On Termination of Probabilistic Programs

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What we all know about termination

The halting problem
— does a program P terminate on a given input state s? —
is semi-decidable.

The universal halting problem
— does a program P terminate on all input states? —
is undecidable.

Alan Mathison Turing
On computable numbers,
with an application to the Entscheidungsproblem
1937

What if programs roll dice?

A radical change

- A program either terminates or not (on a given input)
- Terminating programs have a finite run-time
- Having a finite run-time is compositional

All these facts do not hold for probabilistic programs!

Certain termination

```
while (x > 0) {
    x := x-1 [1/2] x := x-2
}
```

This program never diverges.

For all integer inputs x.

Almost-sure termination

For $0 < p < 1$ an arbitrary probability:

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

This program does not always terminate.
It diverges with probability zero.
It almost surely terminates.

Non almost-sure termination

```
P :: skip [1/2] { call P; call P; call P }
```

$$x_P = \frac{1}{2} \cdot 1 + \frac{1}{2} x_P^3$$

Non almost-sure termination

```
P :: skip [1/2] { call P; call P; call P }
```

This program terminates with probability $\frac{\sqrt{5}-1}{2} < 1$.

Positive almost-sure termination

For $0 < p < 1$ an arbitrary probability:

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

$Pr\{i = N\} = (1-p)^{N-1} \cdot p$

↓

finite expectation

This program almost surely terminates.

In finite expected time.

Despite its possible divergence.

Null almost-sure termination

Consider the symmetric one-dimensional random walk:

```
int x := 10; while (x > 0) { x-- [1/2] x++ }
```

This program almost surely terminates.
But:
It requires an infinite expected time to do so.

Nuances of termination

Olivier Bournez Florent Garnier

...... certain termination

...... termination with probability one

$\Longrightarrow$ almost-sure termination

...... in an expected finite number of steps

$\Longrightarrow$ "positive" almost-sure termination

...... a.s.-termination in an expected infinite number of steps

$\Longrightarrow$ "null" almost-sure termination

Three contributions

The hardness of the various notions of termination.

[MFCS 2015, Acta Informatica 2019]

A powerful proof rule for almost-sure termination.

[POPL 2018]

Proving positive almost-sure termination using weakest pre-conditions.

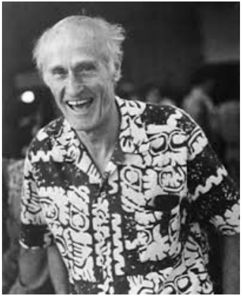
[ESOP 2016, J. ACM 2018]

Part 1: Hardness of termination

It is a known fact that deciding termination of ordinary programs is undecidable.

Our aim is to classify "how undecidable" (positive) almost-sure termination is.

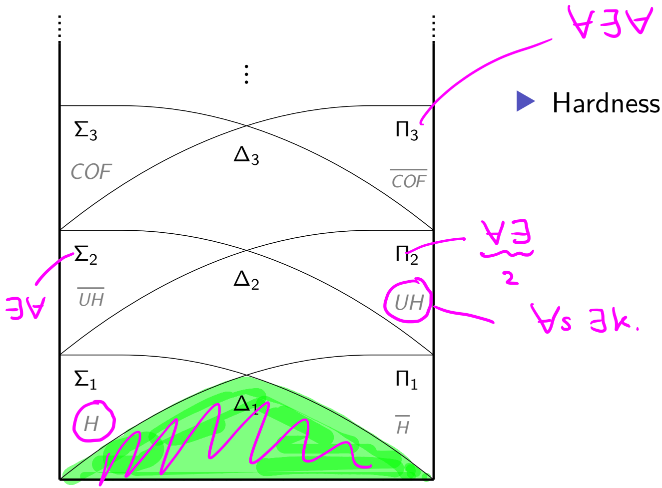
Kleene and Mostovski



Stephen Kleene (1909–1994)

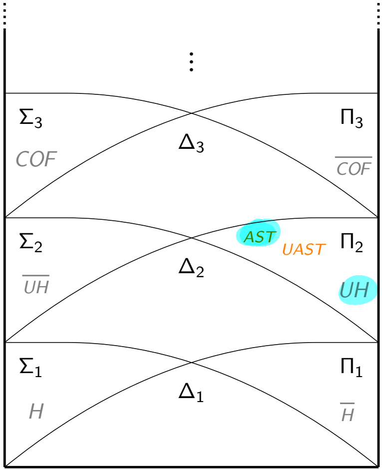
Andrzej Mostovski (1913–1975)

Hardness of almost-sure termination



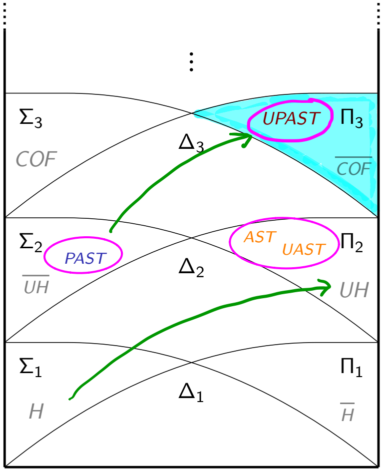
- Hardness landscape

Hardness of almost-sure termination



- Hardness landscape
- AST for one input is as hard as ordinary termination for all inputs

Hardness of almost-sure termination



- Hardness landscape
- AST for one input is as hard as ordinary termination for all inputs
- Finite termination is even "more undecidable"

Proof idea: hardness of positive as-termination

Reduction from the complement of the universal halting problem

For an ordinary program Q, provide a probabilistic program P (depending on Q) and an input s, such that

P terminates in a finite expected number of steps on s
if and only if
Q does not terminate on some input

$\overline{UH} \longmapsto PAST$

Q prob. program P_Q

Let's start simple

```
bool c := true;
int nrflips := 0;
while (c) {
    nrflips++;
    (c := false [1/2] c := true);
}
```

Expected runtime (integral over the bars):

The nrflips-th iteration takes place with probability $1/2^{\texttt{nrflips}}$.

