Monadic
Monadic Second Order Logic

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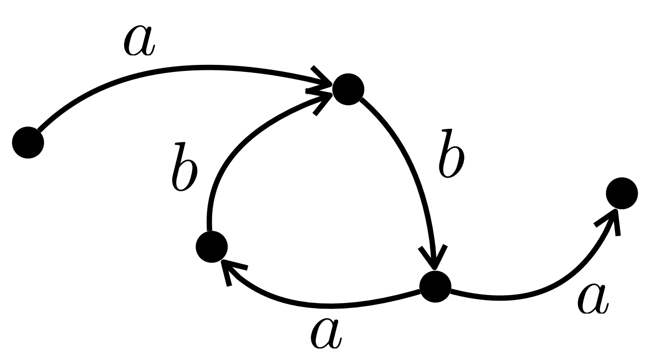
University of Warsaw

13 May 2020

Languages of finite words

accepted by finite automata



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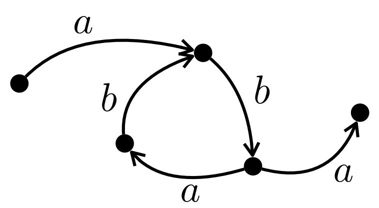

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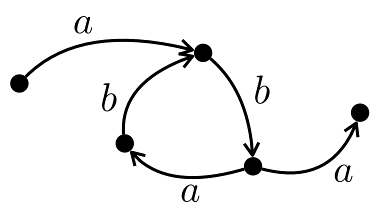

defined by regular expressions

$$E ::= \epsilon \mid a \mid E + E \mid EE \mid E^*$$

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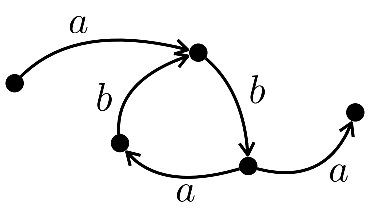

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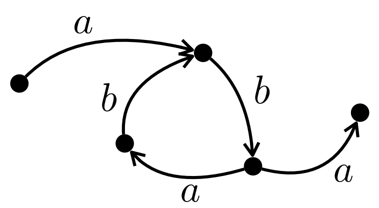

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MSO-definable

$$x < y \qquad Q_a(x) \qquad x \in X$$

$$\phi \vee \psi \qquad \neg\phi \qquad \exists X.\phi$$

Monadic Second-Order Logic

- words as relational structures:

$$a \quad c \quad b \quad c \quad a \quad c \quad b \quad a \quad c \quad a \quad c$$

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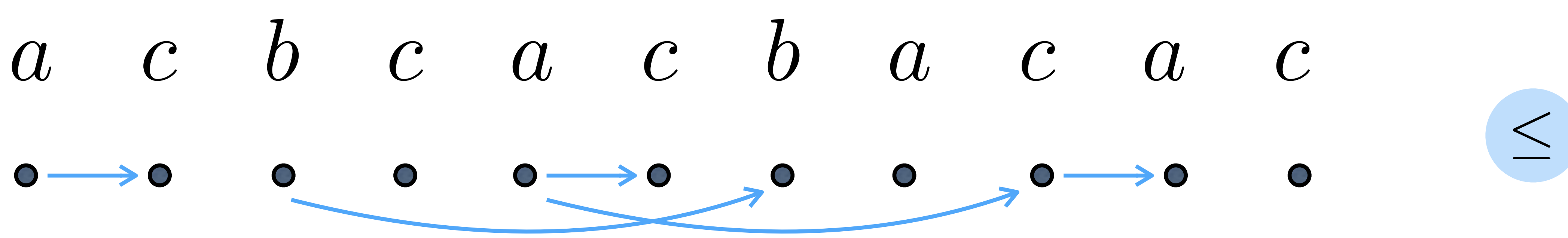

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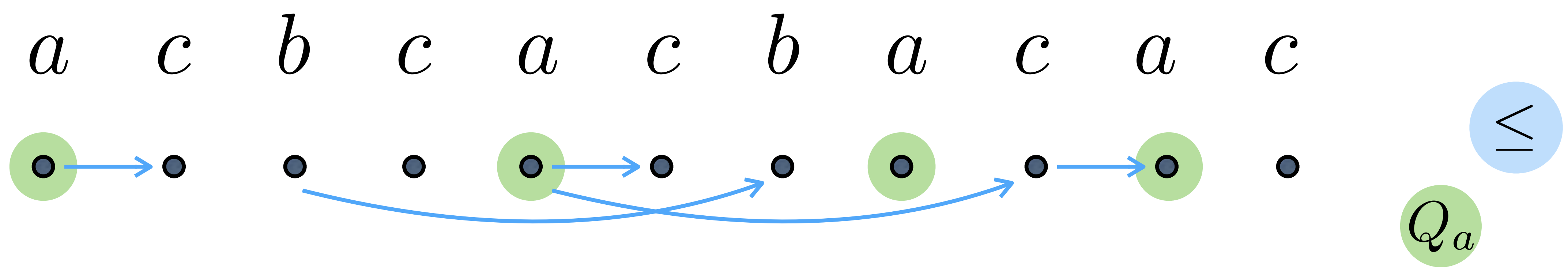

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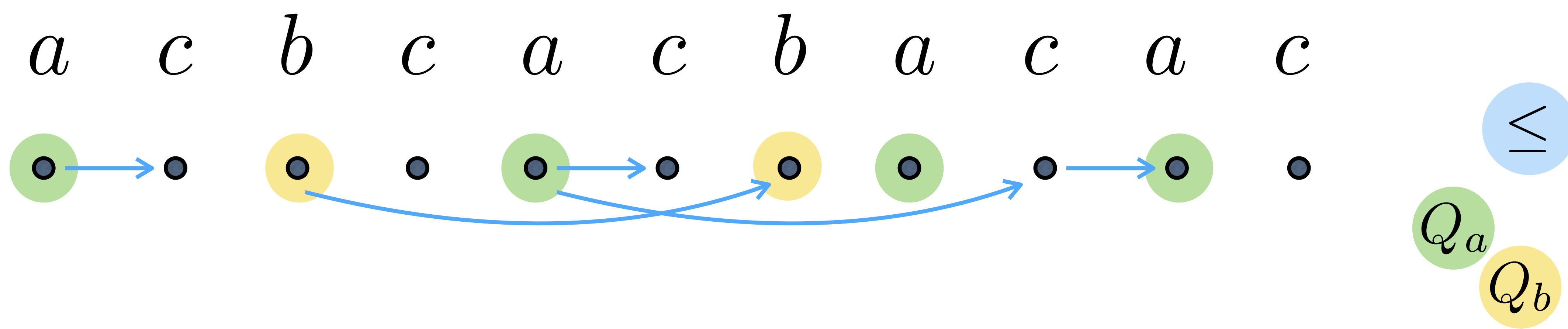

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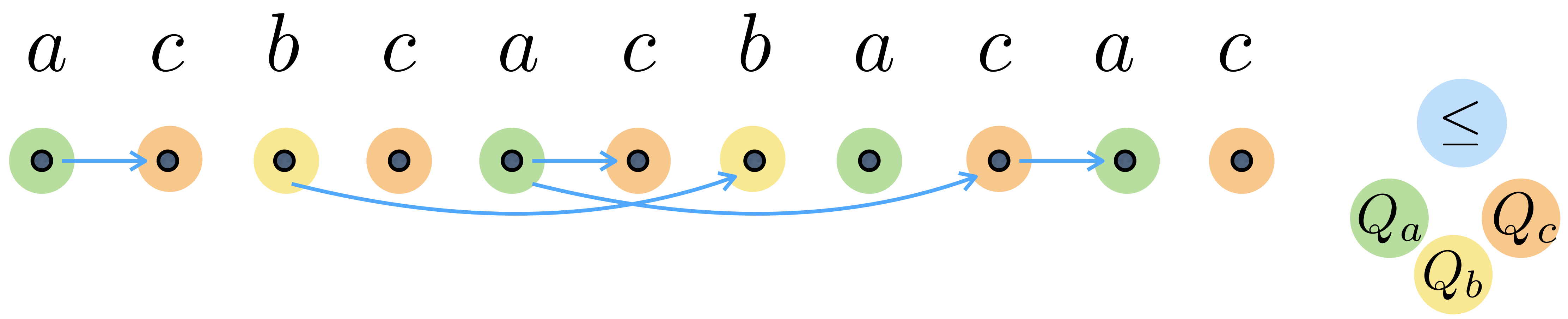

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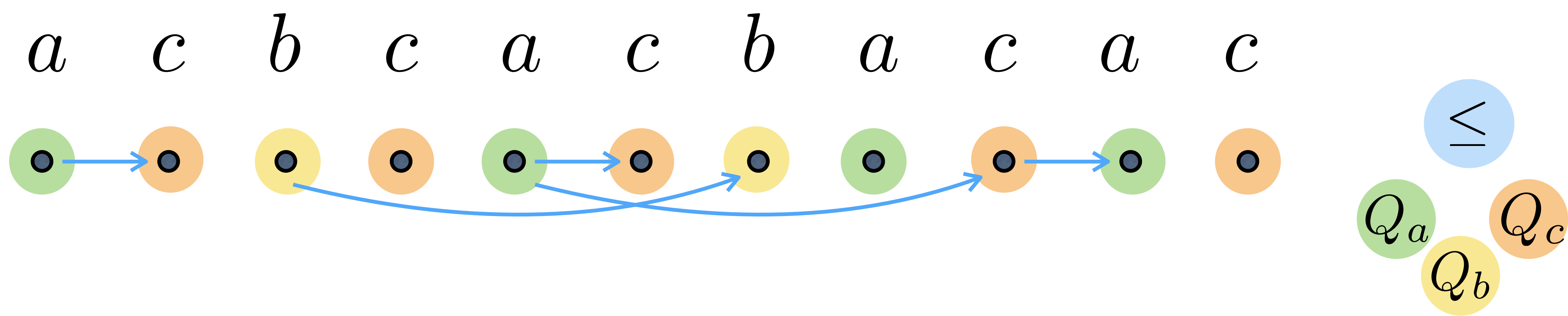

- examples:

$$\forall x. Q_a(x) \Rightarrow \exists y. x < y \wedge Q_c(y)$$

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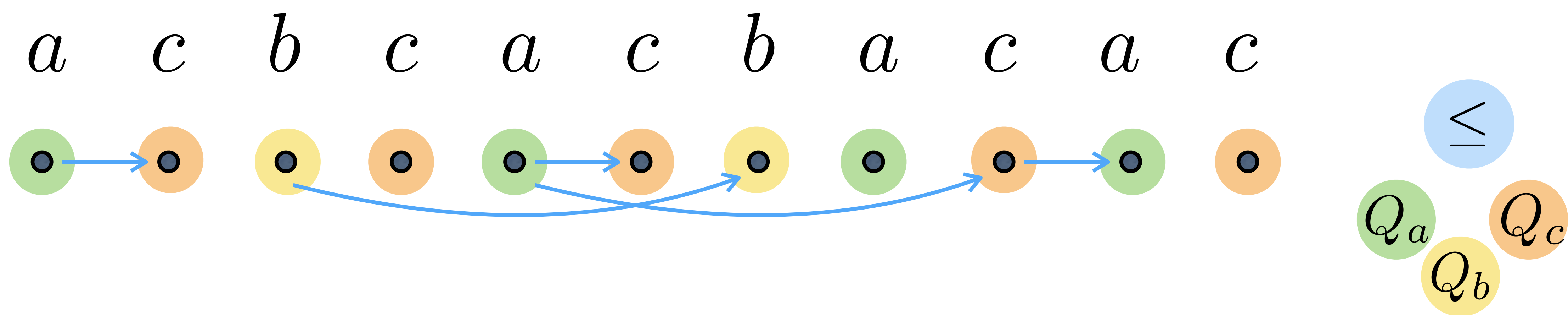

- examples:

$$\forall x. Q_a(x) \Rightarrow \exists y. x < y \wedge Q_c(y)$$

$$\begin{aligned}\exists X.(&\forall x\ \exists y\ y \leq x \wedge y \in X) \wedge \\ (&\forall x\ \exists y\ y \geq x \wedge y \in X) \wedge \\ (&\forall x\ \forall y\ (x < y \wedge \neg(\exists z\ x < z < y)) \Rightarrow (x \in X \Leftrightarrow y \notin X)).\end{aligned}$$

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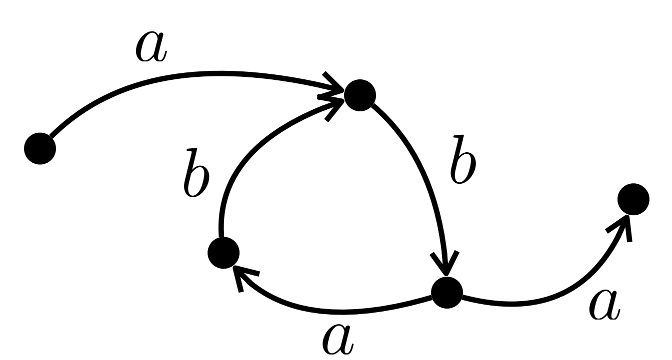

