$\mathsf{FO} = \mathsf{FO}^3$ for Linear Orders with Monotone Binary Relations

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University of Liverpool

YR-OWLS, June 16, 2020

The k-variable property

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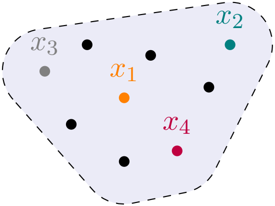
- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$



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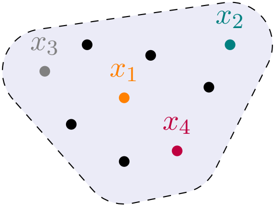

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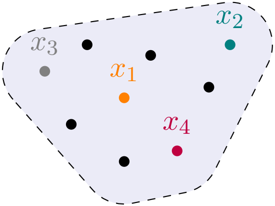

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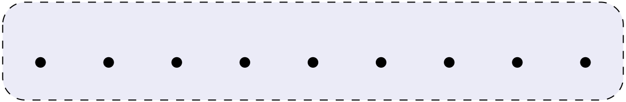

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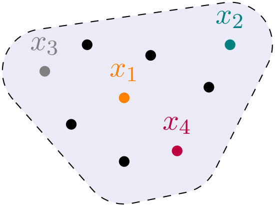

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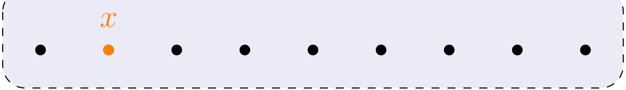

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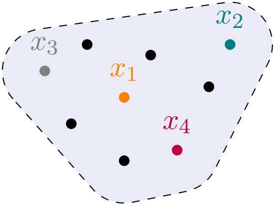

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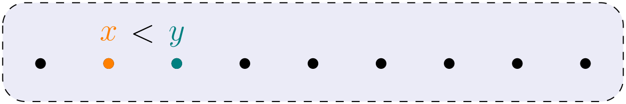

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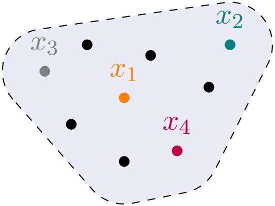

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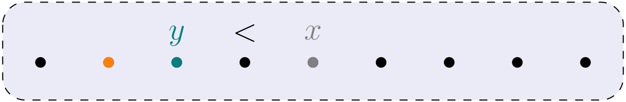

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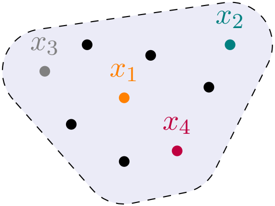

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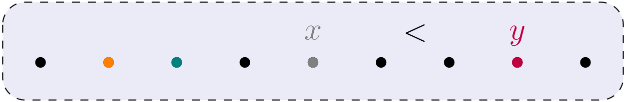

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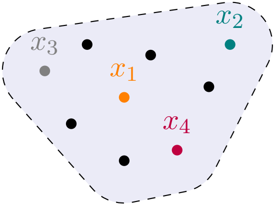

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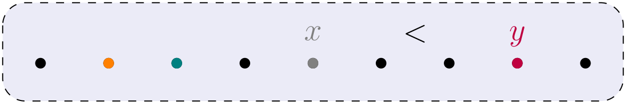

Over linear orders, FO $=$ FO^3.

Bounded variable logics

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[Gabbay 1981] In any class of time flows, TFAE:

- There exists an expressively complete finite set of FO-definable (multi-dimensional) temporal connectives
- There exists k such that every first-order sentence is equivalent to one with at most k variables

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Over (arbitrary) linear orders,
$\mathsf{FO}^3 \subseteq \mathsf{FO} = \mathsf{LTL}$ with Stavi connectives $\subseteq \mathsf{FO}^3$
[Gabbay, Hodkinson, Reynolds 1993]

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1. Corollary of expressive completeness of a temporal logic
   0 or 1 free variables
2. Ehrenfeucht-Fraïssé games with k pebbles
   up to k free variables

Known results (non-exhaustive)

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[Gastin-Mukund'02]

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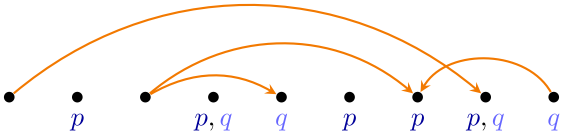
What do these 4 positive results have in common?

