Grounding Game Semantics
in Categorical Algebra

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4 Conclusion: Towards algebraic game semantics
Section 1

Context: Building reliable computer systems
Making computers reliable is hard

In modern computer systems:

- Layers upon layers of complex hardware and software components
- A bug in any one of them could prevent the system from working
- Components may break in subtle ways through their interactions
Making computers reliable is hard

In modern computer systems:

- Layers upon layers of complex hardware and software components
- A bug in any one of them could prevent the system from working
- Components may break in subtle ways through their interactions

Thankfully, there are ways to control this:

- Precise specifications for each component
- Careful and systematic testing
- Formal verification
Formal verification of software components

To prove a program correct:

- Start with a model of the programming language
- Make the specification mathematically precise
- Write a proof showing that the program indeed meets the specification
Formal verification of software components

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- Make the specification mathematically precise
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Mechanizing this process in a proof assistant has many advantages:
- Almost no possibility of mistake in the proof
- Can scale up the methodology to complex programs
- The proof can easily be checked by a third-party (certified software)
State of the art and next steps

Over the past $\sim$10 years, verification has become increasingly tractable:

- Researchers have verified complex components of various kinds
- Industrial-strength verification tools exist

The next step is end-to-end verification:

- Until then, bugs can sneak into the "gaps" between correctness proofs
- Solution: use formal specifications as interfaces to connect proofs

Challenge: existing projects use different models and proof methods

This is by necessity and not by accident.
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- Challenge: existing projects use different models and proof methods
- This is by *necessity* and not by accident
End-to-end verification using a hierarchy of models

This suggests we should organize semantic models into a *hierarchy*:

- Individual components are verified using specialized models
- Embed these models into increasingly general ones where
certified component can be assembled into certified systems
End-to-end verification using a hierarchy of models

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Category theory provides a unified framework to:
- Characterize existing models
- Establish connections between them
- Guide the design of more general ones
I will present a very small step in this direction, looking at connections between two important lines of work:

- *Game semantics* expresses the behavior of program components as *strategies* in games derived from their types;

- *Algebraic effects* model computations with side-effects as *terms* in an algebraic theory.
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I will show that:

- Simple strategies can be used to construct interesting models of algebraic effects
- Conversely, we can take inspiration from algebraic effects to characterize these simple strategies categorically.

I hope this correspondence can be extended in the future to formulate a more general algebraic approach to game semantics.
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Section 2

Background: Game semantics and algebraic effects
Game Semantics

*Game semantics* is a general approach to programming language semantics:

- Types are two-player *games* between a component and its environment.
- Programs of a given type are *strategies* for the corresponding game.
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This approach is very compelling for heterogeneous verification:
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However there are challenges to overcome:
- Huge variety of constructions for games and strategies
- Often too complex to formalize in a proof assistant
- Existing work rarely focuses on specifications and verification
Algebraic effects address the narrower problem of computational side-effects:

- The available side-effects are given by an algebraic theory
- Terms in the theory represent computations, which proceed inwards.
- Operations represent effects, their arguments are the possible continuations.
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- *Terms* in the theory represent computations, which proceed inwards.
- *Operations* represent effects, their arguments are the possible continuations.

Advantages:

- Composing effect theories is easier than in the monadic approach
- The framework is simple and systematic, grounded in categorical algebra

Limitations:

- Narrower scope than game semantics, less generality
In the algebraic framework, a program with side-effects:

\[
greeting(\ast) := (\text{if readbit then print "Hi" else print "Hello"}) \; ; \; \text{stop}
\]

is modeled in the following way:

\[
\Sigma := \{ \text{readbit} : 2, \; \text{print}[s] : 1, \; \text{done} : 0 \mid s \in \text{string} \}
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\[
t := \text{readbit}(\text{print}["Hi"](\text{stop}), \; \text{print}["Hello"](\text{stop}))
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Algebraic signatures can be read as games

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Reading the signature $\Sigma$ as a game, the term $t$ becomes a strategy tree:
Effect signatures

Definition (Effect signature)

An *effect signature* is a set $E$ of operations together with a map $\text{ar} : E \rightarrow \text{Set}$ which assigns an *arity set* to each one.
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Definition (Algebraic interpretation)

The *terms* in $E$ with variables in the set $X$ are defined by the grammar:

$$ t \in E^*X ::= x \mid m(t_n)_{n \in \text{ar}(m)} \quad (m \in E, \ x \in X) $$
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**Definition (Game interpretation)**

The *plays* over an effect signature $E$ with results in $X$ are defined by the grammar:

$$s \in P_E(X) ::= x \mid m \mid mns \quad (x \in X, \ m \in E, \ n \in \text{ar}(m))$$
Categorical characterization of terms and strategies

An effect signature can be interpreted as a polynomial endofunctor $E : \text{Set} \to \text{Set}$ constructing the terms of depth one:

$$EX := \sum_{m \in E} \prod_{n \in \text{ar}(m)} X$$

An algebra for $E$ is a set $A$ with a function $\alpha : EA \to A$; they form a category $\text{Set}^E$. 

