Rewriting Graphically with Cartesian Traced Categories

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Symmetric traced monoidal categories

\[ f : X \rightarrow Y, \quad \text{Tr}_{X,Y}^X(f) : A \rightarrow B \]
Symmetric traced monoidal category

**Tightening**

\[ A \xrightarrow{g, f, h} D = A \xrightarrow{g, f, h} D \]

**Yanking**

\[ X \xrightarrow{X, X} X = X \xrightarrow{X, X} X \]
Symmetric traced monoidal category

Superposing

\[
\begin{array}{c}
A \\
\downarrow C \uparrow A \\
\end{array}
\quad f
\quad
\begin{array}{c}
B \\
\downarrow C \uparrow B \\
\end{array}
\quad =
\quad
\begin{array}{c}
A \\
\downarrow C \uparrow A \\
\end{array}
\quad f
\quad
\begin{array}{c}
B \\
\downarrow C \uparrow B \\
\end{array}
\]

Exchange

\[
\begin{array}{c}
A \\
\downarrow f \uparrow B \\
\end{array}
\quad =
\quad
\begin{array}{c}
A \\
\downarrow f \uparrow B \\
\end{array}
\]

Cartesian categories

The tensor is a **product** and the unit object is **terminal**.

\[ \Delta_A : A \rightarrow A \otimes A \]

\[ \Diamond_A : A \rightarrow I \]
Cartesian categories – axioms

Naturality

\[ A \xrightarrow{f} B = A \]

\[ f \]

\[ B \]

among others...
Cartesian traced categories

Product + trace = fixpoint operator (Hasegawa 1997)

Also known as dataflow categories.
Applying Cartesian axioms require a **rewriting** of the graph.

‘Only connectivity matters’ no longer applies!

Combinatorial **graph language** required.
Graphical languages for Cartesian categories

Can we use something off the shelf?

**String graphs** (Dixon, Kissinger)

**Hypergraphs** (Bonchi, Gadducci, Kissinger, Sobociński, Zanasi)
These frameworks are based in **compact closed** categories.

It is possible to construct a trace using the **compact closed** structure.

But finite products become **biproducts** in a compact closed category.

And if we add a Cartesian structure to a compact closed category it becomes **trivial** anyway.
The compact closed problem

The trace must be constructed as an atomic operation.

\[ X \xrightarrow{f} X, \quad \Tr_{A,B}^X(f) \]

Goal: define a sound and complete graph language for STMCs with atomic trace.
Hypergraphs
Definition

$\text{Hyp}_\Sigma$ is the category with objects the labelled hypergraphs over a signature $\Sigma$ and morphisms the labelled hypergraph homomorphisms.
Hypergraphs are not enough

We could **rule out** the ones that don’t fit our criteria, but this might not be compositional.
**Definition**

$L\text{Hyp}_\Sigma$ is the category with objects the linear hypergraphs labelled over a signature $\Sigma$ and morphisms the labelled linear hypergraph homomorphisms.
Cospans of linear hypergraphs

A cospan $M \rightarrow H \leftarrow N$ is **discrete** if $M$ and $N$ contain no edges.

An **monogamous** cospan only picks the ‘open’ vertices.

**Definition**

$MCsp_D(LHyp_\Sigma)$ is the category of monogamous cospans over $LHyp_\Sigma$. 
Are linear hypergraphs a suitable graph language for STMCs?

We need to define the operations of an STMC.

Most are fairly obvious...

\[ F \cdot G \]

\[ F \otimes G \]

\[ \text{id}_1 \]

\[ \sigma_{1,1} \]
Interpreting terms as graphs

We fix a traced PROP $\text{Term}_\Sigma$ generated over some signature $\Sigma$.

$$[-] : \text{Term}_\Sigma \rightarrow \text{MCsp}_D(\text{LHyp}_\Sigma)$$
Soundness

Equal terms in the category $\Rightarrow$ Isomorphic interpretations as hypergraphs
Theorem (Soundness)

For any morphisms $f, g \in \text{Term}_\Sigma$, if $f = g$ under the equational theory of the category then their interpretations as cospans of labelled linear hypergraphs are isomorphic, $\llbracket f \rrbracket \equiv \llbracket g \rrbracket$.
A cospan of labelled linear hypergraphs $\Rightarrow$ A set of corresponding terms in the category
Definability

\[ \text{Tr}^3(\sigma_{2,1} \otimes \text{id}_2 \cdot \text{id}_2 \otimes \sigma_{1,1} \otimes \text{id}_1 \cdot \phi \otimes \psi \otimes \text{id}_2) \]
Definability

\[ \langle - \rangle : MCsp_D(LHyp_\Sigma) \to \text{Term}_\Sigma \]

**Proposition (Definability)**

*For any* \( F \in LHyp_\Sigma \) *and edge order* \( \leq \), *then* \( m \rightarrow F \leftarrow n \equiv [\langle m \rightarrow F \leftarrow n \rangle]_\leq \).
But we cannot conclude completeness yet!

A labelled linear hypergraph $\Rightarrow$ **Unique** morphism in the category, up to the equational theory

**Proposition (Coherence)**

*For all orderings of edges $\leq_x$ on some $F \in LHyp_\Sigma,$*

$$\langle\langle m \to F \gets n\rangle\rangle_{\leq_1} = \langle\langle m \to F \gets n\rangle\rangle_{\leq_2} = \cdots = \langle\langle m \to F \gets n\rangle\rangle_{\leq_x}$$
Theorem (Completeness)

For any cospan of linear hypergraphs \( m \rightarrow F \leftarrow n \in MCsp_D(LHyp_\Sigma) \) there exists a unique morphism \( f \in \text{Term}_\Sigma \), up to the equations of the STMC, such that \( \llbracket f \rrbracket = F \). Moreover, for any \( f \in \text{Term}_\Sigma \), \( \llangle \llbracket f \rrbracket \rrangle = f \).
Graph rewriting
DPO rewriting

\[
\begin{align*}
L & \quad \quad K \quad \quad R \\
G & \quad \quad C \quad \quad H \\
J
\end{align*}
\]
We need a guarantee that this pushout complement is unique.
In an adhesive category, if we have

- a rewrite rule $L \xleftarrow{p} K \rightarrow R$ where $p$ is mono,
- a matching $L \rightarrow G$

then the pushout complement $K \rightarrow C \rightarrow R$ is unique (if it exists).

We have already met an adhesive category:

**Proposition**

$\text{Hyp}_\Sigma$ is an adhesive category.
Unfortunately $\text{LHyp}_\Sigma$ is not adhesive.

**Definition (Partial adhesive categories (Kissinger))**

A category $\mathcal{P}$ is called a partial adhesive category if it is a full subcategory of an adhesive category $\mathcal{A}$ and the inclusion functor $I : \mathcal{P} \to \mathcal{A}$ preserves monomorphisms.
Partial adhesive categories

Proposition
$L\text{Hyp}_\Sigma$ is a full subcategory of $\text{Hyp}_\Sigma$.

Proposition
The inclusion functor $I: L\text{Hyp}_\Sigma \to \text{Hyp}_\Sigma$ preserves monomorphisms.

Corollary
$L\text{Hyp}_\Sigma$ is a partial adhesive category.

So for matchings that are mono, graph rewriting is well-defined.
DPO rewriting example
Conclusion

• Sound and complete graph language for symmetric traced monoidal categories with a **Cartesian structure**
• This is by defining the trace as an **atomic operation**
• Linear hypergraphs form a **partial adhesive category**
• So graph rewriting can be performed as with regular hypergraphs!