Network Sheaves Valued in Categories of Adjunctions & Their Laplacians

ACT 2021

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Theorem ([J. Curry]). Suppose $\mathcal{D}$ is a complete category, and $P_X$ is the poset of face relations of a (say) simplicial complex $X$. Then, there is an equivalence of categories

$$\text{Sh}_D(\text{Alex}(P_X)) \simeq D^{P_X}.$$
Theorem ([J. Curry]). Suppose $\mathcal{D}$ is a complete category, and $P_X$ is the poset of face relations of a (say) simplicial complex $X$. Then, there is an equivalence of categories

$$\text{Sh}_\mathcal{D}(\text{Alex}(P_X)) \simeq \mathcal{D}^{P_X}.$$

Alexandrov topology on the poset $P_X$  
functor category
Theorem ([J. Curry]). Suppose $\mathcal{D}$ is a complete category, and $P_X$ is the poset of face relations of a (say) simplicial complex $X$. Then, there is an equivalence of categories

$$Sh_{\mathcal{D}}(\text{Alex}(P_X)) \cong \mathcal{D}^{P_X}.$$
Cellular Sheaf Theory

**Theorem** ([J. Curry]). Suppose $\mathcal{D}$ is a complete category, and $P_X$ is the poset of face relations of a (say) simplicial complex $X$. Then, there is an equivalence of categories

$$\mathcal{S}h_{\mathcal{D}}(\text{Alex}(P_X)) \simeq \mathcal{D}^{P_X}.$$ 

A **cellular sheaf** is a functor 

$$\mathcal{F}: P_X \rightarrow \mathcal{D}$$

where $\mathcal{D}$ is an arbitrary category.
Cellular Sheaf Theory

In this talk, we consider cellular sheaves where

- $X$ is a graph, $G = (V_G, E_G)$
- $\mathcal{D}$ is a category of categories and adjunctions

Our motivation is to compute limits i.e. **global sections**—consistent assignments of data to a sheaf.
Example
Categories of “Adjunctions”

- preHilb
- Ltc
- CatAdj
- MAdj
Laplacians

Hodge (sheaf) Laplacian → Tarski Laplacian

→ Cat-Laplacian
→ Weighted Tarski Laplacian

generality
Graph Laplacian
Graph Laplacian

\( G = (V_G, E_G, W_G) \) is a weighted graph
\( w_{ij} \) is weight of \( ij \in E_G \)
\( d_i \) is degree of \( i \in V_G \)
\( n = |V_G| \)
The graph Laplacian is a \( n \times n \) matrix,
\[
[L]_{ij} = \begin{cases} 
-w_{ij}, & ij \in E_G \\
 d_i, & i = j \\
 0, & \text{else}
\end{cases}
\]
\( \dim(\ker L) = \# \text{ connected components} \)
\( \dot{x} = -Lx \) where \( x: V_G \to \mathbb{R} \) exponentially stable
Tarski Laplacian

\( G = (V_G, E_G) \), a graph.
\( \mathcal{F}: P_G \to \mathcal{Ltc} \), a functor.

Can we compute

\[ \lim \left( \mathcal{F}: P_G \to \mathcal{Supt} \right) \]

where \( \mathcal{F}: P_G \to \mathcal{Supt} \) forgets right adjoints?

**Definition.** The Tarski Laplacian is an order preserving map

\[
\prod_{v \in V_G} \mathcal{F}(v) \to \prod_{v \in V_G} \mathcal{F}(v)
\]

\[ (Lx)_v = \bigwedge_{e \in \delta v} \mathcal{F}^R_{v < e} \left( \bigwedge_{w \in \partial e} \mathcal{F}^L_{w < e}(x_w) \right) \]
A Fixpoint Theorem

**Theorem.** Let \( \mathcal{F}: \mathcal{P}_G \rightarrow \mathcal{Ltc} \) be a Tarski sheaf over \( G \). Then,
\[
\text{Post}(L) = \lim \mathcal{F}
\]

\[
\text{Post}(L) = \{ x : L(x) \succeq x \}
\]

\[
\lim \mathcal{F} = \left\{ x \in \prod_{v \in V_G} \mathcal{F}(v) : \mathcal{F}_{v<e}^L(x_v) = \mathcal{F}_{w<e}^L(x_w) \right\}
\]

Mimics **Hodge Theorem:** \( H^k(C^\cdot) \cong \ker L_k \)

