Multi-stage programming
Part one: static vs dynamic

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> “**All** problems in computer science can be solved by another level of indirection

Multi-stage programming: a complement to abstraction

> “**All** problems in computer science can be solved by another level of indirection
>
> (…**except** for the problem of too many layers of indirection.)”

Mechanics: quotes and splices

MetaOCaml, **Template Haskell**, &c.: multi-stage programming with **code quoting**.

Stages: current (available now) and delayed (available later).
(Also double-delayed, triple-delayed, etc.)

Brackets

```
.< e >.
```

Escaping (within brackets)

```
.~e
```

Running code

```
!. e
```

Cross-stage persistence

```
.< x >.
```

Goal: generate a specialized program with better performance

Mechanics: evaluation

`.< e >.`

do not reduce `e`

`.~e`

(inside `.< ... >.`)
reduce `e`

Multi-stage programming guarantees

$$\Gamma \vdash^{n} \texttt{e} : \tau$$

$$\frac{\Gamma \vdash^{n+} \texttt{e} : \tau}{\Gamma \vdash^{n} \texttt{.<e>.} : \tau \text{ code}} \text{ T-bracket}$$

$$\frac{\Gamma \vdash^{n} \texttt{e} : \tau \text{ code}}{\Gamma \vdash^{n} \texttt{!. e} : \tau} \text{ T-run}$$

$$\frac{\Gamma \vdash^{n} \texttt{e} : \tau \text{ code}}{\Gamma \vdash^{n+} \texttt{.\~{}e} : \tau} \text{ T-escape}$$

$$\frac{\Gamma(x) = \tau^{(n-m)}}{\Gamma\ x \vdash^{n} : \tau} \text{ T-var}$$

Guarantee: well-typed generating programs generate well-typed programs

Guarantee: what you quote is what you get

Self-optimizing libraries

Stream Fusion, to Completeness

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Staged stream processing (POPL 2017)

Staged Generic Programming

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Generic programming libraries such as *Scrap Your Boilerplate* eliminate the need to write repetitive code, but typically introduce significant performance overheads. This leaves programmers with the regrettable choice between writing succinct but slow programs and writing tedious but efficient programs.

Applying structured multi-stage programming techniques transforms *Scrap Your Boilerplate* from an inefficient library into a typed optimising code generator, bringing its performance in line with hand-written

Staged *Scrap Your Boilerplate* (ICFP 2017)

A Typed, Algebraic Approach to Parsing

NEELAKANTAN R. KRISHNASWAMI, University of Cambridge
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Staged parser combinators (PLDI 2019)

Implementations

1998		MetaML		
2000	MetaOCaml			
2002			Template Haskell	
2010	BER MetaOCaml			
2013			Typed Template Haskell	
2015	Modular macros			
2018				Dotty macros

`power` in one stage:

```
power :: Int → Int → Int
power x 0 = 1
power x n = x * power x (n - 1)
```

```
λ> power 2 6
64
```

The power of multi-stage programming

`power` in multiple stages (first exponent, then base)

```
power :: Code Int → Int → Code Int
power x 0 = ⟦1⟧
power x n = ⟦$x * $(power ⟦x⟧ (n - 1))⟧
```

```
λ> ⟦\x → $(power ⟦x⟧ 6) ⟧
⟦\x → x * (x * (x * (x * (x * (x * 1))))) ⟧
```

Terminology: values of type `Code t` are **dynamic**. Other values are **static**.

The power of multi-stage programming

Generated code:

```
λ> ⟦\x → $(power ⟦x⟧ 6) ⟧
⟦\x → x * (x * (x * (x * (x * (x * 1))))) ⟧
```

Problem: generated code rather inefficient. Better:

```
⟦\x → x * (x * (x * (x * (x * x)))) ⟧
```

Even better:

```
⟦\x → let y = x * x in let z = y * y in z * y ⟧
```

How should we fix `power`? (first attempt)

Solution one: rewrite `power` to handle `n = 1`:

```
power :: Code Int → Int → Code Int
power x 0 = ⟦1⟧
power x 1 = x
power x n = ⟦$x * $(power ⟦x⟧ (n - 1))⟧
```