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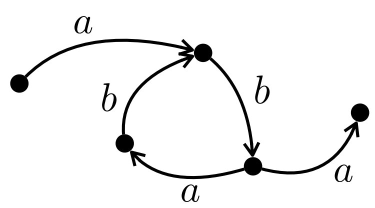

defined by regular expressions

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MSO-definable

$x < y$ $Q_a(x)$ $x \in X$

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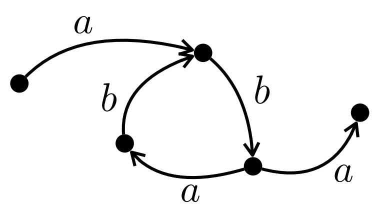

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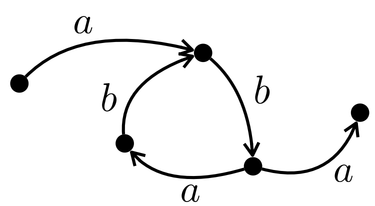

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recognized by finite monoids

$$\begin{array}{ccccc} \overleftarrow{h}(A) = & L & & & A \\ & \cap\!\!\!\mid & & & \cap\!\!\!\mid \\ & \Sigma^* & \xrightarrow[h]{\quad\quad} & & M \end{array}$$

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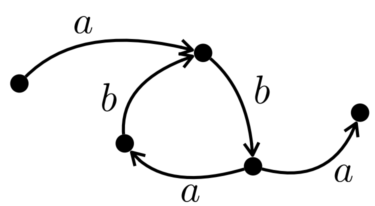

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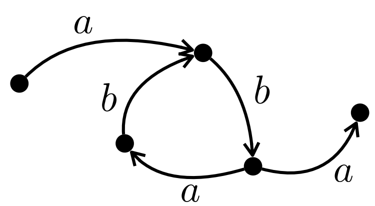

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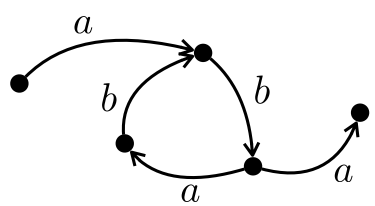

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Things in this talk

- finite words
- ω-words
- countable total orders
- scattered total orders
- total orders of size $\leq \mathfrak{c}$
- finite trees
- infinite trees
- graphs of bounded treewidth
- graphs of bounded cliquewidth
- ...
- ...

Our focus

MSO-definable

$x < y \qquad Q_a(x) \qquad x \in X$

$\phi \vee \psi \qquad \neg\phi \qquad \exists X.\phi$

$\Leftrightarrow$

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$\Longleftarrow$

- quite easy for finite words or trees
- difficult (or open) for other structures
- structure-specific arguments

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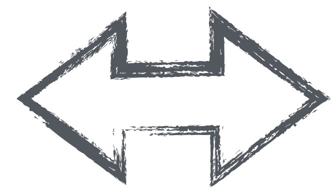

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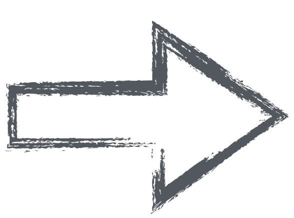

- relatively easy for all cases
- the arguments look generic

Our focus

MSO-definable

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least class closed under:

- $0^*1^* \subseteq \{0,1\}^*$
- boolean combinations
- inv. images along $h : \Sigma \to \Gamma^*$
- dir. images along $h : \Sigma \twoheadrightarrow \Gamma$

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Definable implies recognizable, for finite words

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- L_i rec. by $h_i : \Sigma^* \to M_i$ (for $i = 1, 2$)
  implies $L_1 \cap L_2$ rec. by $\langle h_1, h_2 \rangle : \Sigma^* \to M_1 \times M_2$

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  $\Sigma^* \setminus L_i$ rec. by h_i

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  $\Sigma^* \setminus L_i$ rec. by h_i
- L rec. by $h : \Gamma^* \to M$, $\quad g : \Sigma \to \Gamma^*$
  implies $\overleftarrow{g}(L)$ rec. by $h \circ \hat{g}$ $\quad \hat{g} : \Sigma^* \to \Gamma^*$

Closure under direct images, for finite words

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recognized by a finite monoid

$$\overleftarrow{h}(A) = L \quad L \subseteq \Sigma^* \xrightarrow{h} M \supseteq A$$

- let $L \subseteq \Sigma^*$ be recognized by $h : \Sigma^* \to M$

Closure under direct images, for finite words

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$$\overleftarrow{h}(A) = L \subseteq \Sigma^* \xrightarrow{h} M \supseteq A$$

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- let $L \subseteq \Sigma^*$ be recognized by $h : \Sigma^* \to M$
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- define a monoid on $\mathcal{P}M$:

$$S \cdot T = \{s \cdot t \mid s \in S, t \in T\}$$

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- put $k : \Gamma^* \to \mathcal{P}M$ s.t. $k(c) = \{h(a) \mid g(a) = c\}$

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  $B \subseteq \mathcal{P}M$ s.t. $B = \{S \mid S \cap A \neq \emptyset\}$
- then k and B recognize $g^*(L)$

Definable implies recognizable, for finite words

We have just shown:

The class of languages
recognized by finite monoids
is closed under:

- boolean combinations
- inverse images along homomorphisms,
- direct images along (surjective) letter-to-letter homomorphisms.

Definable implies recognizable, for finite words

We have just shown:

The class of languages
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We want to generalize this to other things.

Monads

Monads are *ways to collect stuff*

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A monad T:

- given a set X, returns a set TX

Monads

Monads are *ways to collect stuff*

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

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Monads

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Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

- given a set X, returns a set TX
- given a function $f : X \to Y$, returns a function $Tf : TX \to TY$

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so that:

- $T(\mathrm{id}_X) = \mathrm{id}_{TX}$, and
- $T(g \circ f) = Tg \circ Tf$

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functor

to be ctd...

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

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unit

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$ unit
- $\mu_X : TTX \to TX$

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$ — unit
- $\mu_X : TTX \to TX$ — multiplication

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$ — unit
- $\mu_X : TTX \to TX$ — multiplication

such that (for every $f : X \to Y$):

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ Y & \xrightarrow[\eta_Y]{} & TY \end{array} \quad \text{and} \quad \begin{array}{ccc} TTX & \xrightarrow{\mu_X} & TX \\ {\scriptstyle TTf}\downarrow & & \downarrow{\scriptstyle Tf} \\ TTY & \xrightarrow[\mu_Y]{} & TY \end{array}$$

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$ unit
- $\mu_X : TTX \to TX$ multiplication

such that (for every $f : X \to Y$):

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ Y & \xrightarrow[\eta_Y]{} & TY \end{array} \quad \text{and} \quad \begin{array}{ccc} TTX & \xrightarrow{\mu_X} & TX \\ {\scriptstyle TTf}\downarrow & & \downarrow{\scriptstyle Tf} \\ TTY & \xrightarrow[\mu_Y]{} & TY \end{array}$$

naturality

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

A monad T comes with (for every set X):

- $\eta_X : X \to TX$ — unit
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naturality

to be ctd...

Monads ctd.

Examples: X^*, X^ω, X^∞, $\mathcal{P}X$, $\mathbb{N}^X$

Further axioms on $\eta_X : X \to TX$

$\mu_X : TTX \to TX$:

$$\begin{array}{ccccc} TX & \xrightarrow{\eta_{TX}} & TTX & \xleftarrow{T\eta_X} & TX \\ & \searrow^{=} & \downarrow{\scriptstyle \mu_X} & \swarrow_{=} & \\ & & TX & & \end{array}$$

Monads ctd.

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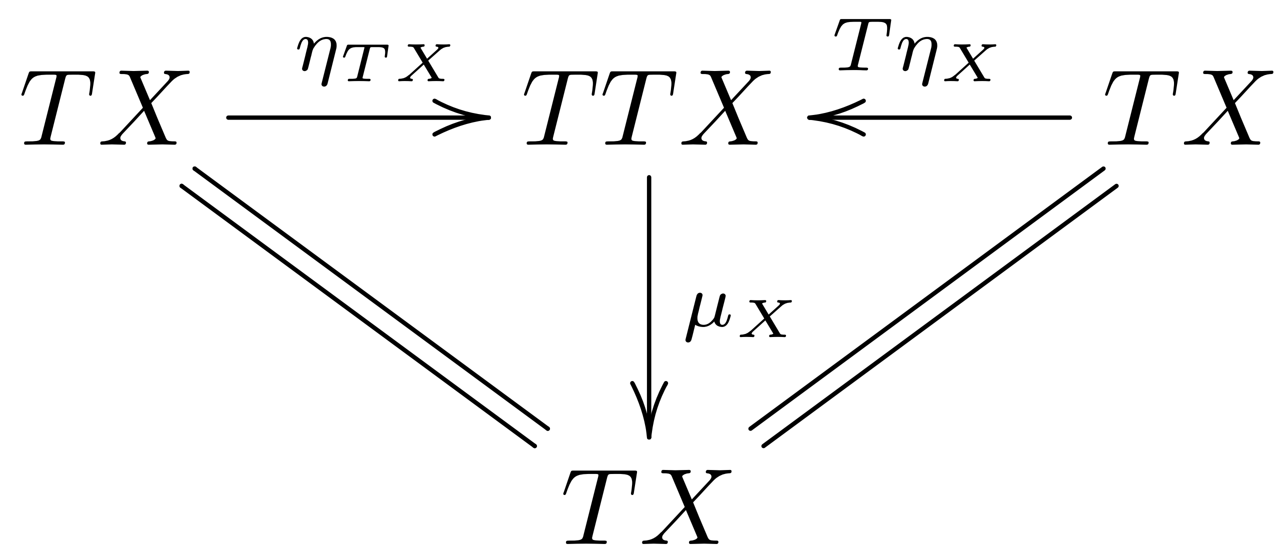

