are functions $x : 1 \to X$, or .

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or .

So the three components of Kripke frames and models become



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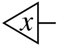
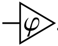
Worlds $x \in X$ are functions $x : 1 \to X$, or .

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or .

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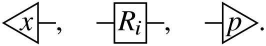
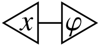


The truth of $x \vDash \varphi$ is given by the generalized Born rule:



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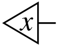
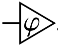
Worlds $x \in X$ are functions $x : 1 \to X$, or .

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or .

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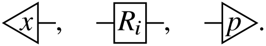

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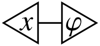

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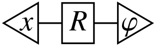

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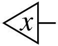
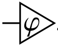
Worlds $x \in X$ are functions $x : 1 \to X$, or .

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or .

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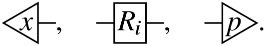

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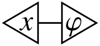

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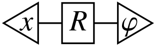
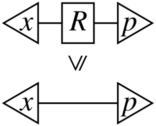


Validity of $p \vdash \Diamond p$ in a Kripke frame is

$\forall x, p$



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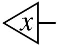
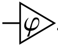
Worlds $x \in X$ are functions $x : 1 \to X$, or .

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or .

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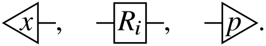

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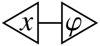
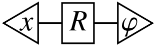


$x \vDash \Diamond\varphi$ is 

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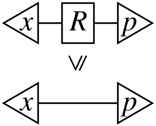
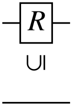
$\forall x, p$  $\iff$ 

Allegories

There are many categorical generalizations of **Rel**. Which of them admits the foregoing approach to modal logic? — Allegories!

Def. An allegory $\mathcal{A}$ is a **Pos**-enriched †-category in which

- each $\mathcal{A}(X,Y)$ has a binary meet,
- † preserves $\subseteq$ and $\cap$,
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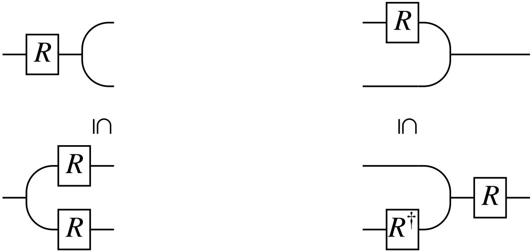

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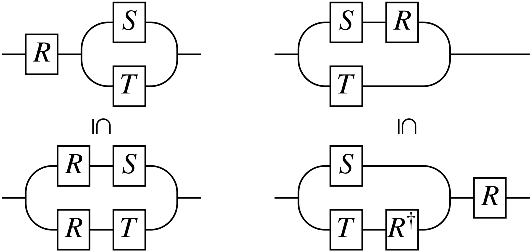

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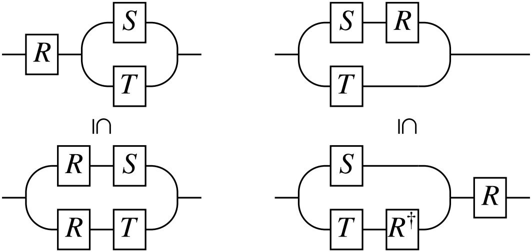

We write $\top_{(X,Y)}$ for the top element of $\mathcal{A}(X,Y)$ if it exists.

$R : X \nrightarrow X$ is
- reflexive if $1_X \subseteq R$,
- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

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unital and tabular	regular	$\top, \wedge, \exists, =$
+ "distributive"	coherent (pre-logoi)	$\bot, \vee$
+ "division"	Heyting (logoi)	$\Rightarrow, \forall$
+ "power"	topoi	$\in$

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Two allegorical expressions for $\mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$:

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Then the diagram commutes, so the bottom edge is isomorphisms.
If moreover $\mathcal{A}$ is tabular, $\mathrm{Cor}(X) \cong \mathcal{A}(X, 1) \cong \mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$.

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E.g.

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We extend this and write

$$\mathcal{A}(Y,1) \underset{\forall_R \,=\, R\backslash -}{\overset{\exists_{R^\dagger} \,=\, R;-}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{A}(X,1) \qquad \mathcal{A}(X,1) \underset{\forall_{R^\dagger} \,=\, R^\dagger\backslash -}{\overset{\exists_R \,=\, R^\dagger;-}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{A}(Y,1)$$

On $\mathcal{A}(X, 1)$, the interpretation on $\mathrm{Cor}(X)$ becomes

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To this, add, for each $R_i : X \nrightarrow X$,

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\begin{aligned}
[\![\Diamond_i\varphi]\!] &= R_i;[\![\varphi]\!], \\
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Syntax

- Basic types τ.
- Each prop. variable p has a basic type $p : \tau$.
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Frames and Models

Generate a category **D** from basic types τ and labels $i : \tau \to \tau'$.