Generated code:

```
λ> ⟦\x → $(power ⟦x⟧ 6) ⟧
⟦\x → x * (x * (x * (x * (x * x)))) ⟧
```

Objection: changing code **structure** to help staging is undesirable

How should we fix `power`? (second attempt)

Solution two: introduce a **type that subsumes static & dynamic**

```
data SD a = Sta :: a → SD a
          | Dyn :: Code a → SD a
```

and a function that **converts** `sd` **values to** `code`

```
cd :: Lift a ⇒ SD a → Code a
cd (Sta s) = ⟦ s ⟧  -- (cross-stage persistence)
cd (Dyn d) = d
```

and **multiplication** for `sd` values that special-cases `1` and `0`:

```
(⊛) :: SD Int → SD Int → SD Int
Sta x ⊛ Sta y = x * y
Sta 0 ⊛ _     = Sta 0
_ ⊛ Sta 0     = Sta 0
Sta 1 ⊛ y     = y
y ⊛ Sta 1     = y
x ⊛ y         = ⟦ $(cd x) * $(cd y) ⟧
```

Finally, **rewrite** `pow` to use `sd`:

```
power x 0 = Sta 1
power x n = x ⊛ pow x (n - 1)
```

How should we fix `power`? (second attempt: problems)

The `sd` type **fixes `pow`** (somewhat) without changing code structure:

```
λ> ⟦\x → $(cd (power (Dyn ⟦x⟧) (Sta 6))) ⟧
⟦\x → x * (x * (x * (x * (x * x)))) ⟧
```

However, `sd` is **not a complete solution**.

Consider the generated code for the following expression:

```
(Sta 2 ⊛ Dyn ⟦x⟧) ⊛ Sta 3

⇝ ⟦(2 * x) * 3 ⟧
```

We could simplify further (since $*$ is **associative** & **commutative**).

Example: dot product

`dot`, **unstaged**:

```
dot :: [Int] → [Int] → [Int]
dot [] [] = 0
dot (x:xs) (y:ys) = (x * y) + dot xs ys
```

`dot`, **staged** (assuming vector structure known, values of one vector unknown):

```
dot :: [Int] → [Code Int] → [Code Int]
dot [] [] = ⟦ 0 ⟧
dot (x:xs) (y:ys) = ⟦ (x * $y) + $(dot xs ys) ⟧
```

Generated code:

```
dot [1,0,2] [⟦ x ⟧, ⟦ y ⟧, ⟦ z ⟧]
⇝
⟦ (1 * x) + (0 * y) + (2 * z) ⟧
```

Desired code:

```
⟦ x + (2 * z) ⟧
```

Example: printf

sprintf, **unstaged**:

```
lit x = \k s → k (s ++ x)
f `cat` g = f · g
```

```
int = \k s x → k (s ++ show x)
sprintf p = fmt p id ""
```

Typical use:

```
sprintf ((int `cat` lit "a") `cat` (lit "b" `cat` int))
```

sprintf, **staged**:

```
lit x = \k s → k ⟦ $s ++ x ⟧
f `cat` g = (f · g)
```

```
int = \k s x → k ⟦ $s ++ show $x ⟧
sprintf p = p id ⟦ "" ⟧
```

Generated code:

```
⟦ λx y →((("" ++ show x) ++ "a") ++ "b") ++ show y ⟧
```

Desired code:

```
⟦ λx y →show x ++ ("ab" ++ show y) ⟧
```

Small suspicion

Might these common problems
share a common solution?

Remainder of today

Partially-static data, motivated

With **control over** β only, generated code is inefficient:

```
λ> ⟦\x → $(power ⟦x⟧ 6) ⟧
⟦\x → x * (x * (x * (x * (x * (x * 1)))))) ⟧
```

With support for **algebraic laws** we can generate better code:

```
⟦\x → let y = x * x in let z = y * y in z * y ⟧
```

Partially-static data

Building *equation-aware* structures

Plan: *drop-in* replacements for

$\langle \texttt{String}, \texttt{++} \rangle$

$\langle \texttt{Int}, \texttt{+}, \texttt{*} \rangle$

$\langle \texttt{Bool}, \wedge, \vee \rangle$

etc.!

