A Graphical Calculus for Lagrangian Relations

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Background and abstract

- [BF18, BCR18] have exhibited how certain classes of electrical circuits can be interpreted in (linear/affine) *Lagrangian relations*.
- Lagrangian relations are a subcategory linear relations which model the evolution of a mechanical system.
- [BPSZ19] establish a graphical calculus for affine relations.
  They describe the components of [BF18, BCR18] using this graphical calculus for affine relations.
- We extend [BPSZ19], giving a universal presentation of the category of Lagrangian relations in terms of affine relations.
- We show that affine Lagrangian relations over odd prime characteristic $\mathbb{F}_p$ are equivalent to p-dimensional qudit stabilizer circuits.

Overview

Review: Lagrangian relations

Symplectic vector spaces

Definition

Given a field k and a k-vector space V, a **symplectic form** on V is a bilinear map $\omega : V \times V \to F$ which is also:

Alternating: For all $v \in V$, $\omega(v, v) = 0$.

Nondegenerate: If $\exists v \in V : \forall w \in V$ we have $\omega(v, w) = 0$, then $v = 0$.

A **symplectic vector space** is a vector space equipped with a symplectic form.
A (linear) **symplectomorphism** is a linear isomorphism between symplectic vector spaces that preserves the symplectic form.

Lemma

Every vector space k^{2n} is equipped with a bilinear form given by the following block matrix:

$$\omega := \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

so that $\omega(v, w) := v \omega w^T$. Moreover, every finite dimensional symplectic vector space over k is symplectomorphic to one of the form k^{2n} with such a symplectic form.

Phase space interpretation of symplectic vector spaces

Symplectic vector spaces are interpreted as the allowed configurations of position/momentum.

This is called the *phase space* of a mechanical system.

Over $k = \mathbb{R}$, $k = \mathbb{R}(t)$ we can interpret electrical circuit components:

Grading is the potential and current.

Over $k = \mathbb{F}_p$ we get the qudit stabilizer group:

Grading is the position and momentum observables.

Resistors are symplectomorphisms

Law (Kirchoff's current law/conservation of charge)

The total current entering a circuit is equal to the current leaving a circuit

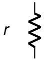
Law (Ohm's law)

Given a circuit with total current z with potential x_0 entering and x_1 leaving, the resistance is $(x_1 - x_0)/z$.

Therefore a resistor with resistance $r \in \mathbb{R}^+$ is given by the symplectomorphism:

$$(x, z) \mapsto (rz + x, z)$$

drawn as



The outgoing potential is determined by Ohm's law.

The outgoing current is determined by Kirchoff's current law.

Linear Lagrangian subspaces

Definition

Let $W \subseteq V$ be a linear subspace of a symplectic space V. The **symplectic dual** of the subspace W is defined to be $W^\omega := \{v \in V \mid \forall w \in W, \omega(v, w) = 0\}$. A linear subspace W of a symplectic vector space V is **isotropic** when $W^\omega \supseteq W$, **coisotropic** when $W^\omega \subseteq W$ and **Lagrangian** when $W^\omega = W$.

Lemma

The following are equivalent for a linear subspace $W \subseteq k^{2n}$

- *W is Lagrangian.*
- *W is coisotropic with dimension n.*
- *W is isotropic with dimension n.*
- *W is maximally isotropic.*
- *W is minimally coisotropic.*

Lemma

Every symplectomorphism $f : V \to V$ induces a Lagrangian subspace $\Gamma_f := \{(fv, v) | v \in V\} \subseteq V \oplus V$.

Lagrangian relations

You can compose Lagrangian subspaces:

Definition

Given a field k, the prop of **Lagrangian relations**, LagRel_k has:

- Objects are symplectic vector spaces k^{2n} with a fixed symplectic form.
- A map from $k^{2n} \to k^{2m}$ is Lagrangian subspaces of $k^{2(n+m)}$.
- The composite of $k^{2n} \xrightarrow{f} k^{2m} \xrightarrow{g} k^{2\ell}$ is given by the relational composite:

$$\{(a,c) \in k^{2n} \oplus k^{2\ell} \mid \exists b \in k^{2m} : (a,b) \in f, (b,c) \in g\}$$

- The identity on k^{2n} is:

$$\{(a,a) | a \in k^n\}$$

- The tensor product of f and g is:

$$f \oplus g := \{((x_0,x_1),(z_0,z_1)) \mid (x_0,z_0) \in f, (x_1,z_1) \in g\}$$

Also known as linear canonical relations/the linear Weinstein category.