Reducing an ordinary program to a probabilistic one

Assume an enumeration of all inputs for Q is given

```
bool c := true;
int nrflips := 0;
int i := 0;
while (c) {
    // simulate Q for one (further) step on its i-th input
    if (Q terminates) {
        cheer; // take 2^nrflips effectless steps
        i++;
        // reset simulation of program Q
    }
    nrflips++;
    (c := false [1/2] c := true);
}
```

Reducing an ordinary program to a probabilistic one

Assume an enumeration of all inputs for Q is given

```
bool c := true;
int nrflips := 0;
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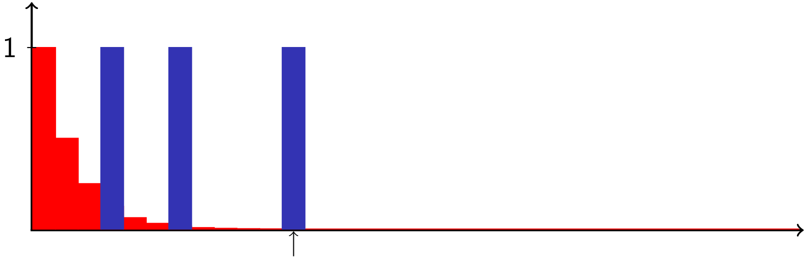
        cheer; // take 2^nrflips effectless steps
        i++;
        // reset simulation of program Q
    }
    nrflips++;
    (c := false [1/2] c := true);
}
```

P looses interest in further simulating Q by a coin flip to decide for termination.

Q does not always halt

Let i be the first input for which Q does not terminate.

Expected runtime of P (integral over the bars):

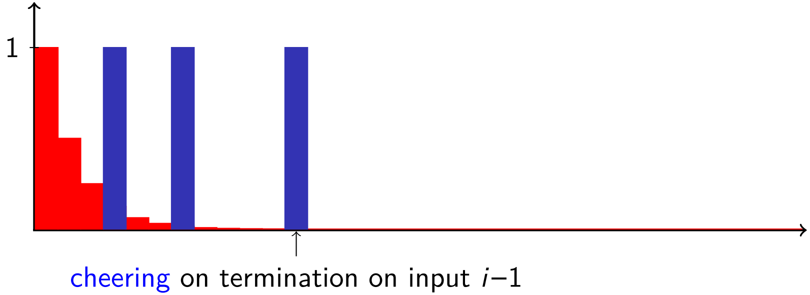


cheering on termination on input $i{-}1$

Q does not always halt

Let i be the first input for which Q does not terminate.

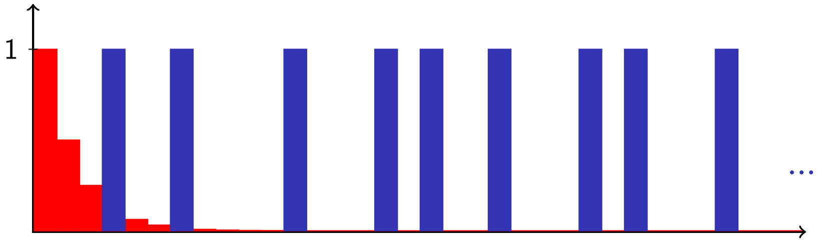
Expected runtime of P (integral over the bars):



Finite cheering — finite expected runtime

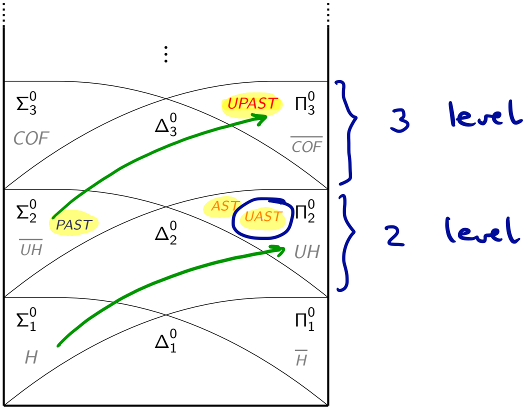
Q terminates on all inputs

Expected runtime of P (integral over the bars):



Infinite cheering — infinite expected runtime

Hardness of almost sure termination



No change for non-deterministic probabilistic programs.
No change when approximating termination probabilities.

Part 2: Proving almost-sure termination

- What? Termination with probability one. For all inputs.
- Why?
  - Reachability can be encoded as termination
  - Often a prerequisite for proving correctness
  - Often implicitly assumed
- Why is it hard in practice?
  - Requires a lower bound 1 for termination probability

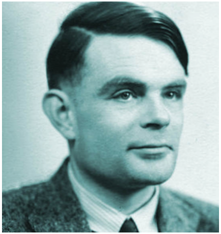
Almost-sure termination

"[Ordinary] termination is a purely topological property [. . .], but almost-sure termination is not. [. . .] Proving almost–sure termination requires arithmetic reasoning not offered by termination provers."

Javier Esparza
CAV 2012

How to prove termination?

Use a variant function on the program's state space whose value — on each loop iteration — is monotonically decreasing with respect to a (strict) well-founded relation.