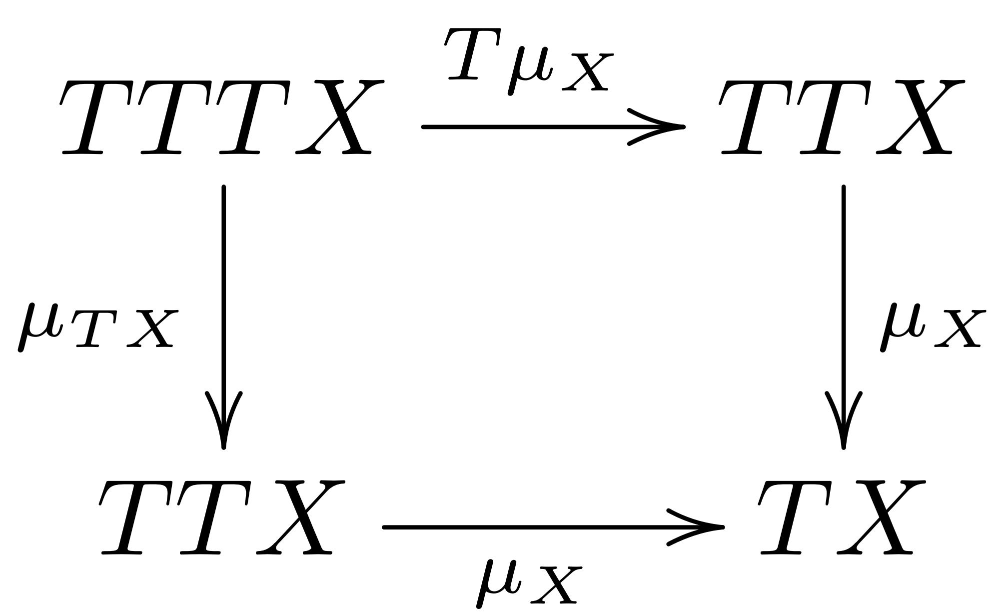

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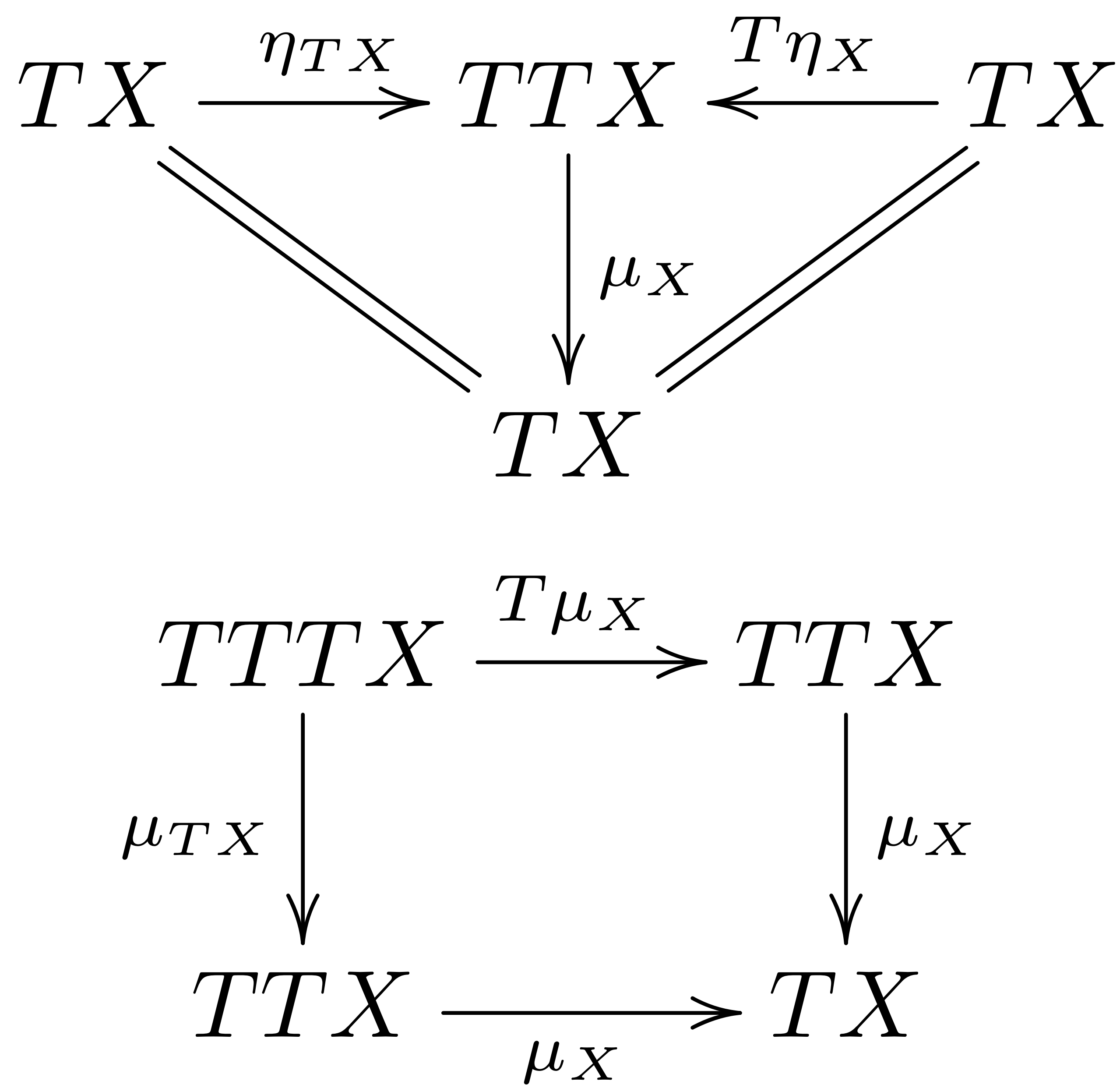

That's it!

Examples

1. The list monad

$$TX = X^*$$

$$Tf(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$$

$$\eta_X(x) = x$$

$$\mu_X(w_1 w_2 \cdots w_n) = w_1{}^\frown w_2{}^\frown \cdots {}^\frown w_n$$

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2. The powerset monad

$$TX = \mathcal{P}X$$

$$Tf = \overrightarrow{f}$$

$$\eta_X(x) = \{x\}$$

$$\mu_X(\Phi) = \bigcup \Phi$$

Examples ctd.

3. The reader monad

$$TX = X^\omega \qquad Tf(x_1x_2\cdots) = f(x_1)f(x_2)\cdots$$

$$\eta_X(x) = xxx\cdots \qquad \mu_X(w_1w_2\cdots) = w_{11}w_{22}w_{33}\cdots$$

Examples ctd.

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4,5,...: term monads

For an equational presentation (Σ, E) , put:

$TX =$ Σ -terms over X modulo the equations

Tf - variable substitution

η - variables as terms

μ - term flattening

What we want to talk about

The class of languages
recognized by finite monoids
is closed under:

- boolean combinations
- inverse images along homomorphisms,
- direct images along (surjective) letter-to-letter homomorphisms.

What we want to talk about

$L \subseteq T\Sigma$

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A T-algebra is:

- a set X and a function $f : TX \to X$

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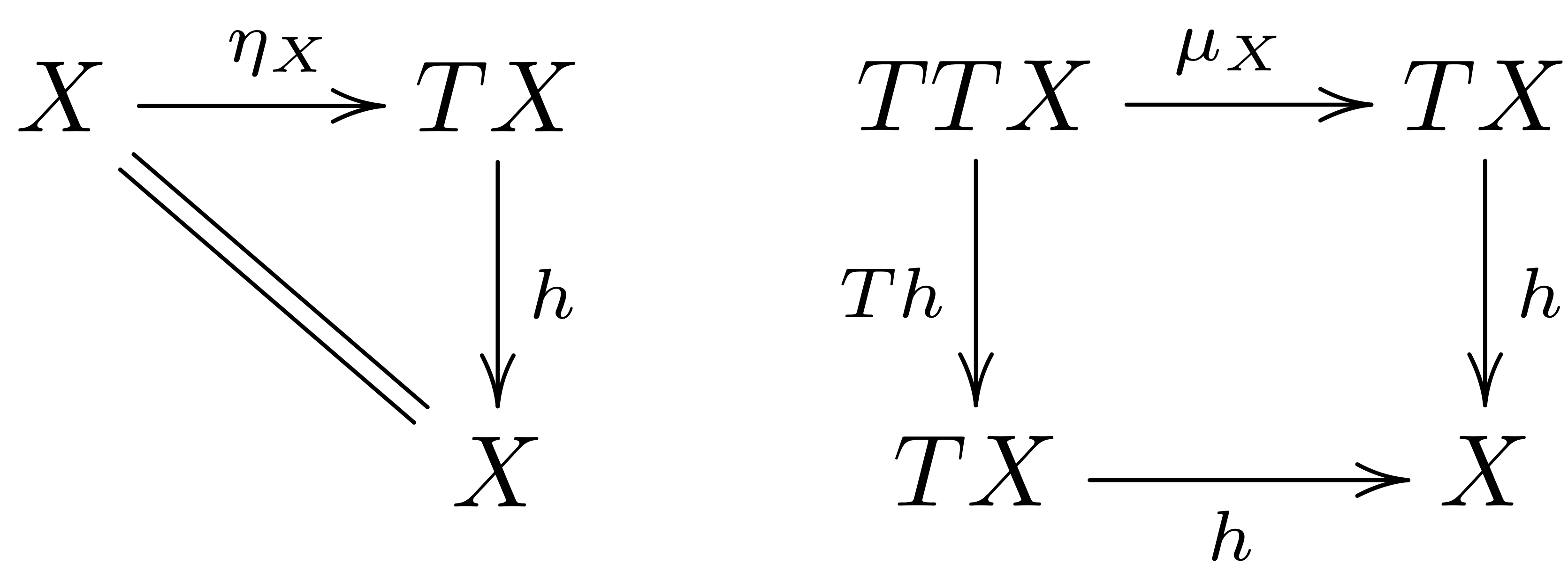

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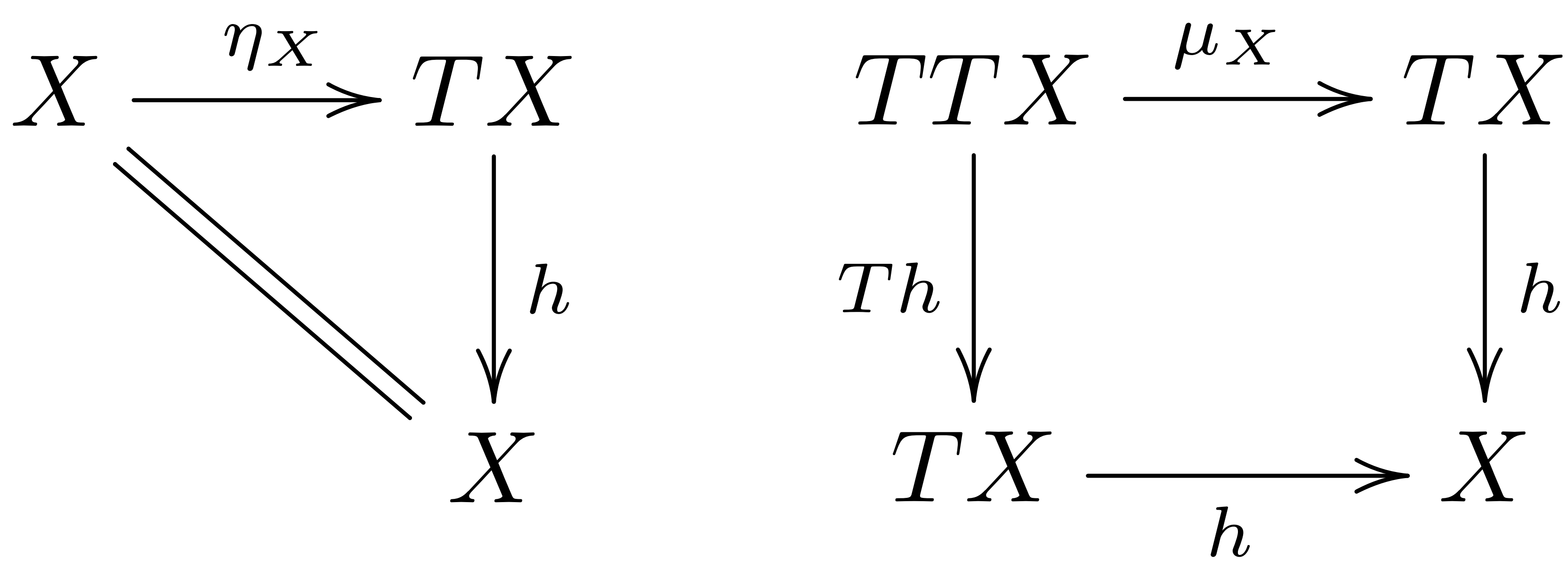

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$(-)^*$-algebras are monoids

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 & \searrow\!\!\!= & \downarrow h \\
 & & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
TTX & \xrightarrow{\mu_X} & TX \\
{\scriptstyle Th}\downarrow & & \downarrow{\scriptstyle h} \\
TX & \xrightarrow[h]{} & X
\end{array}
$$

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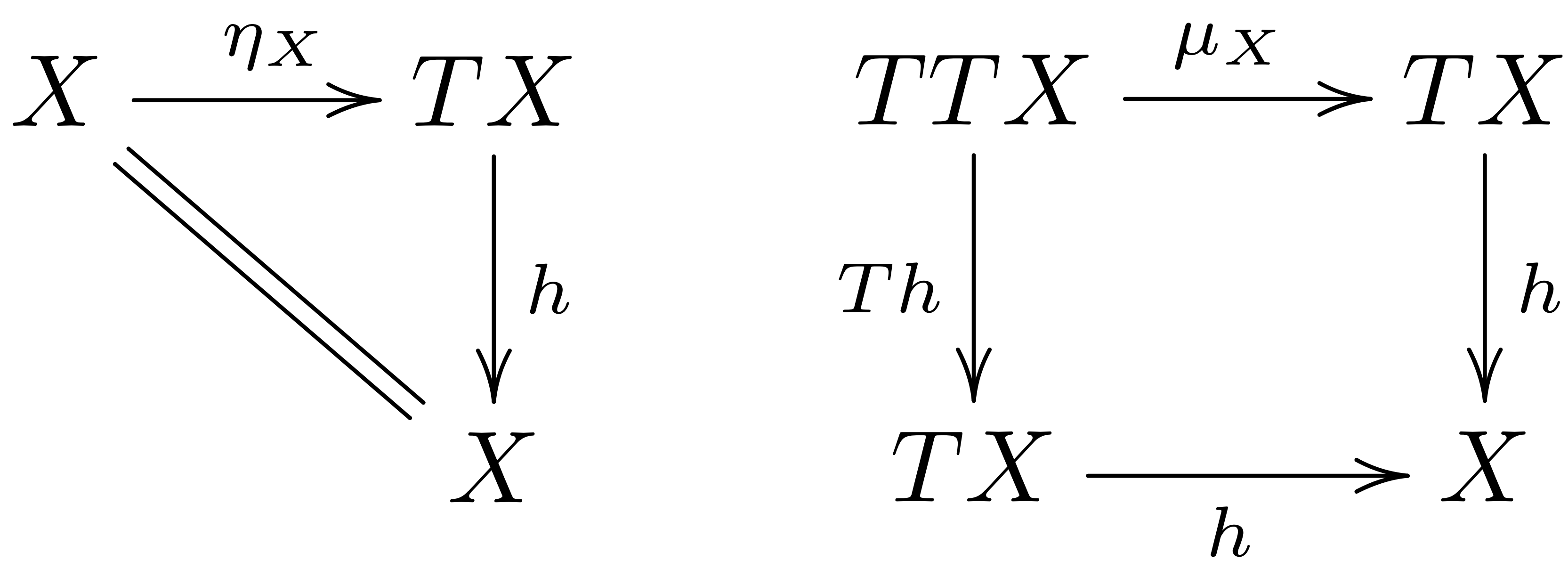

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Term-monad algebras are what you expect

Homomorphisms

A homomorphism from $f : TX \to X$
to $g : TY \to Y$:

a function $h : X \to Y$ such that:

$$\begin{array}{ccc} TX & \xrightarrow{Th} & TY \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y \end{array}$$

Recognizing languages with algebras

Fact: $\mu_X : TTX \to TX$ is always a T-algebra.