Lemma

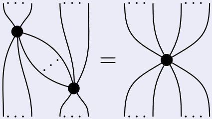
The graph of extends to a faithful symmetric monoidal functor from the prop of symplectomorphisms to Lagrangian relations.

Hypergraph structure of Lagrangian relations

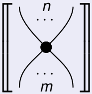
[BF18] investigate the hypergraph structure of LagRel_k

Lemma

LagRel_k *is a hypergraph category*



where the hyper-wires are interpreted as follows:



$$= \left\{ \left(\begin{pmatrix} x_1 \\ z \end{pmatrix} \oplus \overset{n}{\cdots} \oplus \begin{pmatrix} x_n \\ z \end{pmatrix}, \begin{pmatrix} w_1 \\ z \end{pmatrix} \oplus \overset{m}{\cdots} \oplus \begin{pmatrix} w_m \\ z \end{pmatrix} \right) \in k^{2n} \oplus k^{2m} \mid \sum x_i = \sum w_j \right\}$$

In terms of electrical circuits there are idealized wires of resistance 0.

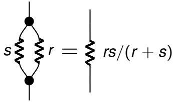
Kirchoff's Law forces the current of all connected wires to be the same.

Ohm's law constrains the resistance:

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{m} x_i - w_j \right) / z = 0 \iff \sum_{i=0}^{n} x_i = \sum_{j=1}^{m} w_j$$

(there is a technical subtlety for $z = 0$)

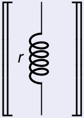
The resistors and hyper-wires interact as they should:



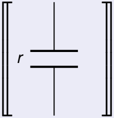
The following observation is due to [BF18]

Observation

Given $r \in \mathbb{R}^+$, inductors with inductance r and capacitors with capacitance r are interpreted as follows in $\mathrm{LagRel}_{\mathbb{R}(t)}$*:*



$$= \left\{ \left(\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x + rzt \\ z \end{pmatrix} \right) \in \mathbb{R}(t)^2 \oplus \mathbb{R}(t)^2 \right\}$$



$$= \left\{ \left(\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x - rzt \\ b \end{pmatrix} \right) \in \mathbb{R}(t)^2 \oplus \mathbb{R}(t)^2 \right\}$$

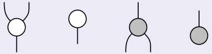
"Multiplying by t is like differentiation with respect to time."

Review: Graphical linear algebra

Graphical linear algebra: Matrices

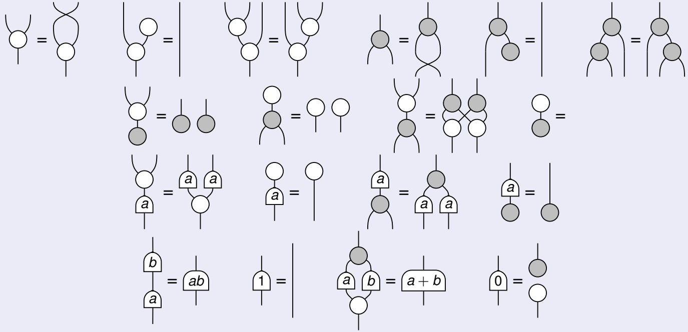
Definition

[Zan18, Defn. 3.4] Given a ring k, let cb_k denote the prop given by the generators:



for all $a \in k$

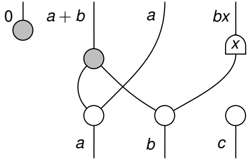
modulo the equations:



Proposition

[Zan18, Prop. 3.9] Given a ring k, cb_k is a presentation for the prop Mat_k, of matrices over k under the direct sum.

The black monoid is interpreted as addition/zero and the comonoid as copying/deleting.



$$\longleftrightarrow \quad \begin{bmatrix} 0 \\ a+b \\ a \\ bx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

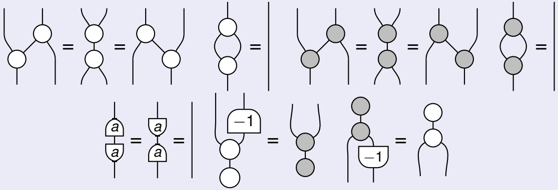
“Copying and addition”

Definition

[Zan18, Defn. 3.42] Given a field k, the prop of **linear relations**, LinRel_k, has morphisms $n \to m$ as linear subspaces of $k^n \oplus k^m$, under relational composition and the direct sum as the tensor product.