Alan Mathison Turing
Checking a large routine
1949

Variant (aka: ranking) functions

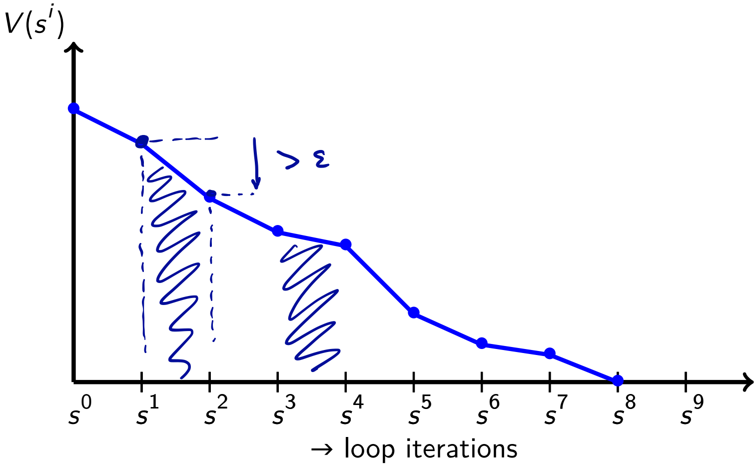
$V : \Sigma \rightarrow \mathbb{R}_{\geq 0}$ is variant function for loop `while`$(G)\ P$ if for every state s:

1. If $s \models G$, then P's execution on s terminates in a state t with:

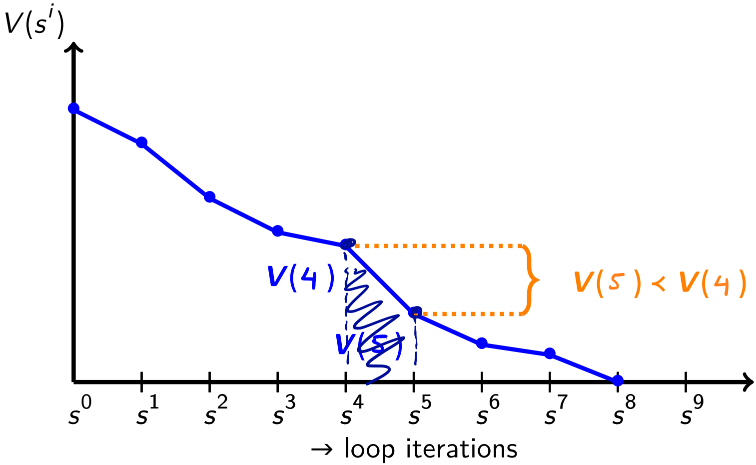
$$V(t) \;\leq\; V(s) - \varepsilon \quad \text{for some fixed } \varepsilon > 0\text{, and}$$

2. If $V(s) \leq 0$, then $s \not\models G$.

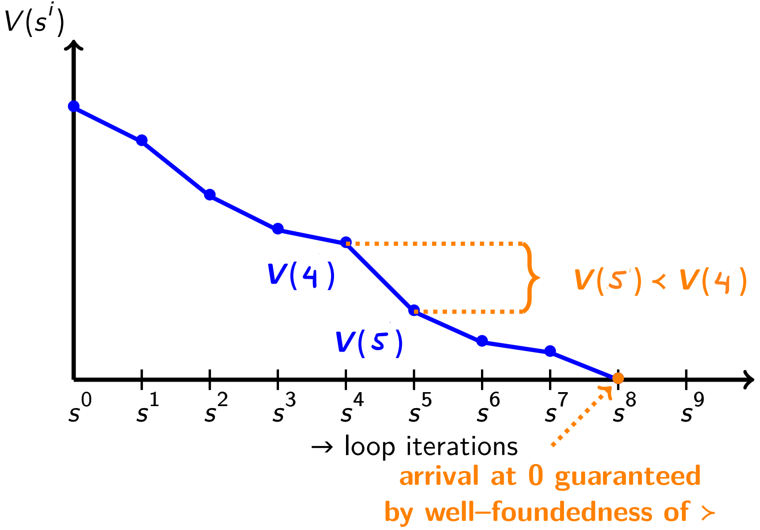
Termination proofs



Termination proofs



Termination proofs



Examples

```
while (x > 0) { x-- }
```

Ranking function $V = x$.

```
x := ... ; y := ... // x and y are positive
while (x != y) {
   if (x > y) { x := x-y } else { y := y-x }
}
```

Ranking function $V = x + y$.

A large body of existing works

Hart/Sharir/Pnueli: Termination of Probabilistic Concurrent Programs. POPL 1982

Bournez/Garnier: Proving Positive Almost-Sure Termination. RTA 2005

McIver/Morgan: Abstraction, Refinement and Proof for Probabilistic Systems. 2005

Esparza *et al.*: Proving Termination of Probabilistic Programs Using Patterns. CAV 2012

Chakarov/Sankaranarayanan: Probabilistic Program Analysis w. Martingales. CAV 2013

Fioriti/Hermanns: Probabilistic Termination: Soundness, Completeness, and Compositionality. POPL 2015

Chatterjee *et al.*: Algorithmic Termination of Affine Probabilistic Programs. POPL 2016

Agrawal/Chatterjee/Novotný: Lexicographic Ranking Supermartingales. POPL 2018

. . . . . .

Key ingredient: super- (or some form of) martingales

On super-martingales

A stochastic process $X_1, X_2, \ldots$ is a martingale whenever:

$$\mathbb{E}(X_{n+1} \mid X_1, \ldots, X_n) \;=\; X_n$$

It is a super-martingale whenever:

$$\mathbb{E}(X_{n+1} \mid X_1, \ldots, X_n) \;\leq\; X_n$$

A historical perspective

A countable Markov process is "non-dissipative"
if almost every infinite path eventually enters
— and remains in — positive recurrent states.

expected return
time $< \infty$

A historical perspective

A countable Markov process is "non-dissipative"
if almost every infinite path eventually enters
— and remains in — positive recurrent states.