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$$
\begin{array}{ccc}
TT\Sigma & \xrightarrow{Th} & TM \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle m} \\
T\Sigma & \xrightarrow[h]{} & M \\
\cup\!| & & \cup\!| \\
L & & A
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$$

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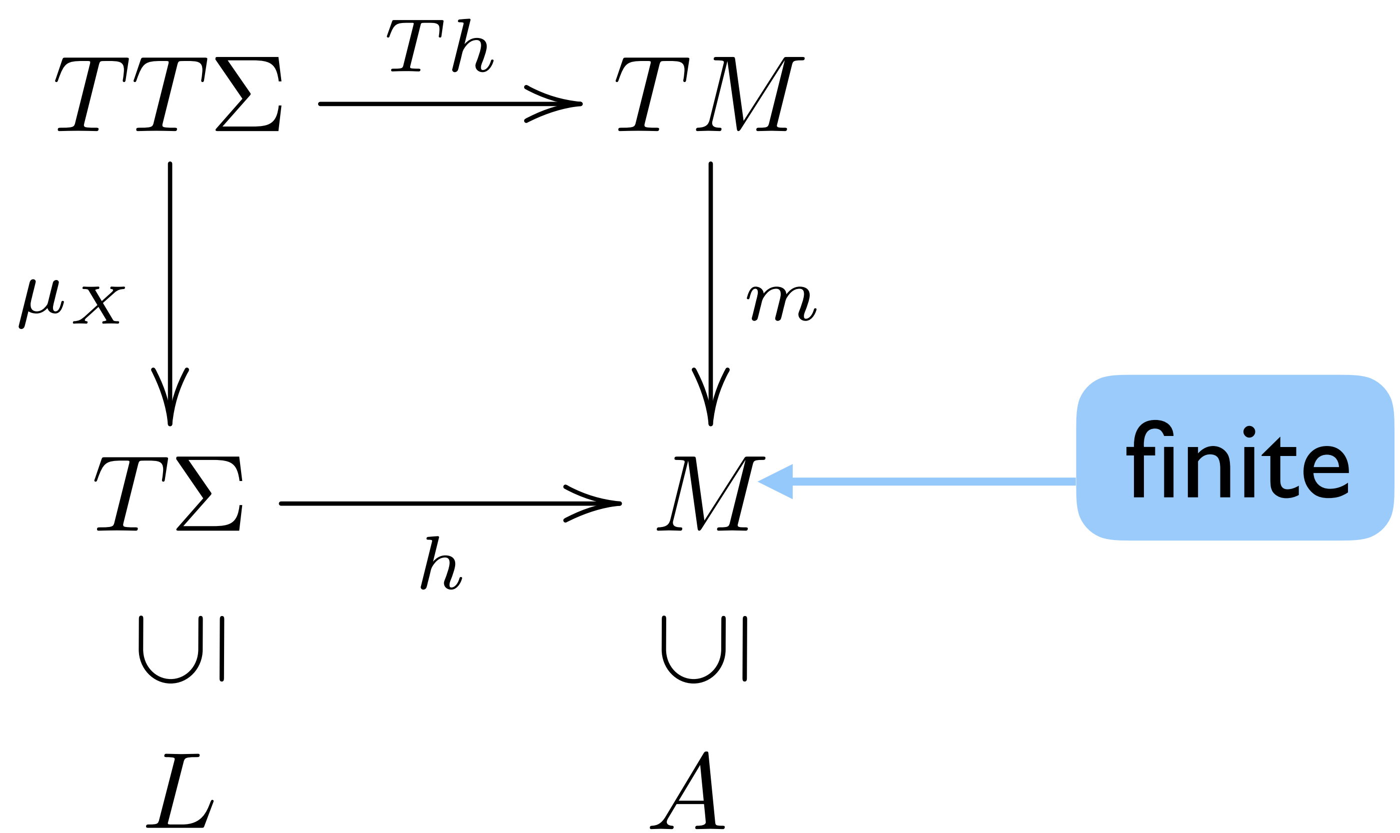

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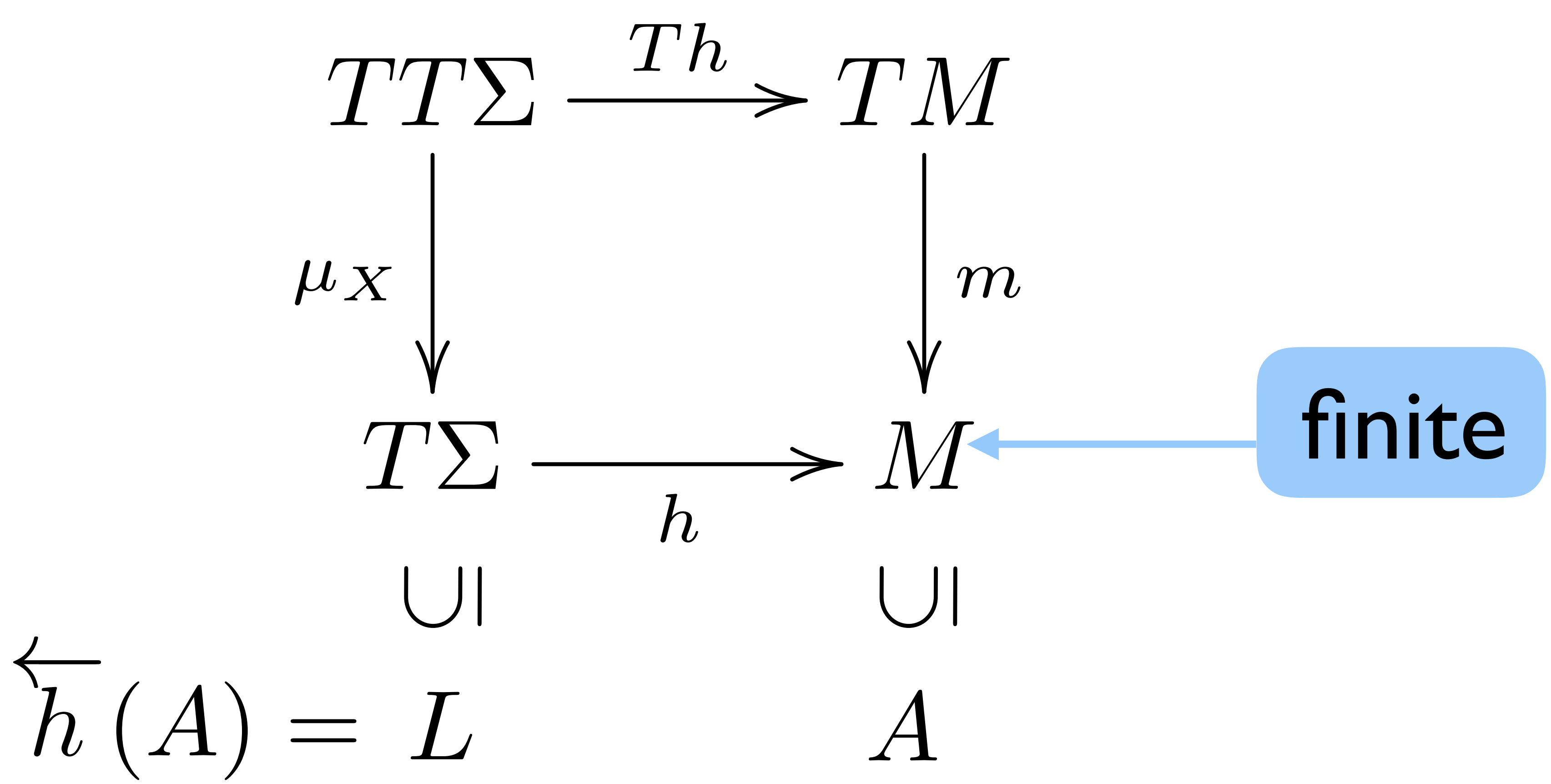

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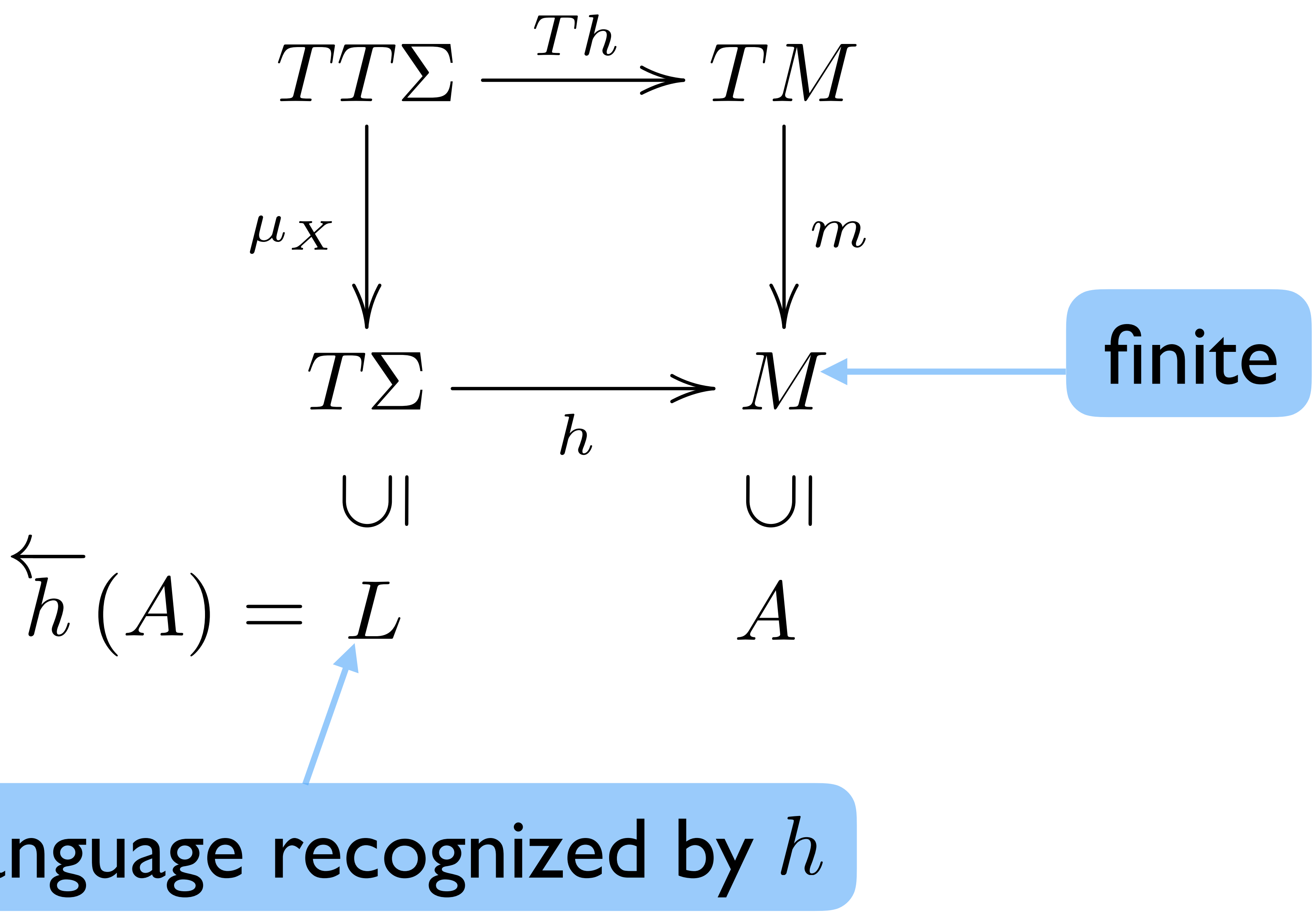

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Fact: $L \subseteq T\Sigma$ is recognizable iff
(the corresponding) $L \subseteq \Sigma^*$ is
regular and closed under $\mathcal{B}$.

Counterexample ctd.