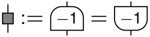
Definition

[Zan18, Defn. 3.44] Given a field k, let ih_k denote the prop given by the quotient of the coproduct of props $\mathrm{cb}_k^{\mathrm{op}} + \mathrm{cb}_k$ by the following equations, for all invertible $a \in k$ (where the generators of $\mathrm{cb}_k^{\mathrm{op}}$ are drawn by reflecting those of cb_k along the x-axis):



Theorem

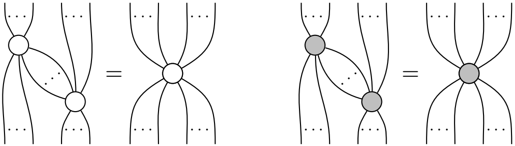
[Zan18, Thm. 3.49] ih_k *is a presentation for* LinRel_k.

Notation for the antipode: 

Special commutative Frobenius algebras correspond to **spiders**.

Connected components of Frobenius algebras with the same arity are equal.

In LinRel_k there are two spiders:



Over $k = \mathbb{Q}$ or $k = \mathbb{F}_p$ these two spiders generate LinRel_k. In particular:

Lemma

For p a prime, $\mathrm{LinRel}_{\mathbb{F}_p}$ *is a presentation for the p-dimensional qudit phase-free fragment of the ZX-calculus up to scalars.*
(p-dimensional qudit means objects are powers of $\mathbb{C}^p$*)*

Over $\mathbb{F}_P$, both spiders correspond to generalized Kronecker-deltas in FHilb, comparing:

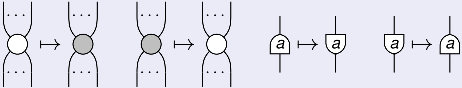
- standard basis elements, for the white spider.
- Fourier basis elements, for the black spider.

Graphical Lagrangian algebra

Graphical Lagrangian algebra

Lemma ([Sob])

The colour swapping functor $(\_)^{\perp} : \mathrm{ih}_k \to \mathrm{ih}_k$*;*

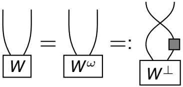


is the isomorphism which takes linear subspaces to their orthogonal complement:

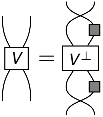
$$V \mapsto V^{\perp} := \{v \in V : \forall w \in V, \langle v, w \rangle = 0\}$$

The symplectic complement is a “twisted” version of the orthogonal complement.

In pictures, a linear subspace $W \subseteq k^{2n}$ is Lagrangian iff:



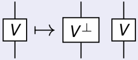
So a linear relation $V \subseteq k^{2n} \oplus k^{2m}$ is a Lagrangian relation iff:



Purification of Lagrangian relation

Lemma

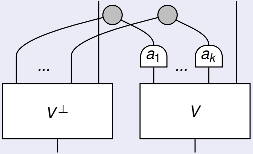
There is a faithful, strong symmetric monoidal functor $L : \mathrm{LinRel}_k \to \mathrm{LagRel}_k$ given by the following action on the generators of ih_k*; doubling, and then changing the colours of one of the copies:*



“Linear relations is a monoidal subcategory of Lagrangian relations”

Theorem (Purification)

Any linear Lagrangian relation can be written in the following form, for V a linear relation:



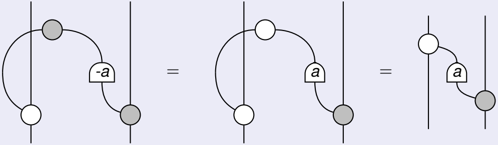
When $k = \mathbb{F}_p$, then $\forall i$ we can fix $a_i = 1$, making $\mathrm{LagRel}_{\mathbb{F}_p} \cong \mathrm{CPM}(\mathrm{LinRel}_k, (\_)^{\perp})$.

Generators of Lagrangian relations

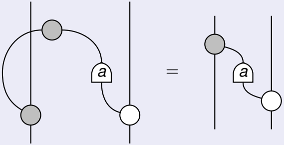
Theorem

LagRel_k *is generated by the image of* $L(\mathrm{LinRel}_k)$ *and the following Lagrangian relations all* $a \in k$*:*

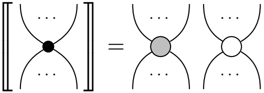
$V_a :=$



$S_a :=$



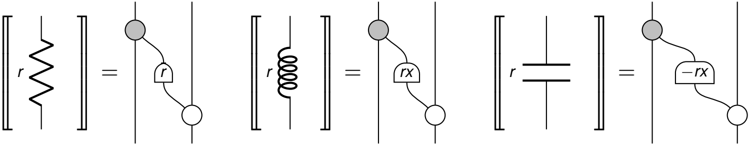
The hypergraph structure of Lagrangian relations can now be seen with string diagrams:



The gradings of the symplectic vector space are pulled to the left and right.