A sufficient condition for being non-dissipative is:

$$\sum_{j \geq 0} j \cdot p_{ij} \;\leq\; i \qquad \text{for all states } i$$

ISBN nr.

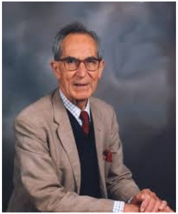
Frederic Gordon Foster
Markoff chains with an enumerable number of states
and a class of cascade processes
1951

Kendall's variation

A Markov process is non-dissipative if for some function $V : \Sigma \to \mathbb{R}$:

$$\sum_{j \geq 0} V(j) \cdot p_{ij} \leq V(i) \quad \text{for all states } i$$

and for each $r \geq 0$ there are finitely many states i with $V(i) \leq r$



David George Kendall

On non-dissipative Markoff chains
with an enumerable infinity of states

1951

Kendall notation

M/G/n

On positive recurrence

Every irreducible positive recurrent Markov chain is non-dissipative.

A Markov process is positive recurrent iff there is a Lyapunov function
$V : \Sigma \to \mathbb{R}_{\geq 0}$ with for finite $F \subseteq \Sigma$ and $\varepsilon > 0$:

$$\sum_j V(j) \cdot p_{ij} \quad < \quad \infty \qquad \text{for } i \in F\text{, and}$$

$$\sum_j V(j) \cdot p_{ij} \quad < \quad V(i) - \varepsilon \qquad \text{for } i \notin F.$$

Markov Chains pp 167-193 | Cite as

Lyapunov Functions and Martingales

Authors   Authors and affiliations

Pierre Brémaud

Pierre Brémaud 1999

Frederic Gordon Foster
On the stochastic matrices associated with certain queuing processes
1953

Our aim

A powerful, simple proof rule for almost-sure termination.

At the source code level.

No “descend” into the underlying probabilistic model.

Proving almost-sure termination

$V = x$

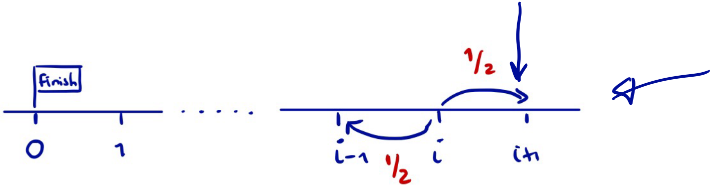
$\mathbb{E}(X_{k+1}) = X_k$

$< X_k - \varepsilon$

does not work

The symmetric random walk:

```
while (x > 0) { x := x-1 [1/2] x := x+1 }
```



Proving almost-sure termination

The symmetric random walk:

```
while (x > 0) { x := x−1 [1/2] x := x+1 }
```

$V = x$

Is out-of-reach for many proof rules.

A loop iteration decreases x by one with probability $^1/_2$

$d = 1$ $p = \frac{1}{2}$

Proving almost-sure termination

The symmetric random walk:

```
while (x > 0) { x := x-1 [1/2] x := x+1 }
```

Is out-of-reach for many proof rules.

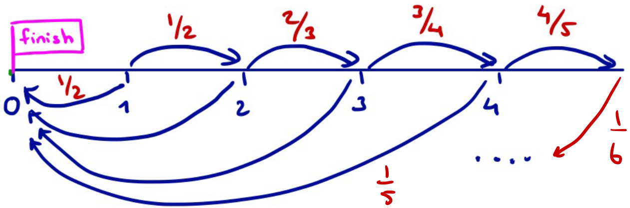
A loop iteration decreases x by one with probability $1/2$

This observation is enough to witness almost-sure termination!

Are these programs almost surely terminating?

- Escaping spline:

```
while (x > 0) { p := 1/(x+1); (x := 0 [p] x++) }
```



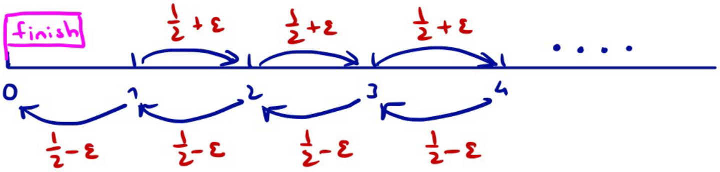
Are these programs almost surely terminating?

- Escaping spline:

```
while (x > 0) { p := 1/(x+1); (x := 0 [p] x++) }
```

- A slightly unbiased random walk:

```
1/2-eps ; while (x > 0) { x--  [p] x++ }
```



Are these programs almost surely terminating?