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Generate a category $\mathbf{D}$ from basic types τ and labels $i : \tau \to \tau'$.

Kripke frames can then be generalized by

Def. A frame diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}^{\mathrm{op}} \to \mathcal{A}$.

$$\begin{array}{cccc} \tau & [\![\tau]\!] & \mathcal{A}([\![\tau]\!], 1) & [\![\varphi]\!] \\ i \big\downarrow & [\![i]\!] \big\uparrow & \big\downarrow [\![i]\!];- & \big\downarrow \\ \tau' & [\![\tau']\!] & \mathcal{A}([\![\tau']\!], 1) & [\![\Diamond_i \varphi]\!] \end{array}$$

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Let $\mathbf{D}_*$ be $\mathbf{D}$ with an object $*$ and labels $p : * \to \tau$ added.

Def. A model diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}_*{}^{\mathrm{op}} \to \mathcal{A}$ s.th. $[\![*]\!] = 1$.

$$\begin{array}{cc}
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$\mathbf{D}$ may have more structure: e.g. $\dagger$ for temporal, $\cup$ for dynamic logics.

Interpretation

For propositions of type τ,

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[\![\Diamond_i \varphi]\!] &= [\![i]\!];[\![\varphi]\!] : [\![\tau']\!] \nrightarrow 1, \\
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$$

Example

Simpson's (1994) semantics in terms of "birelation models":

- A frame is a poset $(X, \leqslant)$ plus $R : X \nrightarrow X$ s.th.

$$
\begin{array}{ccc}
X & \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & X \\
{\scriptstyle\leqslant}\,\Big\uparrow & \rotatebox[origin=c]{-90}{$\subseteq$} & \Big\uparrow\,{\scriptstyle\leqslant} \\
X & \underset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \overset{R^{\dagger}}{\longleftarrow\!\!\!\!\!\!\!+\;\;\;} & X \\
{\scriptstyle\leqslant}\,\Big\uparrow & \rotatebox[origin=c]{-90}{$\subseteq$} & \Big\uparrow\,{\scriptstyle\leqslant} \\
X & \underset{R^{\dagger}}{\longleftarrow\!\!\!\!\!\!\!+\;\;\;} & X
\end{array}
$$

- Each $[\![p]\!] \subseteq X$ is $\leqslant$-upward closed.

This is to take our allegorical semantics in the allegory of posets and bisimulations.

($[\![p]\!] \subseteq X$ is $\leqslant$-upward closed iff $[\![p]\!] : X \nrightarrow 1$ is a bisimulation.)

Maps of diagrams and bisimulations

Def. A map of diagrams is a map-valued natural transformation.

$$
\begin{array}{ccccc}
\tau & & [\![\tau]\!]_1 & \xrightarrow{\alpha_\tau} & [\![\tau]\!]_2 \\
{\scriptstyle i}\big\downarrow & & {\scriptstyle [\![i]\!]_1}\big\uparrow & = & \big\uparrow{\scriptstyle [\![i]\!]_2} \\
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Thm.

$$
\begin{array}{ccc}
 & 1 & \\
[\![\varphi]\!]_1 \nearrow & = & \nwarrow [\![\varphi]\!]_2 \\
[\![\tau]\!]_1 & \xrightarrow[\alpha_\tau]{} & [\![\tau]\!]_2 \\
{\scriptstyle x}\nwarrow & & \nearrow{\scriptstyle y} \\
 & 1 &
\end{array}
$$

Thm. The correspondence below extends to every $\mathcal{A}$.

$$
\begin{array}{ccc}
X & \overset{T}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;\;} & Y \\
R\uparrow\!\!\!+ & \rotatebox{90}{$\subseteq$} & +\!\!\!\uparrow S \\
X & \underset{T}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;\;} & Y
\end{array}
\qquad\qquad
\begin{array}{ccccc}
X & \overset{f}{\longleftarrow} & Z & \overset{g}{\longrightarrow} & Y \\
R\uparrow\!\!\!+ & = & +\!\!\!\uparrow U & = & +\!\!\!\uparrow S \\
X & \underset{f}{\longleftarrow} & Z & \underset{g}{\longrightarrow} & Y
\end{array}
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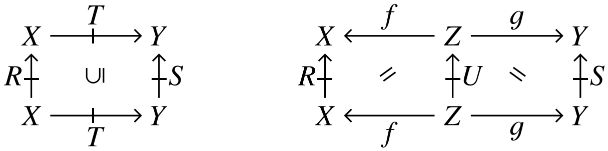