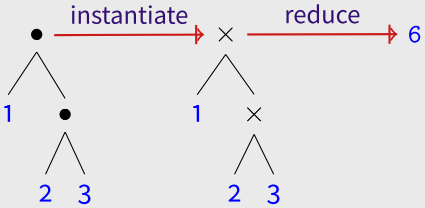
`Magma`, a minimal structure

```
class Magma a where (•) :: a → a → a
```

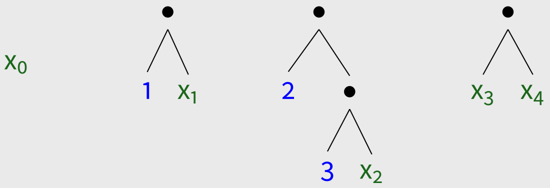
Instances of `Magma`

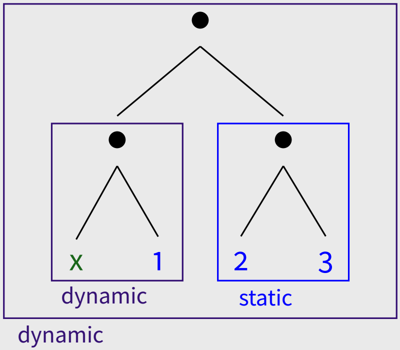
```
newtype Int× = Int× Int

instance Magma Int× where
  Int× x • Int× y = Int× (x × y)
```

Reducing terms

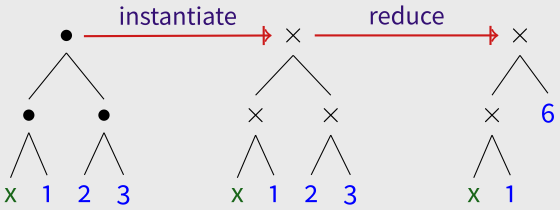


Trees with free variables




x
1
2
3
dynamic
static
dynamic

Reducing terms with free variables



Back to Haskell: binding times

```haskell
data BindingTime =
      Sta -- available now
    | Dyn -- available later
```

```haskell
data BT :: BindingTime → * where
  BTSta :: BT Sta
  BTDyn :: BT Dyn
```

Possibly-static data (for leaves)

```
data SD :: BindingTime → * → * where
  S ::      a → SD Sta a
  D :: Code a → SD Dyn a
```

```
btSD :: SD bt a → BT bt
btSD (S _) = BTSta
btSD (D _) = BTDyn
```

Mixed magmas: binding-time-indexed normal forms

```
data Mag :: BindingTime → * → * where
  LeafM :: SD bt a → Mag bt a
  Br1 :: Mag Sta a → Mag Dyn a → Mag Dyn a
  Br2 :: Mag Dyn a → Mag r   a → Mag Dyn a
```

```
btMag :: Mag bt a → BT bt
btMag (LeafM m) = btSD m
btMag (Br1 _ _) = BTDyn
btMag (Br2 _ _) = BTDyn
```

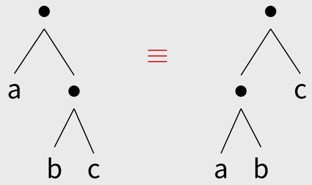
Mag is a Magma

```
instance Magma a ⇒ Magma (Exists Mag a) where
 E a • E b = m (btMag a) (btMag b) a b
 where
  -- leave no static subtrees!
  m BTSta BTSta (LeafM (S a)) (LeafM (S b)) = E (LeafM (S (a • b)))
  m BTSta BTDyn       l             r       = E (Br1 l r)
  m BTDyn   _         l             r       = E (Br2 l r)
```

A general-purpose existential type:

```
data Exists :: (k₁ → k₂ → *) → k₂ → * where
  E :: f b a → Exists f a
```

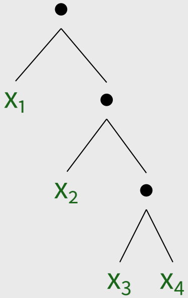
Semigroups (magmas + associativity)




```
class Magma a ⇒ Semigroup a      -- a • (b • c) ≡ (a • b) • c
```