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For $\Delta = \{a, b, c\}$ and $\Sigma = \Delta \cup \{0, 1\}$, let

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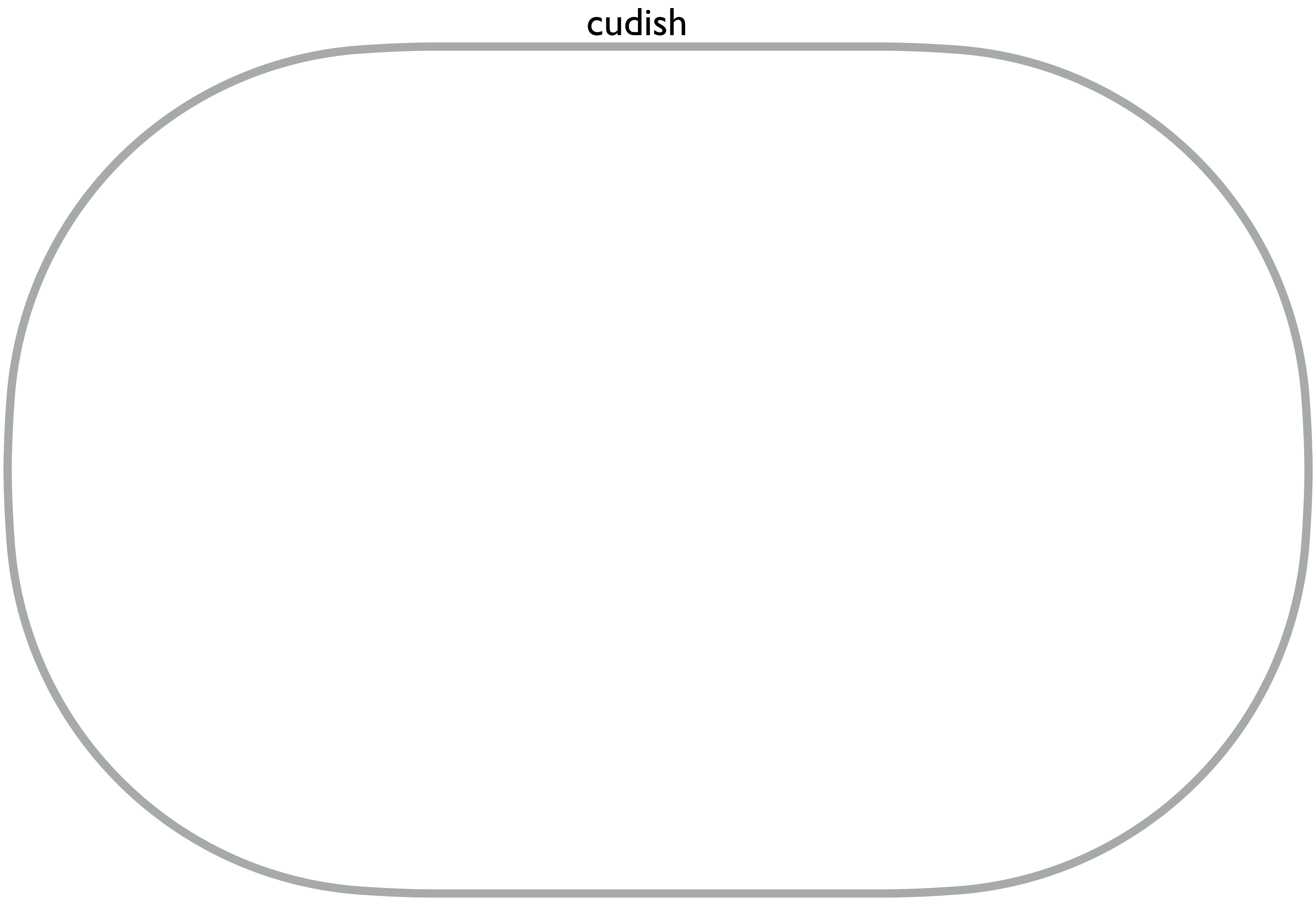
Fact: $\overrightarrow{Th}(L)$ is not regular, so not T-recognizable.

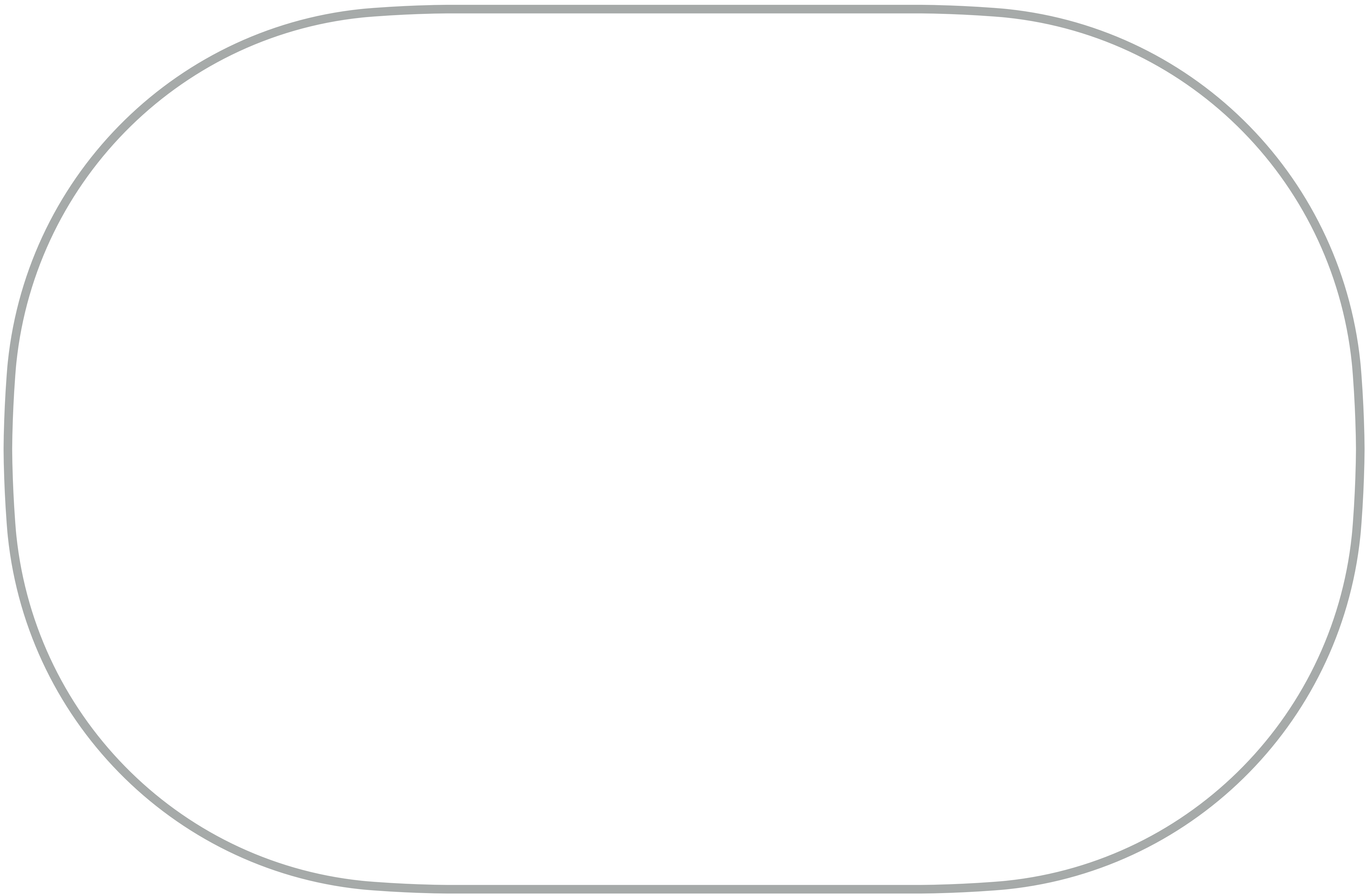
The landscape of monads



The landscape of monads

cudish

$x^3 = x^2$



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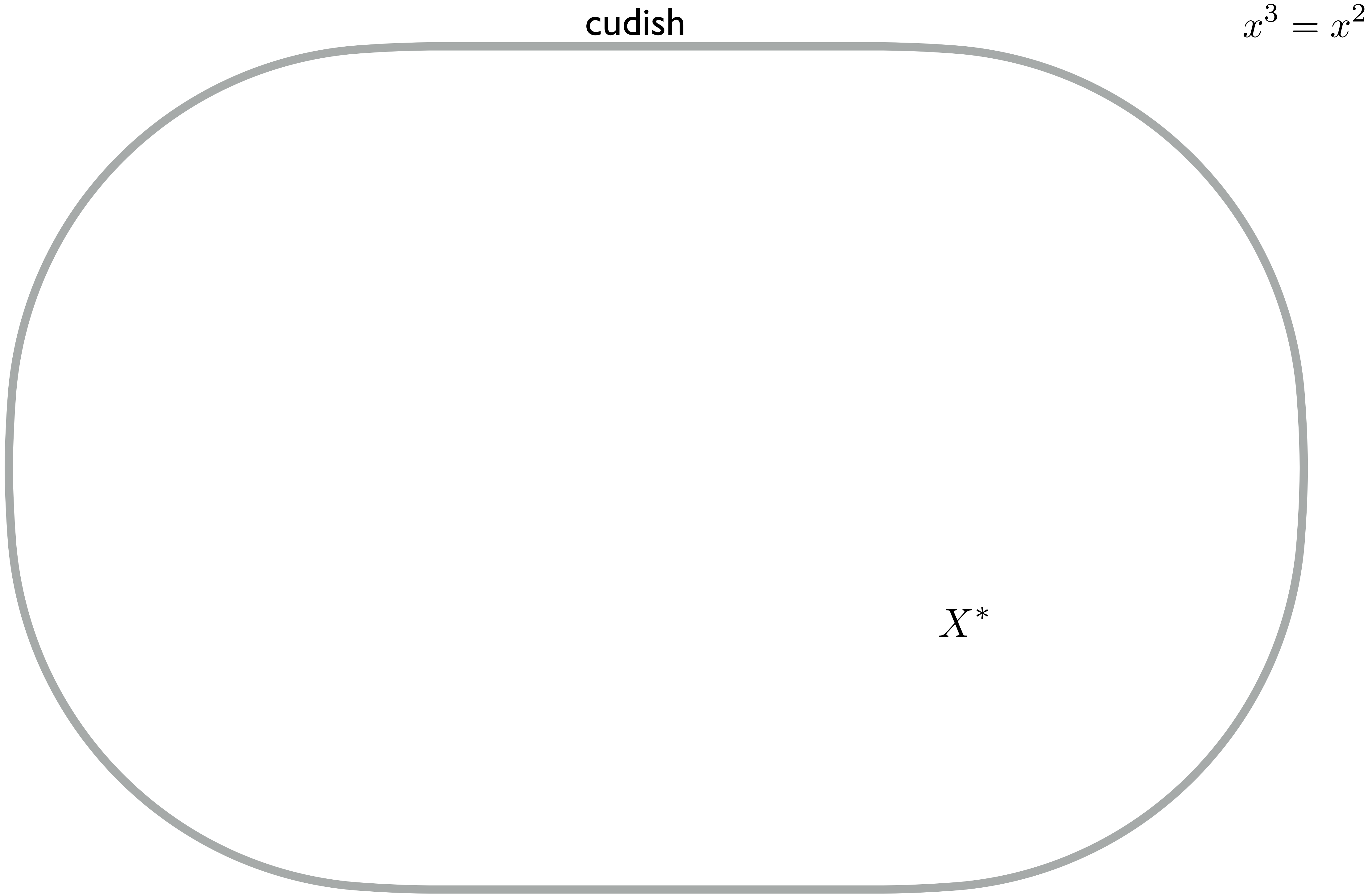

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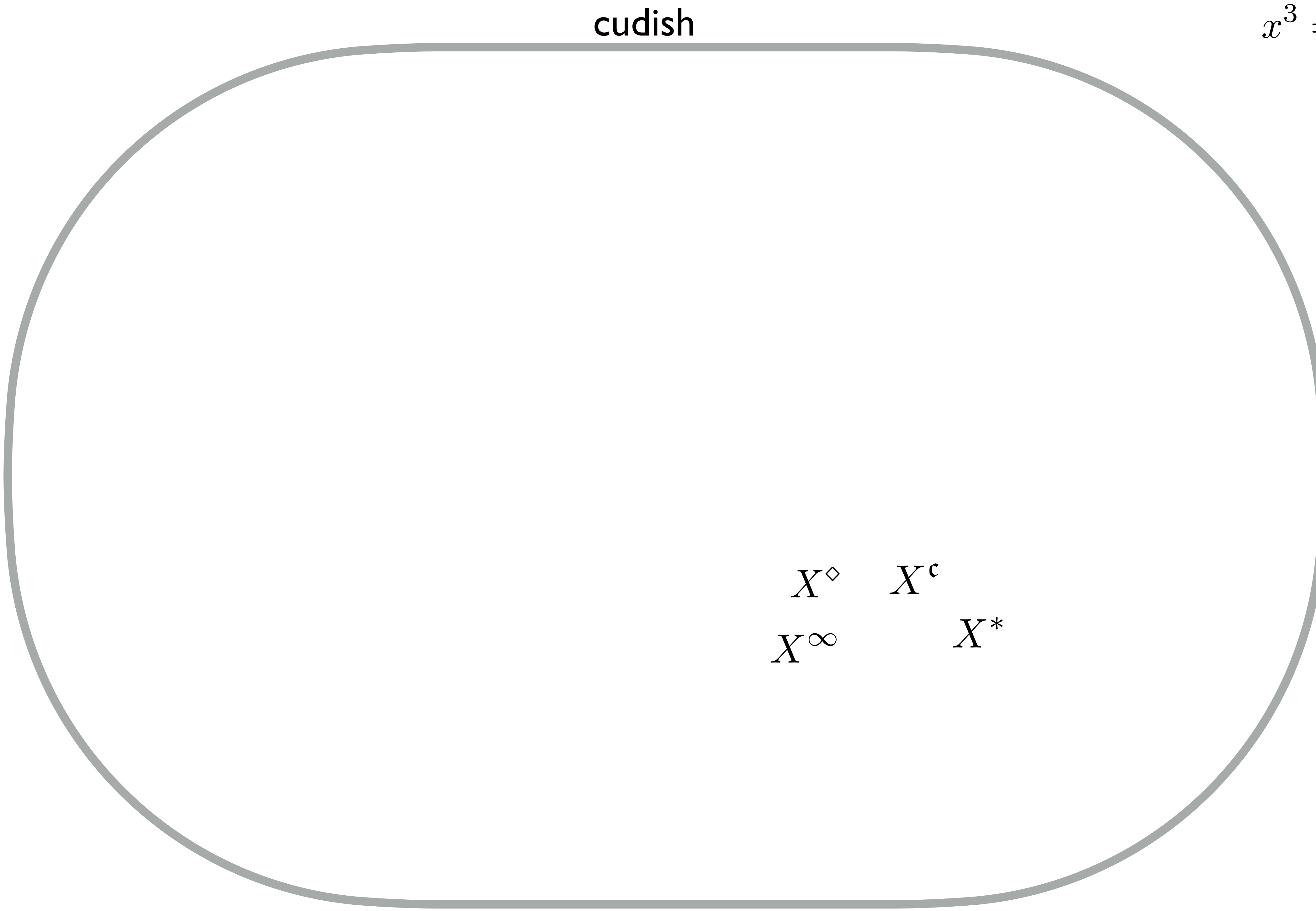