The wires on the left sum the potentials. The wires on the right equate the currents.

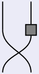
The resistors, inductors and capacitors have the interpretations:

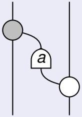


Alternative generators of Lagrangian relations

Corollary

LagRel_k *is generated by the image* $L(\mathrm{LinRel}_k)$ *as well as the following symplectomorphisms, seen as Lagrangian relations for all* $a \in k$*:*

$F :=$ 

$S_a =$ 

Proof.

It is a matter of calculation that $S_1 V_1 S_1 = F$. □

F is interepreted as the symplectic Fourier transform.

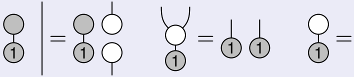
Review: Graphical affine algebra

Graphical affine algebra

We can capture larger fragments with affine behaviour.

Definition ([BPSZ19])

Let aih_k denote the the prop presented by ih_k in addition to the generator (1) and three equations:



The following was stated slightly differently in [BPSZ19, Definition 5]:

Definition

Let AffRel_k denote prop, whose morphisms $n \to m$ are the (possibly empty) affine subspaces of $2^n \oplus 2^m$; with composition given by relational composition and tensor product given by the direct sum.

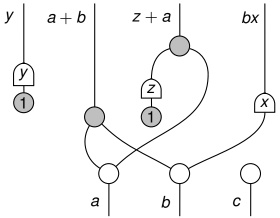
Lemma ([BPSZ19])

aih_k *is a presentation for* AffRel_k.

The generator (1) corresponds to the affine shift.

Translation between affine matrices

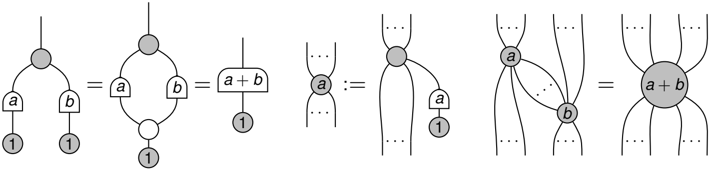
Affine matrices can be translated into string diagrams:



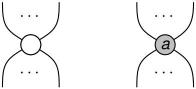
$$\longleftrightarrow \quad \begin{bmatrix} 0 \\ a+b \\ a \\ bx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} y \\ 0 \\ z \\ 0 \end{bmatrix}$$

Phased-Frobenius algebra

There is a “phased-Frobenius algebra” syntax for graphical affine algebra, so that for $a, b \in k$:



So that $\mathrm{AffRel}_{\mathbb{F}_p}$ is generated by the interacting gray phased-Frobenius algebra and white Frobenius algebra:



Lemma

For a prime p, $\mathrm{AffRel}_{\mathbb{F}_p}$ *is the p-dimensional qudit shift operator fragment* ZX*-calculus, up to nonzero scalars.*

Graphical affine Lagrangian algebra

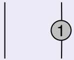
Affine Lagrangian relations

Definition

There is a prop $\mathrm{AffLagRel}_k$ of affine Lagrangian relations, whose morphisms are *affine* Lagrangian subspaces.

Theorem

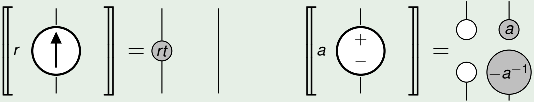
The category $\mathrm{AffLagRel}_k$ *of* **affine Lagrangian relations** *is generated by all the Lagrangian relations seen as affine relations, in addition to the generator:*

$X :=$ 

The following generators were shown to be affine Lagrangian relations by [BCR18]. They were also translated into string diagrams in [BPSZ19].

Example

For any non-negative real r voltage and current sources have the following interpretations in $\mathrm{AffLagRel}_{\mathbb{R}(t)}$:



Stabilizer circuits: quick review

Lemma

For natural numbers $n, d \geq 2$ the n-dimensional qudit **Clifford group** *modulo invertible scalars is generated under tensor and composition of I_d, the boost operator $\mathcal{X}$, the controlled-boost operator $\mathcal{C}$, the Fourier transform $\mathcal{F}$ and the phase-shift operator $\mathcal{S}$:*

$$\mathcal{X}; |a\rangle \mapsto |a+1\rangle \qquad \mathcal{C}; |a\rangle \otimes |b\rangle \mapsto |a\rangle \otimes |a+b\rangle$$

$$\mathcal{F}; |a\rangle \mapsto \frac{1}{\sqrt{d}} \sum_{b=0}^{d-1} e^{2\pi i ab/d} |b\rangle \qquad \mathcal{S}; |a\rangle \mapsto e^{\pi i a(a+d)/d} |a\rangle$$

Where $\{|a\rangle\}_{a \in \mathbb{F}_p}$ is a chosen orthonormal basis for the Hilbert space $\mathbb{C}^p$.