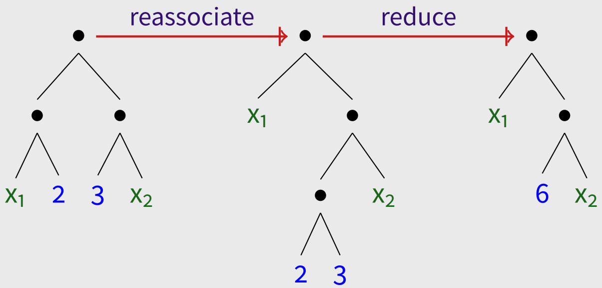
```
instance Semigroup Int×
```

Plain semigroups: fully right-associated



Mixed semigroups: also, no adjacent static data

Normalizing mixed-stage semigroup trees



Mixed semigroups: binding-time-indexed normal forms

```
data Semi :: BindingTime → * → * where
  LeafS :: SD bt a → Semi bt a
  ConsS ::       a → Semi Dyn a → Semi Dyn a
  ConsD ::  Code a → Semi r   a → Semi Dyn a
```

cons a static element:

```
consS :: Magma a ⇒ a → Exists Semi a → Exists Semi a
consS h (E (LeafS (S s)))   = E (LeafS (S (h • s)))
consS h (E t@(LeafS (D _))) = E (ConsS h t)
consS h (E (ConsS s t))     = E (ConsS (h • s) t)
consS h (E t@(ConsD _ _))   = E (ConsS h t)
```

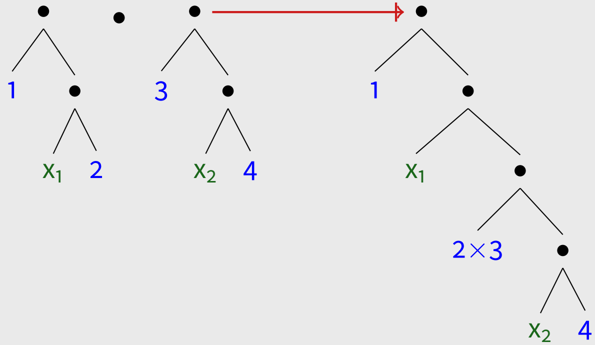
cons a dynamic element:

```
consD :: Code a → Exists Semi a → Exists Semi a
consD h (E t) = E (ConsD h t)
```

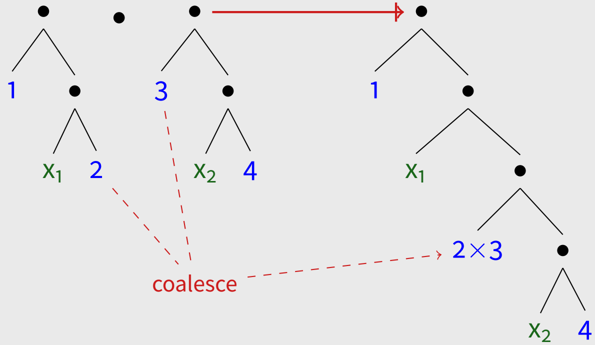
```
instance Semigroup a ⇒ Magma (Exists Semi a)
 -- • traverses the entire left operand
 where E (LeafS (S s)) • l = consS s l
       E (LeafS (D d)) • l = consD d l
       E (ConsS h t) • l = consS h (E t • l)
       E (ConsD h t) • l = consD h (E t • l)
```

```
instance Semigroup a ⇒ Semigroup (Exists Semi a)
```

• maps normal forms to normal forms



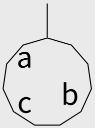
• maps normal forms to normal forms



Adding commutativity

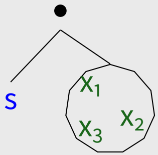

```
class Semigroup a ⇒ CSemigroup a     -- a • b ≡ b • a
```

A new *n*-ary constructor: unordered children



Partially-static commutative semigroups: normal forms

Group together all static data & all dynamic data:




```
data CSemi a = CSemi (Maybe a) (MultiSet (Code a))
```

```
instance CSemigroup a ⇒ Magma (CSemi a) where
  CSemi s₁ d₁ • CSemi s₂ d₂ = CSemi (s₁ •? s₂) (union d₁ d₂)
   where Nothing •? m = m
         m •? Nothing = m
         Just m •? Just n = Just (m • n)

instance CSemigroup a ⇒ Semigroup (CSemi a)
instance CSemigroup a ⇒ CSemigroup (CSemi a)
```