Sufficient condition 1

Fact: if T preserves finiteness
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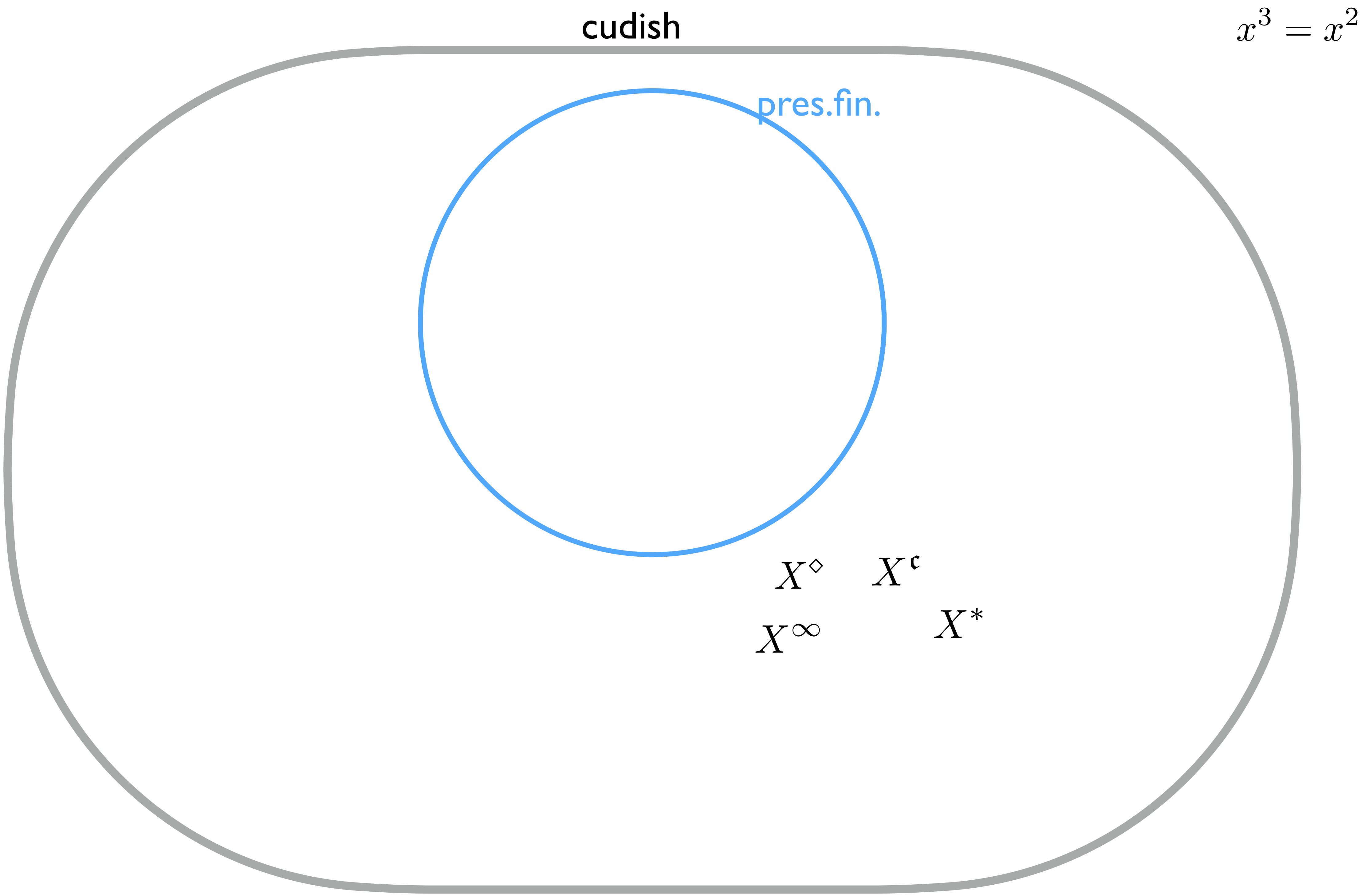
Examples:

- $\mathcal{P}$, $\mathcal{P}^+$, $\mathcal{P}_{\mathrm{fin}}$
- idempotent monoids/semigroups
- distributive lattices
- Boolean algebras

The landscape of monads



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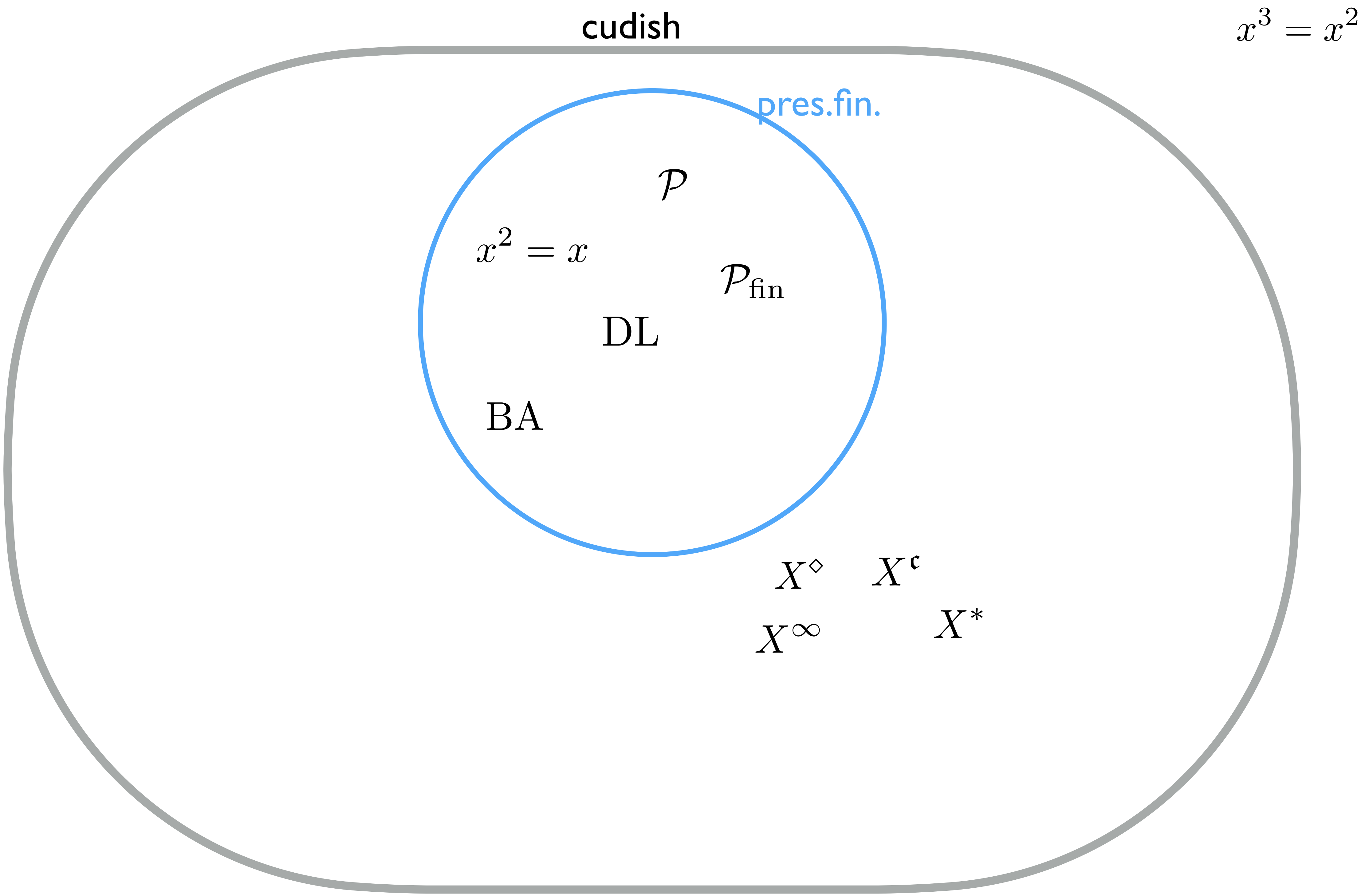

Sufficient condition II

Def.: a monad is Malcevian
if it admits (an eq. presentation with)
a ternary term $t(x, y, z)$ such that

$$t(x, x, y) = y = t(y, x, x)$$

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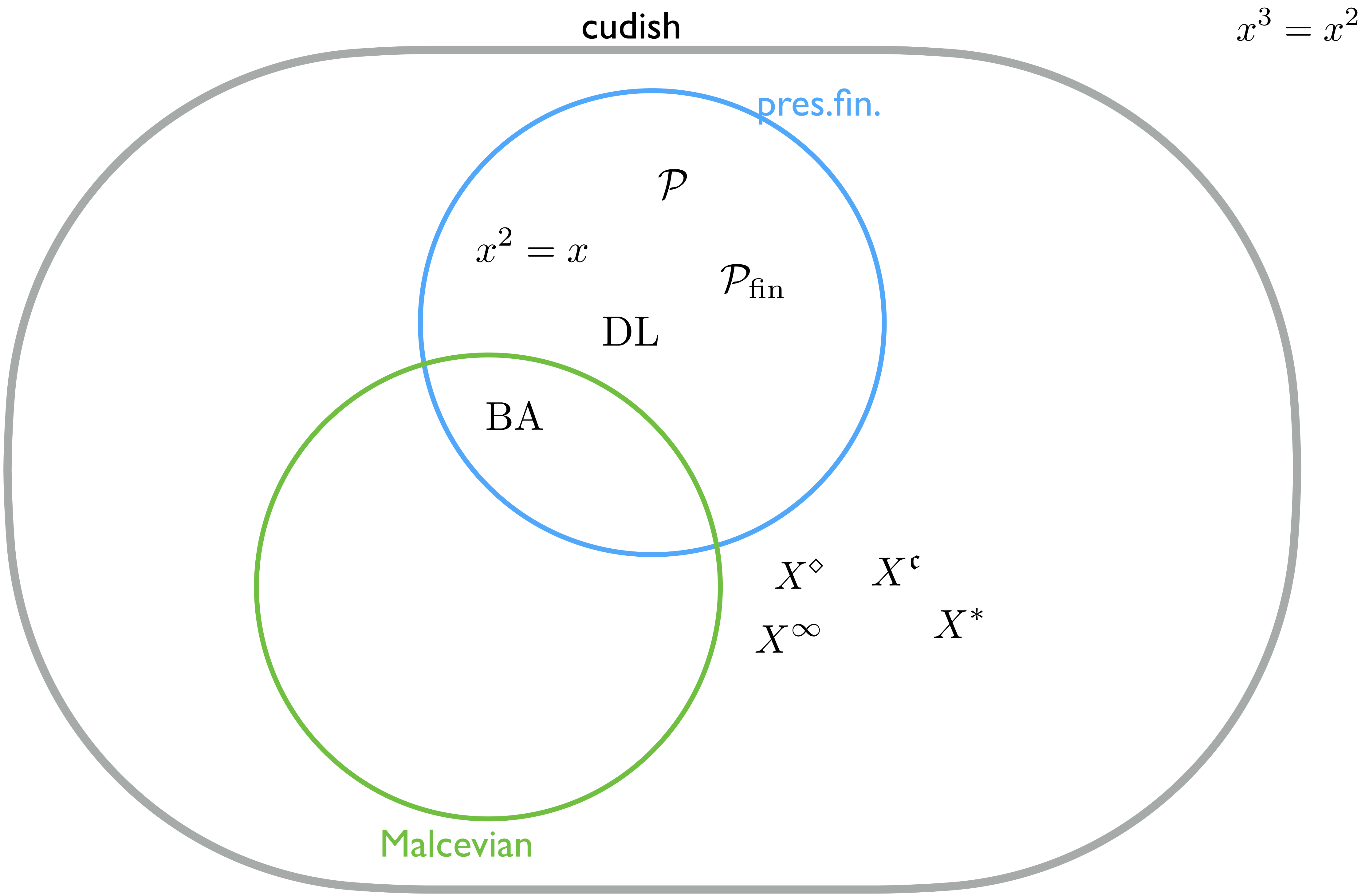
- Heyting algebras