Generalisation [F.'19]

$\mathsf{FO} = \mathsf{FO}^3$ over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



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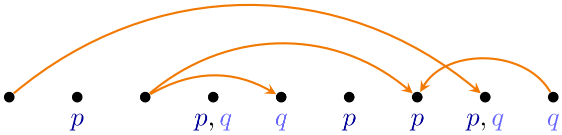

R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
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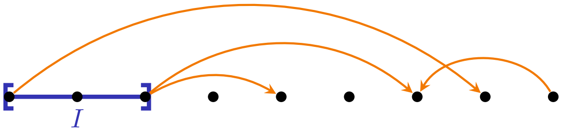

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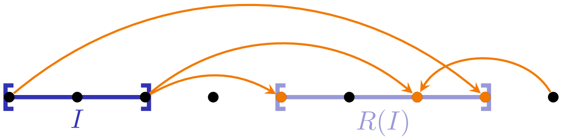

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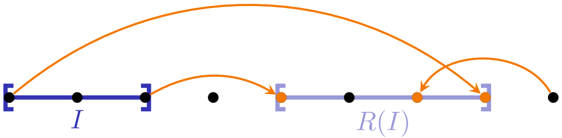

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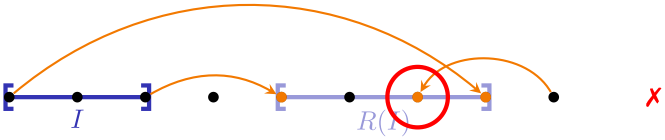

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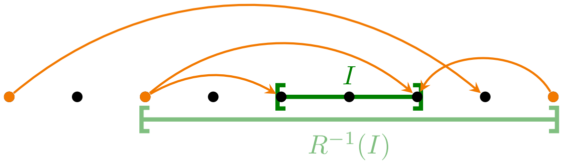

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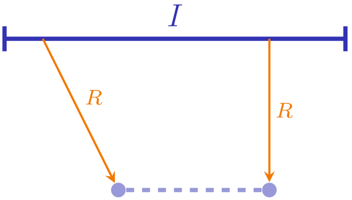

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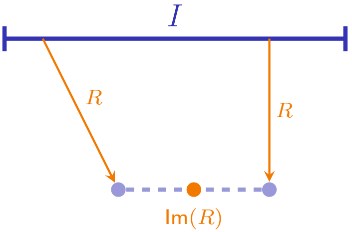

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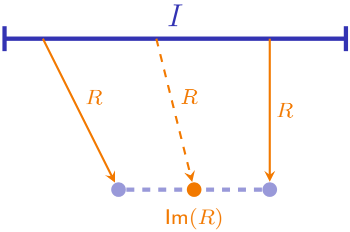

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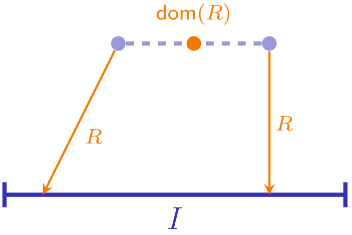

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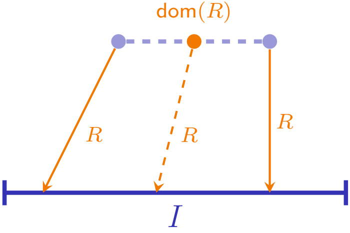

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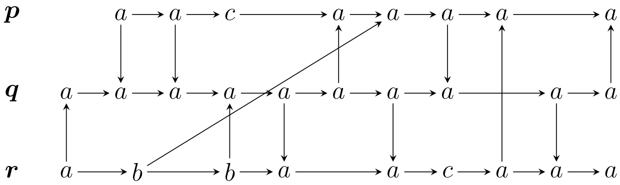

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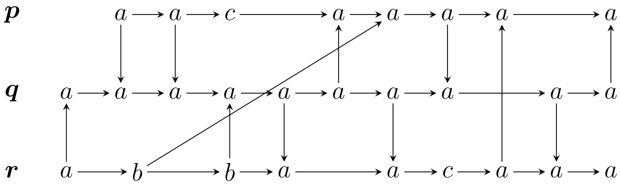

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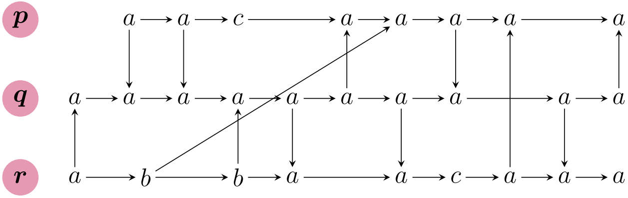

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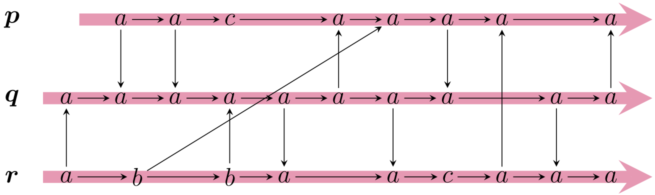