- Escaping spline:

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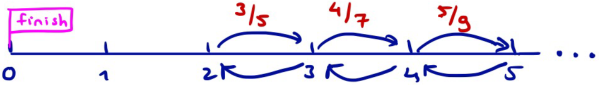
while (x > 0) { p := 1/(x+1); (x := 0 [p] x++) }
```

- A slightly unbiased random walk:

```
1/2-eps ; while (x > 0) { x--  [p] x++ }
```

- A symmetric-in-the-limit random walk:

```
while (x > 0) { p := x/(2*x+1) ; (x-- [p] x++) }
```



Proving almost-sure termination

Goal: prove a.s.–termination of `while(G) P`, for all inputs

Ingredients:

- A supermartingale $V : \Sigma \to \mathbb{R}_{\geq 0}$ with
  - $\mathbb{E}\{V(s_{n+1}) \mid V(s_0), \ldots, V(s_n)\} \leq V(s_n)$
  - Running body `P` on state $s \models$ `G` does not increase $\mathbb{E}(V(s))$
  - Loop iteration ceases if $V(s) = 0$
- ...... and a progress condition: on each loop iteration in s^i
  - $V(s^i) = v$ decreases by $\geq d(v) > 0$ with probability $\geq p(v) > 0$
  - with antitone p ("probability") and d ("decrease")

$x \leq y \longrightarrow f(x) \leq f(y)$ monotone

$x \leq y \longrightarrow f(y) \leq f(x)$ antitone

Proving almost-sure termination

Goal: prove a.s.–termination of `while(G) P`, for all inputs

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- A supermartingale $V : \Sigma \to \mathbb{R}_{\geq 0}$ with
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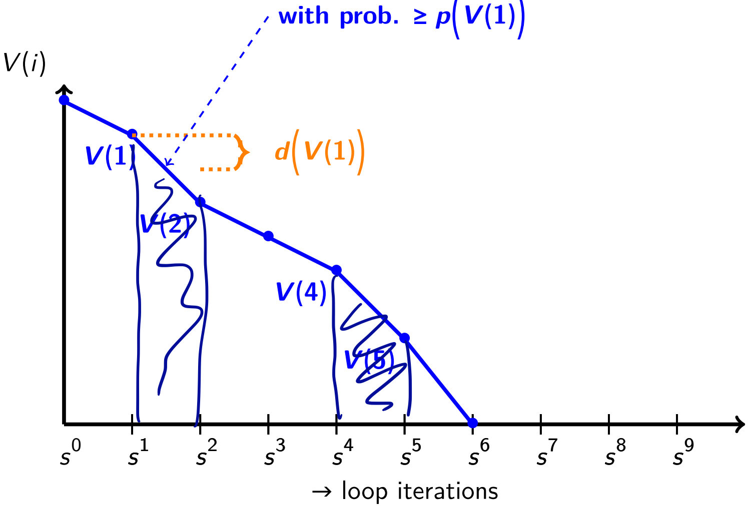
  - Loop iteration ceases if $V(s) = 0$

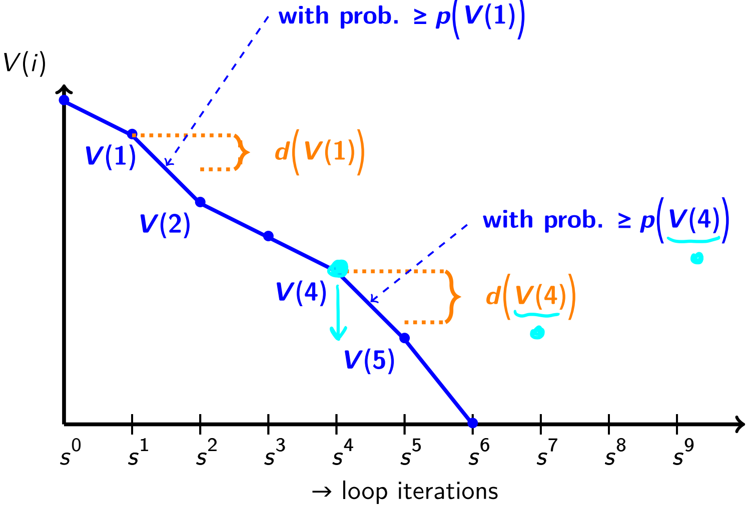
- . . . . . . and a progress condition: on each loop iteration in s^i
  - $V(s^i) = v$ decreases by $\geq d(v) > 0$ with probability $\geq p(v) > 0$
  - with antitone p ("probability") and d ("decrease")

Then: **`while(G) P` is universally almost-surely terminating**

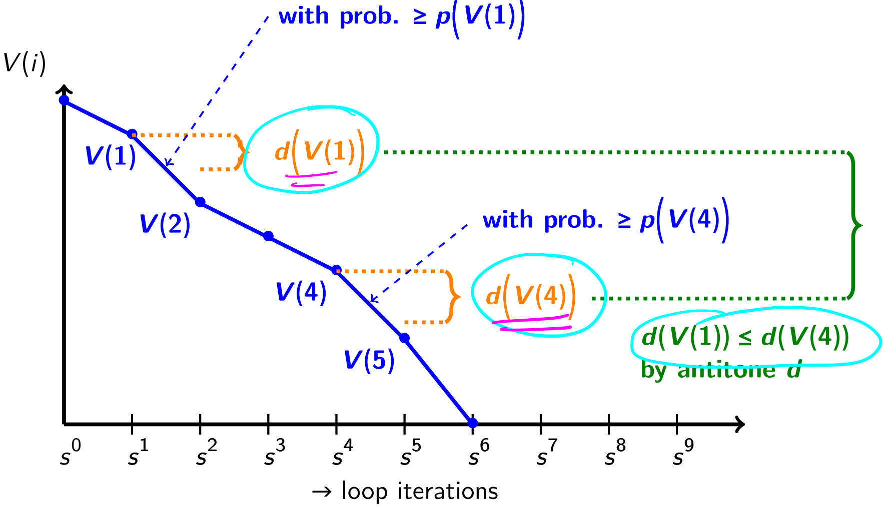
Proving almost-sure termination



Proving almost-sure termination



Proving almost-sure termination



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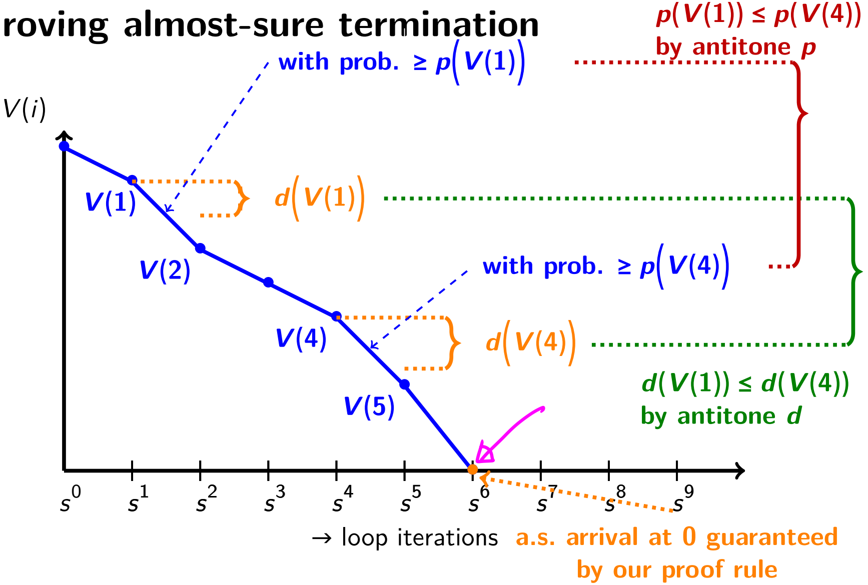
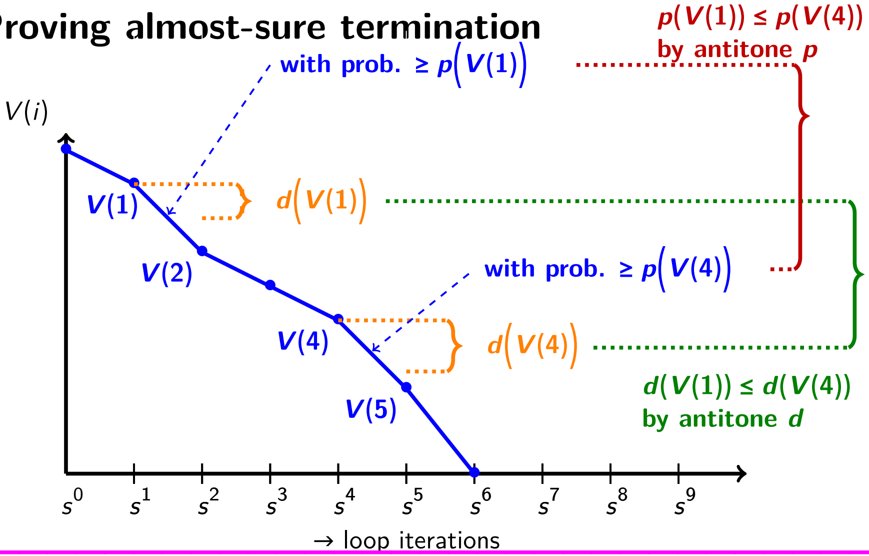


Proving almost-sure termination

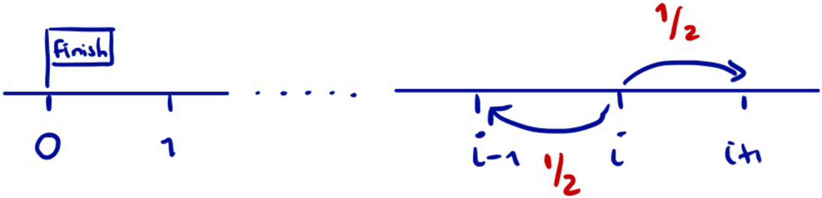


The closer to termination, the more V decreases and this becomes more likely

The symmetric random walk

- Recall:

```
while (x > 0) { x := x-1 [1/2] x := x+1 }
```



The symmetric random walk

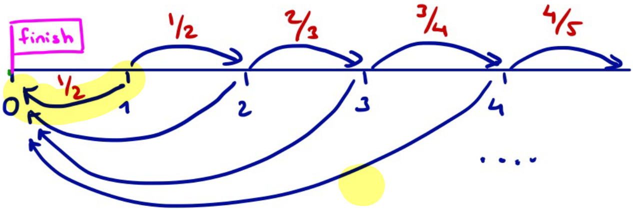
- Recall:

```
while (x > 0) { x := x-1 [1/2] x := x+1 }
```

- Witnesses of almost-sure termination:
  - $V = x$
  - $p(v) = 1/2$ and $d(v) = 1$

That's all you need to prove almost-sure termination!

The escaping spline

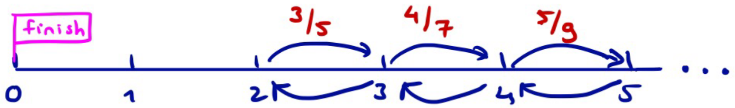


- Consider the program:

```
while (x > 0) { p := 1/(x+1); x := 0 [p] x++}
```

- Witnesses of almost-sure termination:
  - $V = x$
  - $p(v) = \frac{1}{v+1}$ and $d(v) = 1$

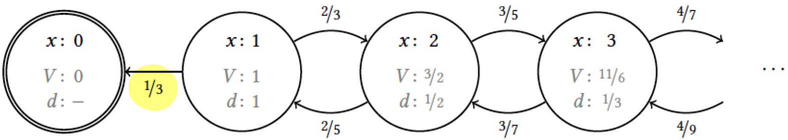
A symmetric-in-the-limit random walk



- Consider the program:

```
while (x > 0) { p := x/(2*x+1) ; x-- [p] x++ }
```

A symmetric-in-the-limit random walk



- Consider the program:

```
while (x > 0) { p := x/(2*x+1) ; x-- [p] x++ }
```

- Witnesses of almost-sure termination:
  - $V = H_x$, where H_x is x-th Harmonic number $1 + 1/2 + \ldots + 1/x$
  - $p(v) = 1/3$ and $d(v) = \begin{cases} 1/x & \text{if } v > 0 \text{ and } H_{x-1} < v \leq H_x \\ 1 & \text{if } v = 0 \end{cases}$

Part 3: Proving positive almost-sure termination

- What? Termination in finite expected time
- How?
  - Weakest-precondition calculus for expected run-times
- Why?
  - Reason about the efficiency of randomised algorithms
  - Reason about simulation (in)efficiency of Bayesian networks
  - Is compositional and reasons at the program's code

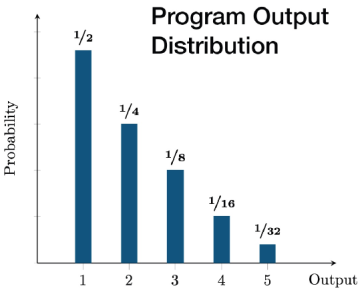
AST by weakest preconditions

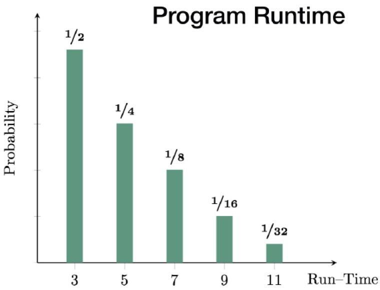
Determine $wp(P, \mathbf{1})$ for program P and postcondition $\mathbf{1}$.

Dexter Kozen
A probabilistic PDL
1983

The run time of a probabilistic program is random

```
int i := 0;
repeat {i++; (c := false [1/2] c := true)}
until (c)
```





The expected runtime is $1 + 3 \cdot 1/2 + 5 \cdot 1/4 + \ldots + (2n{+}1) \cdot 1/2^n \; = \; \ldots$.

Expected run-times

- Expected run-time of program P on input s:

$$\sum_{k=1}^{\infty} k \cdot Pr\left(\begin{array}{c} \text{“}P \text{ terminates after} \\ k \text{ steps on input } s\text{”} \end{array} \right)$$

- Let *ert* be a function $t : \Sigma \to \mathbb{R}_{\geq 0} \cup \{ \infty \}$
- This is called a run-time. Complete partial order ;

$$t_1 \preceq t_2 \quad \text{iff} \quad \forall s \in \Sigma.\ t_1(s) \leq t_2(s)$$

PAST is not compositional

```
int x := 1;
bool c := true;
while (c) {
    c := false [1/2] c := true;
    x := 2*x
}
```

Finite expected termination time

= PAST

PAST is not compositional

Consider the two probabilistic programs:

```
int x := 1;
bool c := true;
while (c) {
    c := false [1/2] c := true;
    x := 2*x
}
```

Finite expected termination time

```
while (x > 0) {
    x--
}
```

Finite termination time

PAST is not compositional

Consider the two probabilistic programs:

$\sum \frac{1}{2^x} \cdot 2^x = \infty$

PAST

```
int x := 1;
bool c := true;
while (c) {
    c := false [1/2] c := true;
    x := 2*x
}
```

;

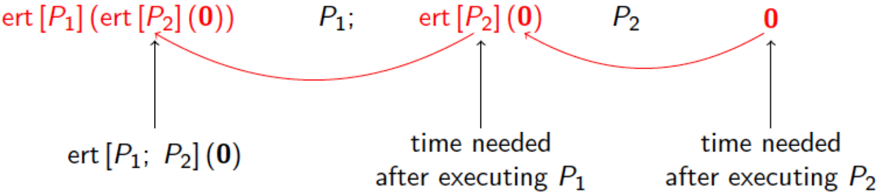
PAST