$$t(x, y, z) = ((x \to y) \to z) \wedge ((z \to y) \to z) \wedge (x \vee z)$$

The landscape of monads



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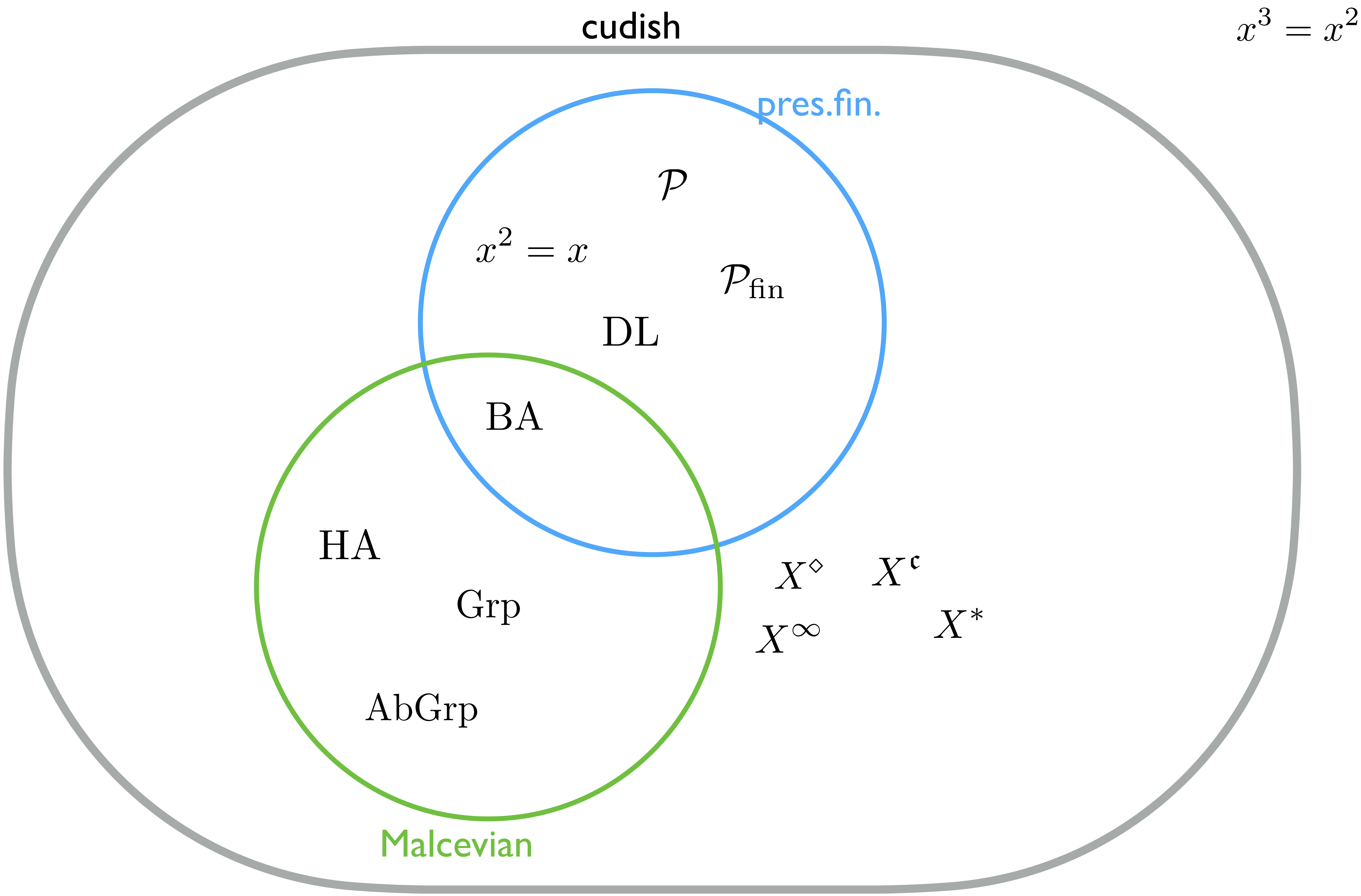

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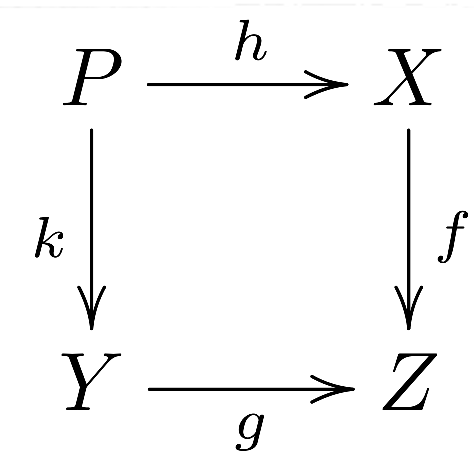
weak pullback:
for all $x \in X$, $y \in Y$ s.t. $f(x) = g(y)$
there is $p \in P$ s.t. $h(p) = x$, $k(p) = y$



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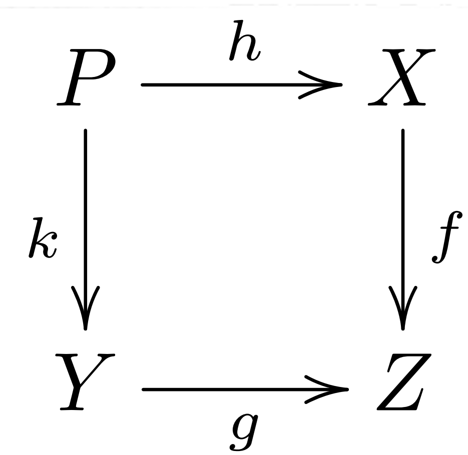
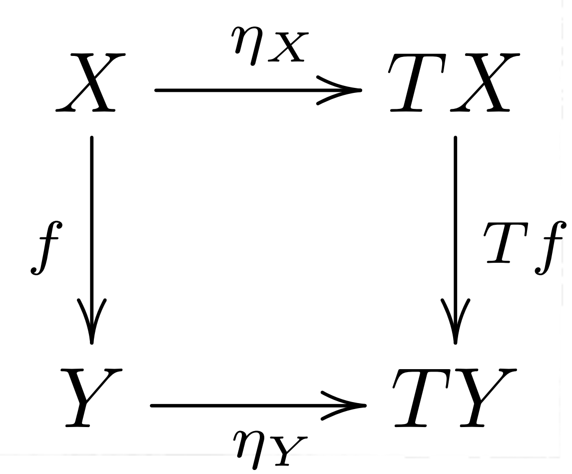


E.g. for η:

"a non-unit element never becomes a unit element after a substitution"



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Fact: weakly Cartesian monads are cudish.

(the powerset construction works)

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Examples:

- any monad presented by *linear regular equations:*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad ✓$$

$$x \cdot y = y \cdot x \quad ✓$$

$$x \cdot x = x \quad ✗$$

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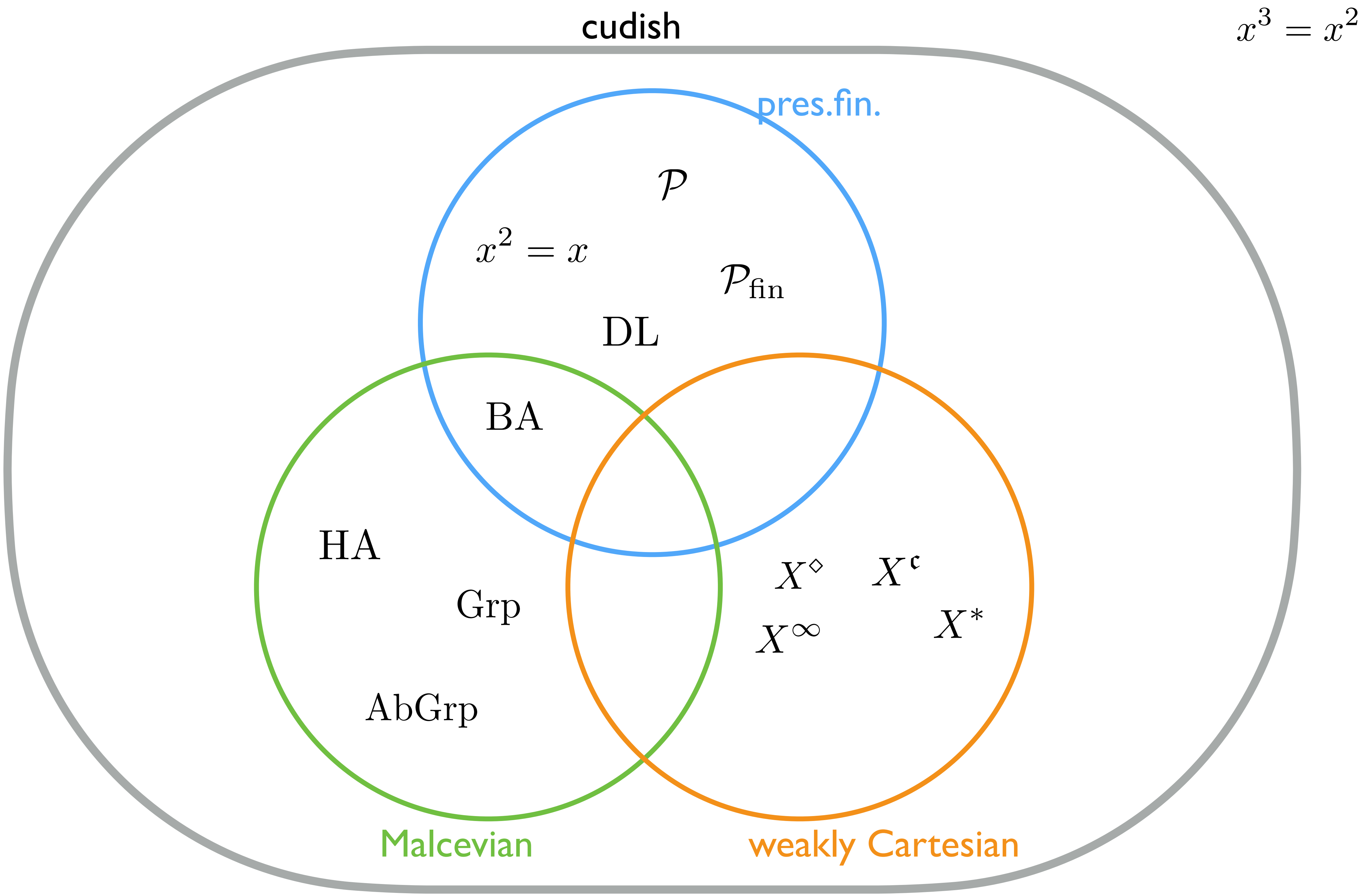
- T presented by a binary operation with:

$$x \cdot (x \cdot y) = x \cdot y$$

The landscape of monads



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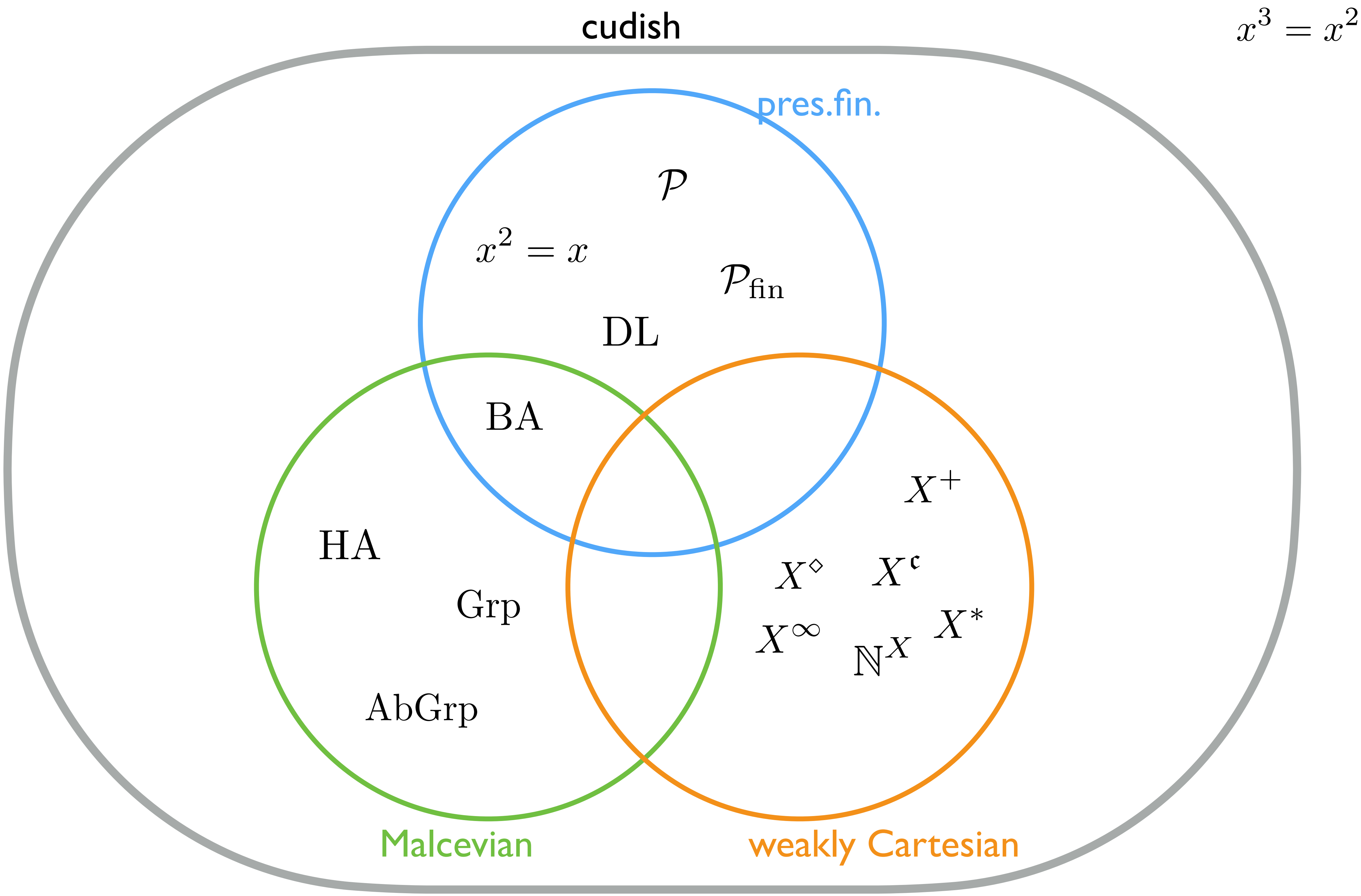