Partially-static data

General structure

Requirements (rough sketch)

```
-- type of partially-static data
--   (parameterised by class)
PS :: (* → Constraint) → * → *

-- injection of static values
sta :: algebra a ⇒ a → PS algebra a

-- injection of dynamic values
dyn :: Code a → PS algebra a

-- turn partially-static values into dynamic
cd :: PS algebra a → Code a
```

Example: `sta` and `dyn` for `CSemigroup`

$$\texttt{sta}_{\text{CS}} = \lambda \texttt{s} \rightarrow \texttt{CSemi (Just s) empty}$$

$$\texttt{dyn}_{\text{CS}} = \lambda \texttt{d} \rightarrow \texttt{CSemi Nothing (singleton d)}$$

Question: How should we define the general `PS`?

Ingredient 1: coproducts

```
class (algebra a, algebra b, algebra (Coprod algebra a b)) ⇒
 Coproduct algebra a b
  where
   -- coproduct representation (varies with algebra)
   data family Coprod algebra a b :: *

   -- injections
   inl :: a → Coprod algebra a b
   inr :: b → Coprod algebra a b

   -- eliminator/fold
   eva :: algebra c ⇒
      (a → c) → (b → c) → Coprod algebra a b → c
```

$$\texttt{eva f g (inl } s_1 \bullet \texttt{inr } d_1 \bullet \ldots) \quad \rightsquigarrow \quad \texttt{f } s_1 \bullet \texttt{g } d_1 \bullet \ldots$$

Ingredient 2: free objects

```
class algebra (FreeA algebra x) ⇒ Free algebra x where
  -- free object representation (varies with algebra)
  data family FreeA algebra x :: *

  -- variable injection
  pvar :: x → FreeA algebra x

  -- eliminator/fold
  pbind :: algebra c ⇒ FreeA algebra x → (x → c) → c
```

pbind f (pvar x_1 • pvar x_2 • ...) ⇝ f x_1 • f x_2 • ...

Free extensions from coproducts & free objects

Free extension constraints:

$$\texttt{FreeExt}_C :: (* \rightarrow \texttt{Constraint}) \rightarrow * \rightarrow \texttt{Constraint}$$

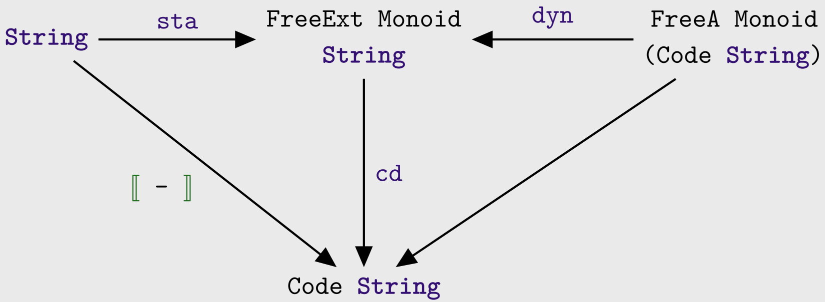
```
type FreeExt_C algebra a =
   Coproduct algebra a (FreeA algebra (Code a))
```

Free extension types:

$$\texttt{FreeExt} :: (* \rightarrow \texttt{Constraint}) \rightarrow * \rightarrow *$$

```
type FreeExt algebra a =
   Coprod algebra a (FreeA algebra (Code a))
```

Free extensions, pictured



frex interface: `sta` and `dyn`

```
-- sta: left injection into the free extension
sta :: (algebra a, FreeExt_C algebra a) ⇒
       a → FreeExt algebra a
sta = inl
```

```
-- dyn: right injection of variables into the free extension
dyn :: (Free algebra (Code a), FreeExt_C algebra a) ⇒
       Code a → FreeExt algebra a
dyn = inr · pvar
```

frex interface: `cd`

```
-- cd: elimination of free extensions into code
cd :: (Lift a, Free algebra (Code a), algebra (Code a),
       FreeExt_C algebra a) ⇒
      FreeExt algebra a → Code a
cd = eva tlift (`pbind` id)
```

```
-- (tlift turns static values into code)
tlift :: Lift a ⇒ a → Code a
tlift = liftM TExp · lift
```