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As is well-known, the “carrier set” functor $U : \text{Set}^E \to \text{Set}$ has a left adjoint, which maps a set $X$ to the term algebra with carrier set $E^*X$. 
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By working in a category of directed-complete partial orders, I obtain a similar characterization for the strategies over $E$. 
Section 3

Result: Strategies for algebraic effects
Strategies over an effect signature

Definition (Coherent plays)

The coherence relation $\subseteq \mathcal{P}E(X) \times \mathcal{P}E(X)$ is the smallest relation satisfying:

- $x \subseteq x$
- $m \subseteq m$
- $m \subseteq m$
- $n_1 = n_2 \Rightarrow s_1 \subseteq s_2 \Rightarrow m n_1 s_1 \subseteq m n_2 s_2$

Definition (Effect strategy)

A strategy $\sigma \in \mathcal{S}_E(X)$ over a signature $E$ with results in $X$ is a prefix-closed set $\sigma \subseteq \mathcal{P}E(X)$ of pairwise coherent plays.
Strategies over an effect signature

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The coherence relation $\sqsubseteq \subseteq P_E(X) \times P_E(X)$ is the smallest relation satisfying:

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(n_1 = n_2 \Rightarrow s_1 \sqsubseteq s_2) & \Rightarrow \quad m n_1 s_1 \sqsubseteq m n_2 s_2
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Strategies over an effect signature

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The coherence relation $\circ \subseteq P_E(X) \times P_E(X)$ is the smallest relation satisfying:

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x & \circ x \\
m & \circ m \\
m & \circ mns \\
(n_1 = n_2 \Rightarrow s_1 \circ s_2) & \Rightarrow mn_1s_1 \circ mn_2s_2
\end{align*}
\]

Definition (Effect strategy)

A strategy $\sigma \in S_E(X)$ over a signature $E$ with results in $X$ is a prefix-closed set $\sigma \subseteq P_E(X)$ of pairwise coherent plays.
Algebraic characterization of strategies

Strategies under set inclusion form a pointed directed-complete partial order:

- The empty strategy is the least element
- Unions of directed sets of strategies preserve the coherence condition
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It turns out the strategies for $E$ can be characterized as free algebras in $\mathbf{DCPO}_{\bot!}$, where the effect signature $E$ is interpreted as the endofunctor:

$$\hat{E}X := \bigoplus_{m \in E} \left( \prod_{n \in \text{ar}(m)} X \right)_{\bot}$$
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**Theorem**

The forgetful functor $\hat{U} : \text{DCPO}_{\hat{E} \bot !} \to \text{Set}$ has a left adjoint. The pointed dcpo $S_E(X)$ carries the corresponding $\hat{E}$-algebra.
Strategies are ideal completions of terms

The ideal completion $\mathcal{I}$ constructs the free dcpo on a poset:

$$
\begin{array}{c}
\mathcal{I} \\
\text{DCPO}_\bot ! \\
\text{Pos}_\bot \\
\end{array}
\xrightarrow{U}

If we order terms with variables in $X_\bot$ using the rules

$$
\begin{align*}
\bot \sqsubseteq t \\
x \sqsubseteq x \\
\forall n \in \text{ar}(m) \cdot t_n \sqsubseteq t'_n \\
\end{align*}
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this provides an alternative construction of strategies as $\mathcal{I}E^*(X_\bot)$. 
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If we order terms with variables in $X_\bot$ using the rules

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\bot \sqsupseteq t \quad \quad x \sqsupseteq x \quad \quad \forall n \in \text{ar}(m) \cdot t_n \sqsupseteq t'_n \quad \quad m(t_n)_{n \in \text{ar}(m)} \sqsupseteq m(t'_n)_{n \in \text{ar}(m)}
$$

this provides an alternative construction of strategies as $\mathcal{I}E^*(X_\bot)$.

**Theorem**

The following partial orders are isomorphic: $S_E(X) \cong \mathcal{I}E^*(X_\bot) \cong \mu Y \cdot \hat{E}Y \ominus X_\bot$
Section 4

Conclusion: Towards algebraic game semantics
Algebraic effects and game semantics have themes in common, but they look at them very differently.
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For example:

- Generalizing to multi-sorted signatures allows for richer games. This could serve as a common low-level algebraic grounding for a variety of sequential game semantics models.
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For example:

- Generalizing to multi-sorted signatures allows for richer games. This could serve as a common low-level algebraic grounding for a variety of sequential game semantics models.

- Effect signatures and natural transformations $\eta_X : EX \to FX \in \textbf{Set}$ form a symmetric monoidal closed category. Endofunctor composition and the free monad construction can be defined directly on signatures. We can carry out a version of Reddy’s object-based semantics in this setting.