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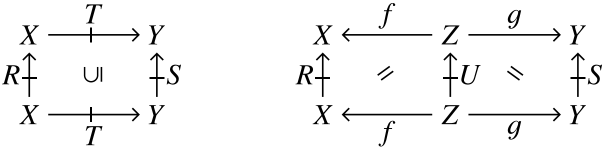

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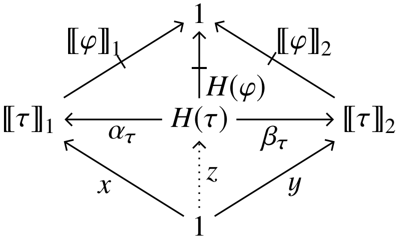

Duality and correspondence

For a nice enough $\mathcal{A}$, we have order embeddings

$$\exists_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)),$$

and order-reversing embeddings

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Thm. In such an $\mathcal{A}$, the condition ${R_1}^\dagger;R_2 \subseteq R_3;{R_4}^\dagger$ corresponds to

$$\Diamond_2\Box_4\varphi \vdash \Box_1\Diamond_3\varphi, \qquad\qquad \Diamond_1\Box_3\varphi \vdash \Box_2\Diamond_4\varphi.$$

Duality and correspondence

For a nice enough $\mathcal{A}$, we have order embeddings

$$\exists_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)),$$

and order-reversing embeddings

$$\forall_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)).$$

Thm. In such an $\mathcal{A}$, the condition $R_1{}^\dagger;R_2 \subseteq R_3;R_4{}^\dagger$ corresponds to

$$\Diamond_2\Box_4\varphi \vdash \Box_1\Diamond_3\varphi, \qquad \Diamond_1\Box_3\varphi \vdash \Box_2\Diamond_4\varphi.$$

Indeed, (the intuitionistic version of) the much stronger "calculus for correspondence" (Conradie et al. 2014) is sound in any division $\mathcal{A}$ s.th. $\mathbf{Map}(\mathcal{A})$ is well-pointed.

Standard translation into categorical logic of **Map**($\mathcal{A}$).

$$
\begin{aligned}
(\, x : T \mid \mathrm{tr}(p : \tau)\,) &= (\, x : T \mid Px \,), \\
(\, x : T \mid \mathrm{tr}(\bot : \tau)\,) &= (\, x : T \mid x \neq x \,), \\
(\, x : T \mid \mathrm{tr}(\varphi \wedge \psi : \tau)\,) &= (\, x : T \mid \mathrm{tr}(\varphi : \tau) \wedge \mathrm{tr}(\psi : \tau)\,), \\
(\, x : T \mid \mathrm{tr}(\Box_i \varphi : \tau)\,) &= (\, x : T \mid \forall y : T' \,(R_i xy \Rightarrow \mathrm{tr}(\varphi : \tau')[y/x]\,), \\
(\, x : T \mid \mathrm{tr}(\Diamond_i \varphi : \tau)\,) &= (\, x : T \mid \exists y : T' \,(R_i xy \wedge \mathrm{tr}(\varphi : \tau')[y/x]\,).
\end{aligned}
$$

Logic of the semantics

Since $\exists_{R^\dagger}$ and $\forall_{R^\dagger}$ are left and right adjoints,

$$\frac{\varphi \vdash_\tau \psi}{\Diamond\varphi \vdash_{\tau'} \Diamond\psi} \qquad \frac{\varphi \vdash_\tau \psi}{\Box\varphi \vdash_{\tau'} \Box\psi}$$

$$\Diamond(\varphi \vee \psi) \vdash_{\tau'} \Diamond\varphi \vee \Diamond\psi \qquad \Box\varphi \wedge \Box\psi \vdash_{\tau'} \Box(\varphi \wedge \psi)$$

$$\Diamond\bot_\tau \vdash_{\tau'} \bot_{\tau'} \qquad \top_{\tau'} \vdash_{\tau'} \Box\top_\tau$$

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Thm. **tIK** is sound and complete w.r.t. all allegorical semantics.

Future Work

- More on bisimulation theorems. In particular, Hennessy-Milner and van Benthem-type theorems.
- More variants of modal logic. E.g. fixed point logic.
- Axiomatization of smaller fragments. E.g. without division structure.
- Axiomatization of particular base logics. E.g. the allegory of fuzzy relations.
- In particular, **Rel**(C) as models of quantum theory (Heunen-Tull 2015).
- Diagrammatic methods for the distribution and division structures.