**Corollary.** \( \lim \mathcal{F} \) is a complete lattice.

**Proof.** Tarski Fixed Point Theorem.
Toward Applications

- Graph Signal Processing (GSP)
- Formal Concepts
- Consensus
**Cat-Laplacian**

*CatAdj* is a 2-category.

\[ \mathcal{F} : P_G \to \text{CatAdj} \], a **cellular adjunction stack**

**Definition.** The *Cat*-Laplacian is a functor

\[
\prod_{v \in V_G} \mathcal{F}(v) \xrightarrow{L} \prod_{v \in V_G} \mathcal{F}(v)
\]

\[
(LX)_v = \prod_{e \in \delta v} \prod_{w \in \delta e} \mathcal{F}_{v < e}^R \mathcal{F}_{w < e}^L (X_w)
\]
A 2-Categorical Fixpoint Theorem

\[ \Delta(X)_v = \prod_{e \in \delta v} F^R_{v<e} F^L_{v<e}(X_v) \]

\[ \eta: 1 \Rightarrow \Delta \]
\[ \mu: L^2 \Rightarrow \Delta \]

\[ \text{Post}(L) = \{ X, f: X \to L(X): \mu_X \circ Lf \circ f = \eta_X \} \]

\[ \mathcal{F}: P_G \to \text{Cat}, \text{a cellular stack} \]

**Theorem.** \( \lim \mathcal{F} \simeq \text{Post}(L) \)
Cellular M-Stacks
**$\mathbb{M}$-Categories**

$\mathbb{M}$, closed monoidal thin (preorder) category whose unit is terminal. 

E.g.

- $\mathbb{B}$, 2-element Boolean
- $\mathbb{I} = ([0, 1], \cdot, 1, \leq)$
- $\mathbb{L} = ([0, \infty], +, 0, \geq)$
- $\mathbb{H}$, Heyting algebra

$\mathbb{M} Adj$, category of $\mathbb{M}$-enriched and $\mathbb{M}$-adjunctions
Weighted Tarski Laplacian

\[ \mathcal{F}: P_G \rightarrow \mathbb{M}Cat, \text{ adjunction } \mathbb{M}\text{-stack} \]

\[ \mathcal{W}: P_G \rightarrow \mathbb{M}, \text{ a weighting} \]

**Definition.** The weighted Laplacian is a \( \mathbb{M} \)-functor

\[ \prod_{v \in V_G} \mathcal{F}(v) \xrightarrow{L} \prod_{v \in V_G} \mathcal{F}(v) \]

\[ (LX)_v = \prod_{e \in \delta v, w \in \partial e}^{\mathcal{W}} \mathcal{F}_{v \prec e}^{R} \mathcal{F}_{w \prec e}^{L}(X_w) \]
An $\mathbb{M}$-Enriched Fixpoint Theorem

$\mathcal{F}: P_G \to \mathbb{M}Cat$, adjunction $\mathbb{M}$-stack

$\mathcal{W}: P_G \to \mathbb{M}$, a weighting

$m \in \mathbb{M}$, choice of object

**Theorem.**

\[\text{hom}(X, L(X)) \succeq m\]

if and only if $\forall \nu < e > w$

\[\left[\mathcal{W}(e), \text{hom}(\mathcal{F}_{w<e}^L(X_w), \mathcal{F}_{v<e}^L(X_v))\right] \succeq m\]
Cellular Sheaves of Lattices and the Tarski Laplacian

Robert Ghrist*  Hans Riess†

Abstract
This paper initiates a discrete Hodge theory for cellular sheaves taking values in a category of lattices and Galois connections. The key development is the Tarski Laplacian, an endomorphism on the cochain complex whose fixed points yield a cohomology that agrees with the global section functor in degree zero. This has immediate applications in consensus and distributed optimization problems over networks and broader potential applications.

1 Introduction
The goal of this paper is to initiate a theory of sheaf cohomology for cellular sheaves valued in a category of lattices. Lattices are algebraic structures with a rich history [41] and a wide array of applications [13, 2, 17, 42, 34, 16]. Cellular sheaves are data structures that stitch together algebraic entities according to the pattern of a cell complex [43]. Sheaf cohomology is a compression that collapses all the data over a topological space — or cell complex — to a minimal collection that intertwines with the homological features of the base space [31].

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• our paper (on Tarski Laplacian): to appear in *Homology, Homotopy, and Applications*:
• preprint w/ Paige North on 𝕀-stacks: TBD
• you can follow me on twitter: [@hansmriess](https://twitter.com/hansmriess)
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Thank you.