Definition

The prop of *n*-dimensional qudit **stabilizer circuits** is generated by the Clifford group, seen as linear maps as well as the vector $|0\rangle$ and its transpose.

Stabilizer circuits are affine Lagrangian relations

The following is due to [Gro06]:

Theorem

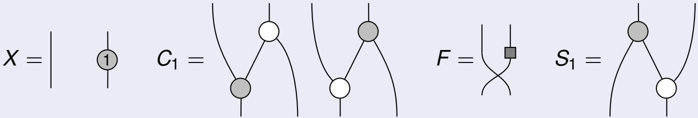
For d an odd prime, the n-dimensional qudit Clifford group is isomorphic to affine symplectomorphisms over $\mathbb{F}_d^n$.

We extend this to Lagrangian relations:

Theorem

*For p an **odd** prime, $\mathrm{AffLagRel}_{\mathbb{F}_p}$ is a presentation for p-dimensional qudit stabilizer circuits (up to nonzero scalars), where:*

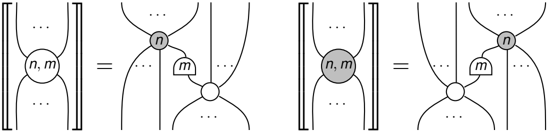
$$\mathcal{X} \leftrightarrow X \qquad \mathcal{C} \leftrightarrow C_1 \qquad \mathcal{F} \leftrightarrow F \qquad \mathcal{S} \leftrightarrow S_1$$



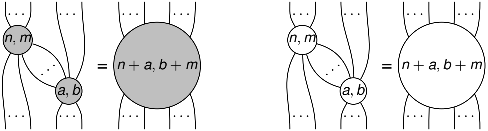
The matrices representing affine Lagrangian subspaces are stabilizer tableaus.

The qudit stabilizer ZX-calculus

For $n, m, a, b \in \mathbb{Z}/p\mathbb{Z}$ for p a prime, the two phased-spiders generate affine Lagrangian relations:



Where we have the following “phased-spider fusion” laws for the group $(\mathbb{Z}/p\mathbb{Z})^2$:



Corollary

$\mathrm{AffLagRel}_{\mathbb{F}_2}$ *is equivalent to Spekkens' toy model [Spe07],* **not** *qubit Stabilizer circuits.*

The grey spiders in the qubit case do not have the same phases, $\mathbb{Z}/4\mathbb{Z} \not\cong (\mathbb{Z}/2\mathbb{Z})^2$.

Recap

- We reviewed Lagrangian relations.
- We reviewed graphical linear/affine algebra.
- We combined these two things to get "graphical linear/affine Lagrangian algebra."
  - We showed that every Lagrangian relation can be purified.
  - We showed that stabilizer circuits are equivalent to affine Lagrangian relations.

Future work

- Work out measurement in (affine) Lagrangian relations.
  - Conjecture that $\mathrm{CPM}(\mathrm{LagRel}_k)$ is coisotropic relations over k.
  - Investigate Karoubi envelope of $\mathrm{CPM}(\mathrm{LagRel}_k)$.
- Give a proper monoidal theory for (affine) Lagrangian relations.
  - Proof would likely involve graph states and local complementation.
- Can this graphical calculus for (affine) Lagrangian relations be generalized for PIDs?
- Is there a deeper connection between stabilizer circuits and electrical circuits?

References

John C Baez, Brandon Coya, and Franciscus Rebro.
Props in network theory.
Theory and Applications of Categories, 33(25):727–783, 2018.

John C Baez and Brendan Fong.
A compositional framework for passive linear networks.
Theory and Applications of Categories, 33(38):pp 1158–1222, 2018.

Filippo Bonchi, Robin Piedeleu, Paweł Sobociński, and Fabio Zanasi.
Graphical affine algebra.
In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2019.

David Gross.
Hudson's theorem for finite-dimensional quantum systems.
Journal of mathematical physics, 47(12):122107, 2006.

Paweł Sobociński.
Graphical linear algebra: Orthogonality and projections.

Robert W Spekkens.
Evidence for the epistemic view of quantum states: A toy theory.
Physical Review A, 75(3):032110, 2007.

Fabio Zanasi.
Interacting Hopf Algebras: the theory of linear systems.
PhD thesis, Université de Lyon, 2018.