Partially-static data

Instances & applications

Algebras and their free extensions

Algebra	Free extension
monoids	alternating static/dynamic sequence
commutative monoids	(static element) $\times$ (bag of names)
commutative rings	multinomial
distributive lattices	multinomial (exponents 0 or 1)
F-algebras	free algebra of coproducts
sets	binary sum

Coproduct of monoids

```
class Monoid t where
  𝟙 :: t
  (⊛) : t → t → t
```

The coproduct is an **alternating sequence**:

```
data AorB = A | B
data Alternate :: AorB → * → * → * where
  Empty :: Alternate any a b
  ConsA :: a → Alternate B a b → Alternate A a b
  ConsB :: b → Alternate A a b → Alternate B a b
```

```
instance (Monoid a, Monoid b) ⇒ Coproduct Monoid a b where
  data Coprod Monoid a b where M :: Alt _ a b → Coprod Monoid a b
  inl a = M (ConsA a Empty)
  inr b = M (ConsB b Empty)
  ...
```

```
instance Free Monoid x where
   newtype FreeA Monoid x = P [x] deriving (Monoid)
   pvar x = P [x]
   P [] `pbind` f = 𝟙
   P xs `pbind` f = foldr ((⊛) · f) 𝟙 xs
```

Using the monoid free extension

`printf`:

$$\texttt{sprintf ((int} \oplus \texttt{lit "a")} \oplus \texttt{(lit "b"} \oplus \texttt{int)}$$

`printf`, staged with $[\![\ ]\!]$ and $\$$:

$$[\![\ \lambda \texttt{x y} \rightarrow \texttt{((("" ++ show x) ++ "a") ++ "b") ++ show y}\ ]\!]$$

`printf`, staged with partially-static data:

$$[\![\ \lambda \texttt{x y} \rightarrow \texttt{show x ++ "ab" ++ show y}\ ]\!]$$

Free extension of commutative monoids

```
class Monoid m ⇒ CMonoid m
```

The coproduct is a product!

```
instance (CMonoid a, CMonoid b) ⇒ Coproduct CMonoid a b where
   data Coprod CMonoid a b = C a b
   inl a = C a 𝟙
   inr b = C 𝟙 b
   eva f g (C a b) = f a ⊛ g b
```

The free object is a bag

```
instance Ord x ⇒ Free CMonoid x where
   newtype FreeA CMonoid x = CM (MultiSet x)
   ...
```

Using the commutative monoid free extension

`power`

$$\texttt{power 5}\ [\![\texttt{x}]\!]$$

`power`, staged with $[\![\ ]\!]$ and $\$$:

$$[\![\ \texttt{1 * (x * (x * (x * (x * x))))}\ ]\!]$$

`power`, with partially-static data:

$$[\![\ \texttt{let y = x * x in let z = y * y in x * z}\ ]\!]$$

Free commutative rings

```
class Ring a where
  (⊕), (⊗) :: a → a → a
  rneg :: a → a
  𝟘, 𝟙 :: a
```

Free rings are **multinomials** with integer coefficients:

```
data Multinomial x a = MN (Map (MultiSet x) a)

instance Ord x ⇒ Free Ring x where
  newtype FreeA Ring x = RingA (Multinomial x Int)
  pvar x = RingA (MN (singleton (singleton x) 𝟙))
  RingA xss `pbind` f = evalMN initMN f xss
```

Free extension of commutative rings

No closed form for coproducts. But can define free extension!