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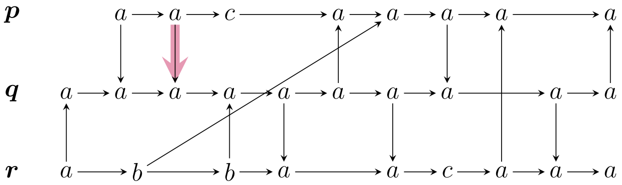

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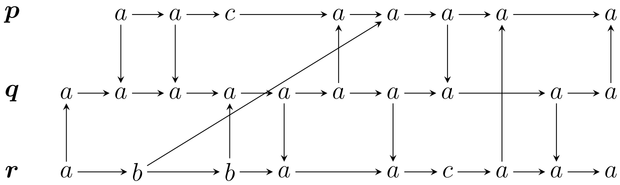

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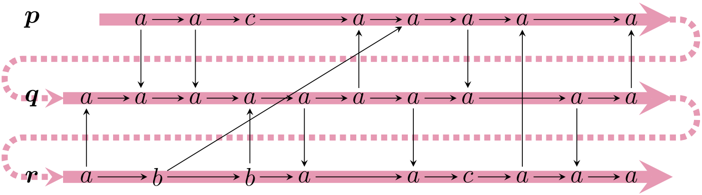

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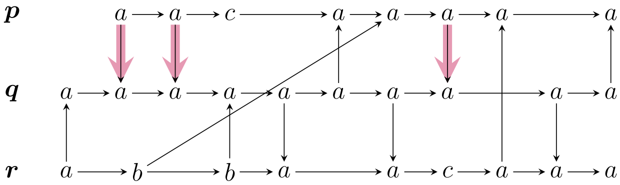
- Fixed, finite set of processes
- Process order $\leq_{\mathsf{proc}}$
- Message relations $\lhd_{p,q}$

Extended to a linear order

Applications

5. Message sequence charts (MSCs)



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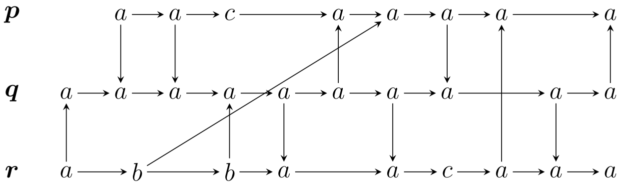

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FIFO $\rightarrow$ monotone

$\rightarrow$ Interval-preserving structure

Applications

FO $=$ FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q \in \mathbb{Q}})$ $\ldots$
4. $(\mathbb{R}, \leq)$ $+$ polynomial functions (new)
5. MSCs
6. Mazurkiewicz traces

How does the interval-preserving assumption help?

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$$\varphi(x_1, x_2, x_3) = \exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y)$$

$$\begin{tikzcd} & x_2 \arrow[dl, "R_2"'] \\ x_1 \arrow[r, "R_1"] & y \\ & x_3 \arrow[ul, "R_3"] \end{tikzcd}$$

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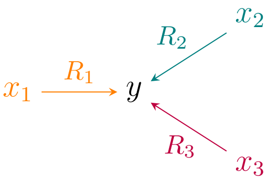

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$$
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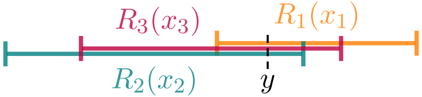

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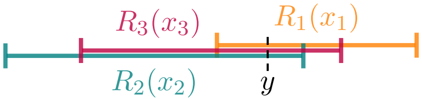

Equivalent FO^3 formula?

The proof

FO $=$ FO^3 over structures with

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Key idea: Go through an intermediate language:
Star-free Propositional Dynamic Logic.

$$\begin{tikzcd}
& \text{FO} \arrow[dl, bend right] & \\
\text{Star-free PDL} \arrow[rr] & & \text{FO}^3 \arrow[ul, bend right]
\end{tikzcd}$$

Star-free Propositional Dynamic Logic

Examples

Star-free Propositional Dynamic Logic

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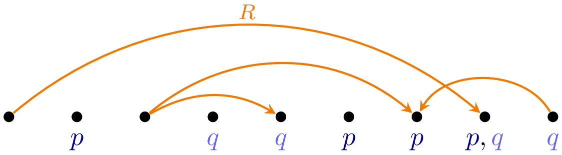

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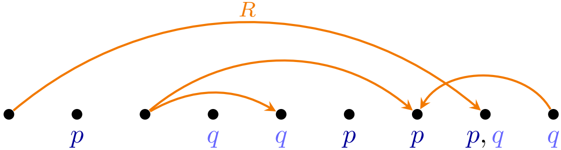

$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓

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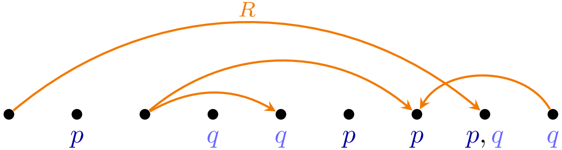