```
while (x > 0) {
    x--
}
```

not PAST

Run-times by program verification

$ert(P, t)(s)$ is the expected run-time of P on input state s
if t captures the run-time of the computation following P.



Expected run-time transformer

Syntax	Run-time $ert(P, t)$
`skip`	$\mathbf{1}+t$
`diverge`	∞
`x := E`	$\mathbf{1} + t[x := E]$
`P1 ; P2`	$ert(P_1, ert(P_2, t))$
`if (G) P1 else P2`	$\mathbf{1} + [G] \cdot ert(P_1, t) + [\neg G] \cdot ert(P_2, t)$
`P1 [p] P2`	$\mathbf{1} + p \cdot ert(P_1, t) + (1-p) \cdot ert(P_2, t)$
`while(G)P`	$\text{lfp } X. \mathbf{1} + ([G] \cdot ert(P, X) + [\neg G] \cdot t)$

lfp is the least fixed point operator wrt. the ordering $\preceq$ on run-times

Plus a set of proof rules to get bounds on run-times of loops

Elementary properties

- Continuity: $ert(P, t)$ is continuous, that is

  for every chain $T = t_0 \preceq t_1 \preceq t_2 \preceq \ldots$: $ert(P, \sup T) = \sup ert(P, T)$

- Monotonicity: $t \preceq t'$ implies $ert(P, t) \preceq ert(P, t')$

- Constant propagation: $ert(P, \mathbf{k} + t) = \mathbf{k} + ert(P, t)$

- Preservation of ∞: $ert(P, \infty) = \infty$

- Relation to wp: $\boxed{ert(P, t) = ert(P, \mathbf{0}) + wp(P, t)}$

- Affinity: $ert(P, r \cdot t + t') = ert(P, \mathbf{0}) + r \cdot wp(P, t) + wp(P, t')$

Elementary properties Isabelle/HOL certified [Hölzl]

- Continuity: $ert(P, t)$ is continuous, that is

  for every chain $T = t_0 \preceq t_1 \preceq t_2 \preceq \ldots : ert(P, \sup T) = \sup ert(P, T)$

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- Affinity: $ert(P, r \cdot t + t') = ert(P, \mathbf{0}) + r \cdot wp(P, t) + wp(P, t')$

Coupon collector's problem

ON A CLASSICAL PROBLEM OF PROBABILITY THEORY

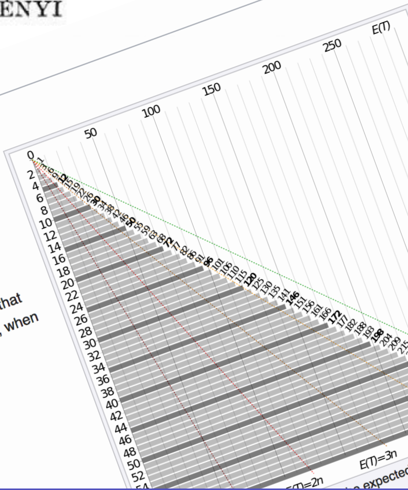
by

P. ERDŐS and A. RÉNYI

Coupon collector's problem

From Wikipedia, the free encyclopedia

In probability theory, the **coupon collector's problem** describes the "collect all coupons and win" contests. It asks the following question: Suppose that there is an urn of n different coupons, from which coupons are being collected, equally likely, with replacement. What is the probability that more than t sample trials are needed to collect all n coupons? An alternative statement is: Given n coupons, how many coupons do you expect you need to draw with replacement before having drawn each coupon at least once? The mathematical analysis of the problem reveals that the expected number of trials needed grows as $\Theta(n \log(n))$.[1] For example, when
...pected number ...bout 225[2] trials to collect all 50 coupons.



Coupon collector's problem

```
cp := [0,...,0]; i := 1; x := 0; // no coupons yet
while (x < N) {
   while (cp[i] != 0) {
       i := uniform(1..N) // next coupon
   }
   cp[i] := 1;  // coupon i obtained
   x++;  // one coupon less to go
}
```

Using the ert-calculus one can prove that:

$$ert(cpcl, \mathbf{0}) \;=\; \mathbf{4} + [N > 0] \cdot 2N \cdot (2 + H_{N-1}) \;\in\; \Theta(N \cdot \log N)$$

By systematic program verification à la Floyd-Hoare. Machine checkable.

How long to sample a Bayes' network?

“the main challenge in this setting [sampling-based approaches] is that many samples that are generated during execution are ultimately rejected for not satisfying the observations." [FOSE 2014]

Andy Gordon

Tom Henzinger

Aditya Nori

Sriram Rajamani

How long to simulate a Bayes network?

evidences

Benchmark BNs from www.bnlearn.com

ert

BN	$\|V\|$	$\|E\|$	aMB	$\|O\|$	EST	time (s)
hailfinder	56	66	3.54	5	$5\ 10^5$	0.63
hepar2	70	123	4.51	1	$1.5\ 10^2$	1.84
win95pts	76	112	5.92	3	$4.3\ 10^5$	0.36
pathfinder	135	200	3.04	7	∞	5.44
andes	223	338	5.61	3	$5.2\ 10^3$	1.66
pigs	441	592	3.92	1	$2.9\ 10^3$	0.74
munin	1041	1397	3.54	5	∞	1.43

aMB = *average Markov Blanket*, a measure of independence in BNs

Epilogue

① Hardness of probabilistic termination.

AST for one input $\equiv_{hard}$ universal halting problem.

Positive almost-sure termination is Π_3-complete.

② Proof rule for almost-sure termination.

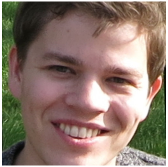
Widely applicable.

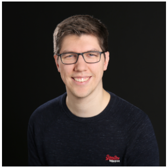
③ Weakest pre-conditions for expected run-time analysis.

To (dis)prove positive almost-sure termination. And more.

A big thanks to my co-authors!


Kevin Batz


Benjamin Kaminski


Christoph Matheja


Annabelle McIver


Carroll Morgan


Federico Olmedo

Further reading

- B. Kaminski, JPK, C. Matheja.
  On the hardness of analysing probabilistic programs. Acta Inf. 2019.
- B. Kaminski, JPK, C. Matheja, and F. Olmedo.
  Expected run-time analysis of probabilistic programs. J. ACM 2018.
- A. McIver, C. Morgan, B. Kaminski, JPK.
  A new proof rule for almost-sure termination. POPL 2018.
- K. Batz, B. Kaminski, JPK, and C. Matheja.
  How long, O Bayesian network, will I sample thee? ESOP 2018.
- K. Chatterjee, H. Fu and P. Novotny.
  Termination analysis of probabilistic programs with martingales.
  In: Found. of Prob. Programming, 2020 (to appear).

Using wp for expected run-times?

```
while(true) { x++ }
```

- Consider the post-expectation x
- Characteristic function $\Phi_x(X) = X(x \mapsto x+1)$
- Candidate upper bound is $I = \mathbf{0}$
- Induction: $\Phi_x(I) = \mathbf{0}(x := x+1) = \mathbf{0} = I \leq I$

We — wrongly — conclude that $\mathbf{0}$ is the runtime.

Using weakest pre-expectations is unsound for expected run-time analysis.