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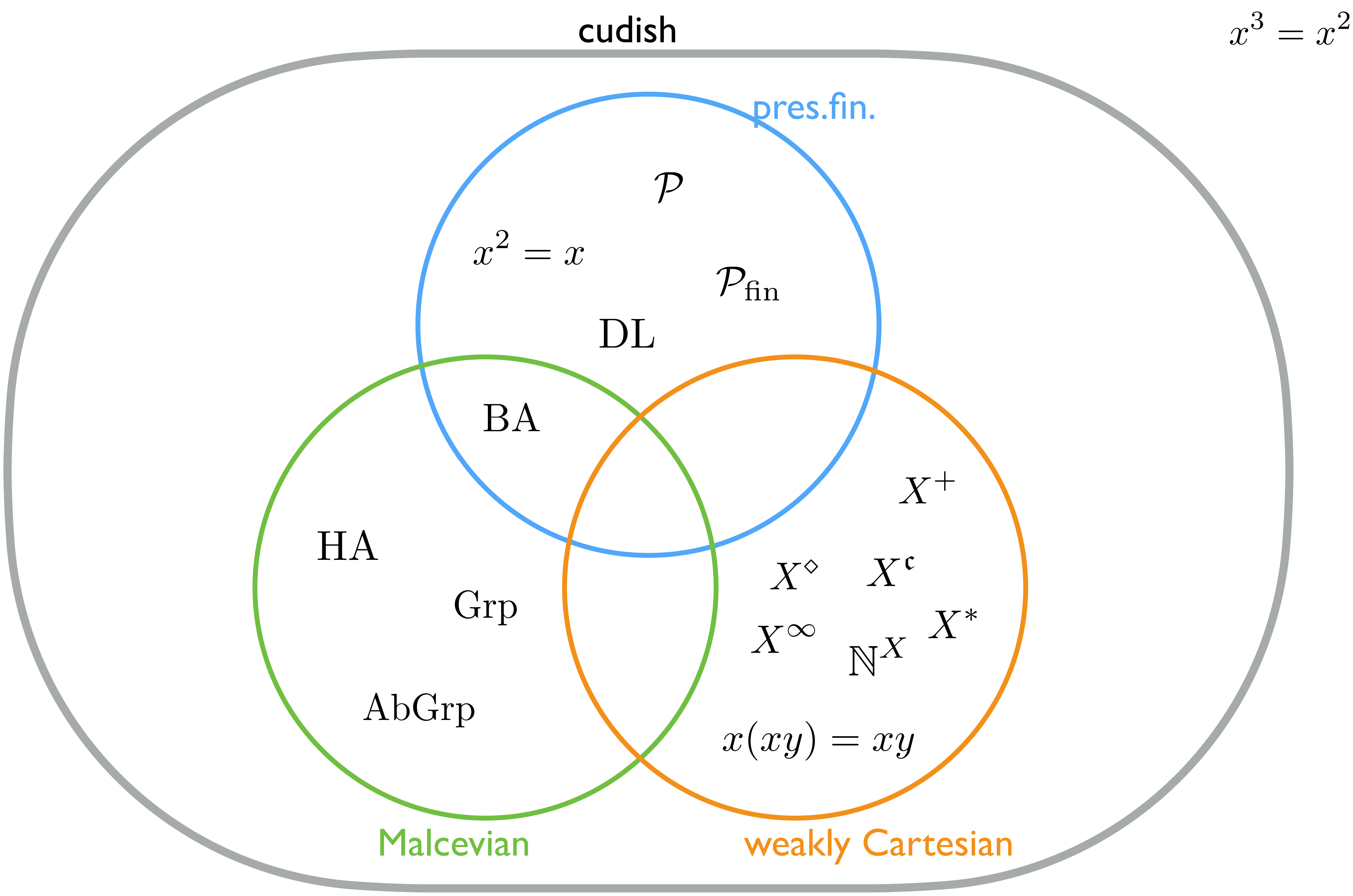

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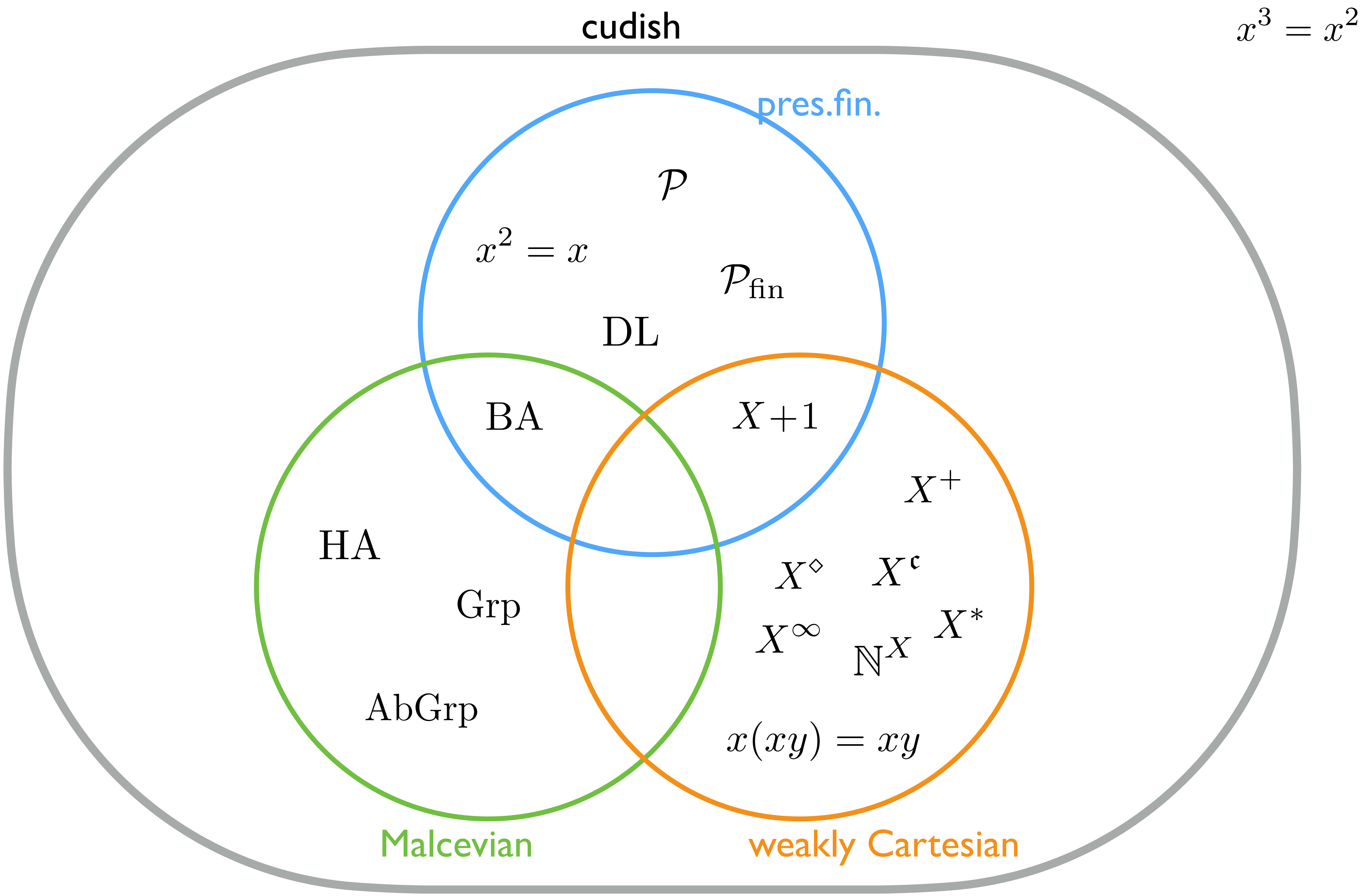

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3. A binary operation with:

$$z \cdot (x \cdot (x \cdot y)) = z \cdot (x \cdot y)$$

(a "powerset squared" construction works)

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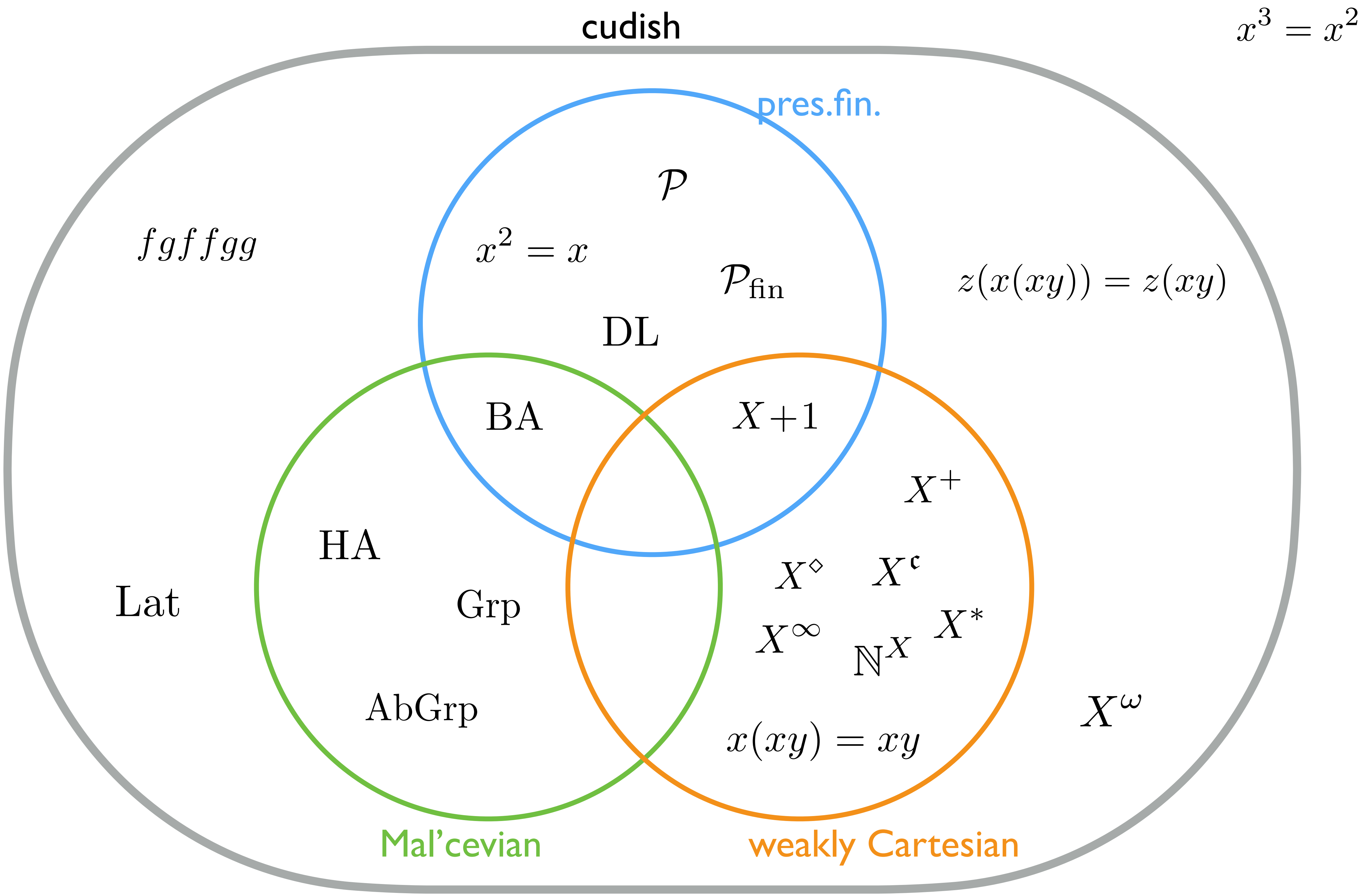
   (a "powerset squared" construction works)

4. Unary operations f, g with:

$$fgfgg(x) = x \qquad fgffgg(x) = fgffgg(y)$$

   (has no nontrivial finite algebras)

The landscape of monads



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4. The "almost Mal'cevian" monad:
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