$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓
$\langle R \rangle\, q$	✓	✗	✓	✗	✗	✗	✗	✗	✗

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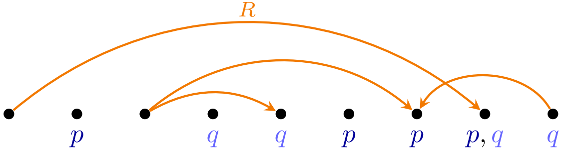

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$\langle \leq \cdot R^{-1} \rangle\, q$	✓	✓	✓	✓	✓	✓	✓	✗	✗

Star-free Propositional Dynamic Logic

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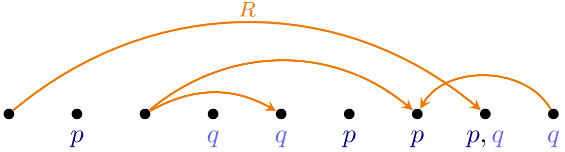

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$\langle \leq \cdot \{\langle R \rangle\, q\}? \cdot \leq \rangle\, p$	✓	✓	✓	✗	✗	✗	✗	✗	✗

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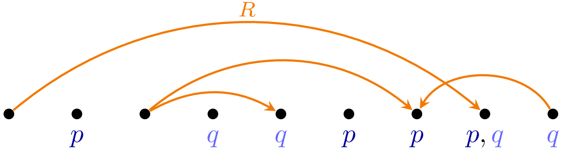

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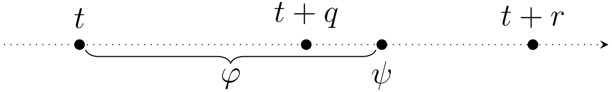
$\langle R^c \cap \leq \rangle\, (p \wedge q)$	✗	✓	✓	✓	✓	✓	✓	✓	✗

Star-free Propositional Dynamic Logic

Examples

Over $(\mathbb{R}, <, \{+q \mid q \in \mathbb{Q}_+\})$,

$\varphi \,\mathsf{U}_{(q,r)}\, \psi \equiv$



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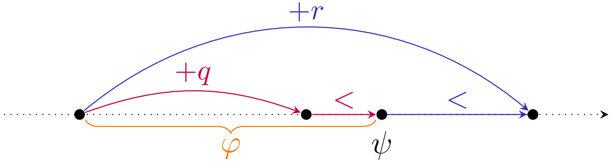
$$\varphi\, \mathsf{U}_{(q,r)}\, \psi \equiv \left\langle (+q \cdot <) \cap (+r \cdot <^{-1}) \cap (< \cdot \{\neg\varphi\}? \cdot <)^{\mathsf{c}} \right\rangle \psi$$



Star-free Propositional Dynamic Logic

Syntax

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\,\varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathsf{PDL}_{\mathsf{sf}}$

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Combines features from

- Propositional Dynamic Logic [Fisher-Ladner 1979]
- Star-free regular expressions
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PDL_{sf}

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- Propositional Dynamic Logic [Fisher-Ladner 1979]
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- The calculus of relations

Theorem: [Tarski-Givant 1987 (calculus of relations)]
PDL_{sf} and FO^3 are expressively equivalent

A fragment of Star-free PDL

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$\mathsf{PDL_{sf}}$

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cap \pi \mid (\leq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\leq \cdot \pi \cdot \geq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \geq)^{\mathsf{c}}$$

$\mathsf{PDL_{sf}^{int}}$

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$\mathsf{PDL_{sf}^{int}}$

Lemma: $\forall \pi \in \mathsf{PDL_{sf}^{int}}$, $[\![\pi]\!]$ is interval-preserving

Equivalences over interval-preserving structures

$$\begin{tikzcd}
\mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d] \\
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Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

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- Atomic formulas, disjunction: easy

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- Negation: Express π^{c} using $(\leq \cdot \pi \cdot \leq)^{\mathsf{c}}$, $(\leq \cdot \pi \cdot \geq)^{\mathsf{c}}$, $(\geq \cdot \pi \cdot \leq)^{\mathsf{c}}$, $(\geq \cdot \pi \cdot \geq)^{\mathsf{c}}$.

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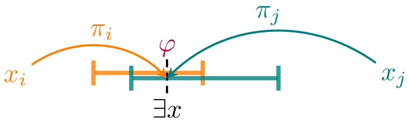

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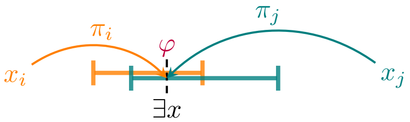

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$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity > 2?
- Uniform approach for proving completeness of temporal logics?

Thank you!