Free extension: **multinomials** with coefficients in a:

```
instance (Ring a,Ord x) ⇒ Coproduct Ring a (FreeA Ring x) where
  newtype Coprod Ring a (FreeA Ring x) = CR (Multinomial x a)
  inl a = CR (MN (singleton empty a))
  inr (RingA (MN x)) = CR (MN (map initMN x))
  eva f g (CR c) = evalMN f (g · pvar) c
```

eva f g (a + bx^2y) ⇝ f a ⊕ (f b ⊗ g x ⊗ g x ⊗ g y)

Using the commutative ring free extension

inner product

```
[1; 0; 2] `dot` [⟦x⟧; ⟦y⟧; ⟦z⟧]
```

inner product, staged with ⟦ ⟧ and $:

```
⟦ (1 * x) + (0 * y) + (2 * z) ⟧
```

inner product, with partially-static data

```
⟦ x + (2 * z) ⟧
```

Lots more examples!
(see the paper[1])

[1] Partially-Static Data as Free Extension of Algebras, J. Yallop, T. von Glehn, O. Kammar (ICFP'18)

Using *frex*

1. write the instance

```
instance Monoid String where
  𝟙 = ""
  (⊛) = (++)
```

2. use `frex`'s `Monoid (FreeExt`$_C$` ...)` instance:

```
(dyn x ⊛ sta "a") ⊛ (sta "b" ⊛ dyn x)
```

3. convert to code:

```
cd ((dyn x ⊛ sta "a") ⊛ (sta "b" ⊛ dyn x))
  ⇝    ⟦x ++ "ab" ++ x ⟧
```

Using `frex` with existing polymorphic code

```
dot :: Ring r ⇒ [r] → [r] → r
dot xs ys = sum (zipWith (×) xs ys)

mmmul :: Ring r ⇒ [[r]] → [[r]] → [[r]]
mmmul m n = [[dot a b | b ← transpose n] | a ← m]
```

Matrix multiplication unfolded

```
cdMtx $ staMtx (V (V 0 1) (V 1 2)) `mmmul` dynMtx ⟦m⟧
```

convert vectors to lists of partially-static values

```
cdMtx $ [[sta 1, sta 0],    `mmmul`  [[dyn ⟦m!0!0⟧, dyn ⟦m!0!1⟧],
         [sta 1, sta 2]]              [dyn ⟦m!1!0⟧, dyn ⟦m!1!1⟧]]
```

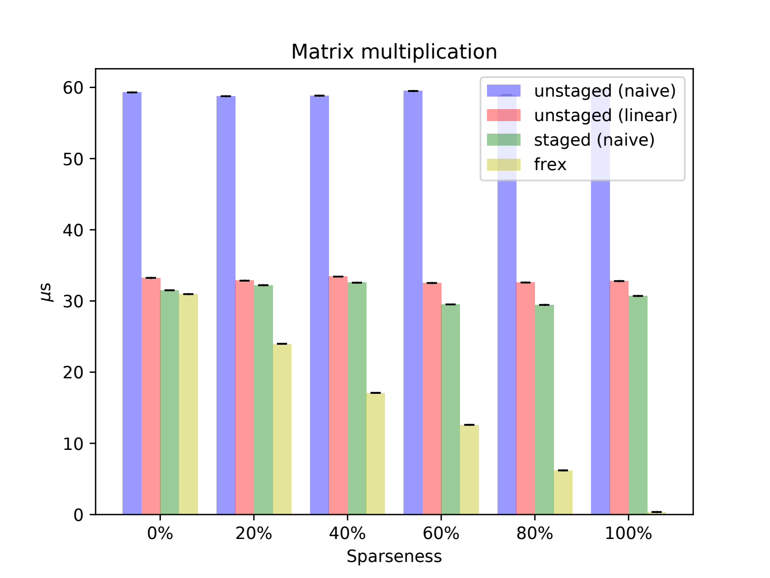
partially-static arithmetic with `mmmul`

```
cdMtx $ [[1×⟦m!0!0⟧ + 0×⟦m!1!0⟧, 1×⟦m!0!1⟧ + 0×⟦m!1!1⟧],
         [1×⟦m!0!0⟧ + 2×⟦m!1!0⟧, 1×⟦m!0!1⟧ + 2×⟦m!1!1⟧]]
```

conversion to optimized code

```
⟦ V (V        m!0!0              m!0!1          )
    (V   m!0!0 + 2×m!1!0   m!0!1 + 2×m!1!1) ⟧
```

Performance improvements



Performance improvements