The Sierpinski Carpet as a Final Coalgebra

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July 16, 2021

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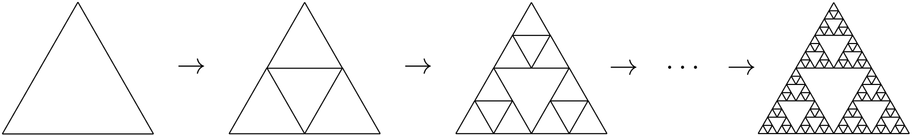
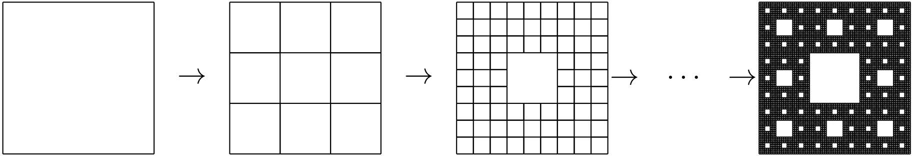
ACT 2021

Background

- Freyd: Bipointed Sets, unit interval as a final coalgebra
- Leinster: Generalized, topological spaces
- Bhattacharya, Moss, Ratnayake, Rose: Metric spaces (Bipointed sets - unit interval, Tripointed sets - Sierpinski Gasket)
  - G the colimit of an initial chain of a functor which corresponds to iterations of the Sierpinski Gasket
  - By Adamek's Theorem, $(G, g : M \otimes G \to G)$ is an initial algebra.
  - Take S to be the Cauchy completion of G and find $(S, s : S \to M \otimes S)$.
  - This is a final coalgebra by a contraction mapping theorem argument.
  - S is Bilipschitz equivalent to the Sierpinski Gasket.

The Sierpinski Carpet





Square Metric Spaces (SquaMS)

Let M_0 be the boundary of the unit square.
A *square set* is a set X with with an injective map $S_X : M_0 \to X$.
(X, S_X) is a *square metric space* if X is a metric space bounded by 2, and the boundary indicated by S_X satisfies the following:

- Distances along sides coincide with the unit interval.
- Distances are bounded below by the taxicab metric (corners do not collapse).

SquaMS is the category whose objects are square metric spaces, and whose morphisms are short maps which preserve S.
Examples of Square Metric Spaces:

- (M_0, id) with the path metric (this is an initial object, it maps into every object uniquely via S_X, there is no final object)
- $([0,1]^2, S)$ where S is the inclusion map, with the taxicab metric metric. That is, for (x_1, y_1) and (x_2, y_2), $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

$M \times X$

Let $M = \{0,1,2\}^2 \setminus \{(1,1)\}$.

$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$		$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

$$d_{M\times X}(((i_0,i_1),x),((j_0,j_1),y)) = \begin{cases} \frac{1}{3}d(x,y) & (i_0,i_1)=(j_0,j_1) \\ 2 & \textit{otherwise} \end{cases}$$

$M \otimes X$

$$
\begin{array}{ccccc}
\boxed{(0,2)\times X} & \sim & \boxed{(1,2)\times X} & \sim & \boxed{(2,2)\times X} \\
\wr & & & & \wr \\
\boxed{(0,1)\times X} & & & & \boxed{(2,1)\times X} \\
\wr & & & & \wr \\
\boxed{(0,0)\times X} & \sim & \boxed{(1,0)\times X} & \sim & \boxed{(2,0)\times X}
\end{array}
$$

The distance between elements of $M \otimes X = M \times X/\sim$ will be defined as the infimum over all finite paths in $M \times X$ of the *score*, where the score is the sum of the distances (in $M \times X$) along the path, but where we count 0 for pairs in the relation $\sim$. Define $S_{M\otimes X}$ in the obvious way around the boundary.

The quotient metric is a metric (not just a pseudometric).

So $M \otimes -$ is a functor on SquaMS.

Colimit, Initial Algebra

Consider the initial chain

$$M_0 \xrightarrow{!} M \otimes M_0 \xrightarrow{M\otimes !} M^2 \otimes M_0 \xrightarrow{M^2\otimes !} \dots \xrightarrow{M^{n-1}\otimes !} M^n \otimes M_0 \xrightarrow{M^n\otimes !} \dots$$

G, the colimit of the initial chain, exists and is preserved by $M \otimes -$. Let $\eta : M \otimes G \to G$ be the isomorphism between them.

Theorem (Adamek)

If $\mathcal{C}$ is a cateogry with an initial object 0, F an endofunctor, and A the colimit of the initial chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} \dots \xrightarrow{F^{n-1}!} F^n 0 \xrightarrow{F^n!} \dots$$

with $i_n : F^n 0 \to A$. Suppose A is preserved by F and let $a : FA \to A$ be the unique morphism such that $a \circ Fi_n = i_{n+1}$.
Then (A, a) is an initial algebra.

Thus, $(G, \eta : M \otimes G \to G)$ is an initial algebra.

$N \otimes -$

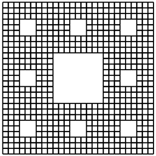
Let $N = \{0,1,2\}^2$ (so $N = M \cup \{(1,1)\}$).

$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$	$(1,1)$	$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

- $N \otimes -$ is a functor, $U = [0,1]^2$ with the taxicab metric is an $N \otimes -$ final coalgebra
- The obvious natural transformation $i : M \otimes - \to N \otimes -$ gives us a functor from the coalgebras of $M \otimes -$ to the coalgebras of $N \otimes -$
- If $(B, \beta : B \to M \otimes B)$ is an $M \otimes -$ coalgebra, then $(B, i_B \circ \beta : B \to N \otimes B)$ is an $N \otimes -$ coalgebra

Q, the Cauchy Completion of G

Let Q be the Cauchy completion of G, and let $\gamma : Q \to M \otimes Q$ be the map corresponding to the completion of η^{-1}.
Goal: $(Q, \gamma : Q \to M \otimes Q)$ is a final $M \otimes -$ coalgebra. (Main task: for an $M \otimes -$ coalgebra $(B, \beta : B \to M \otimes B)$, there is a short map $h : B \to Q$. Uniqueness will follow from the contraction mapping theorem.)
We restrict our attention to corner points and show that $M^k \otimes U \to M^k \otimes M_0$ restricted to the corner points is an isometry.



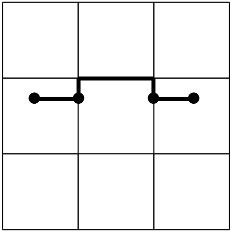
Using the map $B \to U$ (final $N \otimes -$ coalgebra), we can define approximate maps $B \to M^k \otimes U$, then to $M^k \otimes M_0 \to G \to Q$.
These maps may not be short, but they will approximate the required short map.

Bilipschitz Equivalence

Let $\mathbb{SC}$ be the Sierpinski Carpet in the plane with the Taxicab metric. There is a bijection $f : Q \to \mathbb{SC}$ (since $\mathbb{SC}$ is a final coalgebra in the category of square *sets*). For $x, y \in Q$,

$$\frac{1}{2} d_Q(x, y) \leq d_{\mathbb{SC}}(f(x), f(y)) \leq 2 d_Q(x, y)$$

Idea: Navigating around the center hole for a horizontal (or vertical) segment at most doubles the distance.



Conclusion

Theorem (N., Moss)

Q, the Cauchy completion of the initial $M \otimes -$ algebra in SquaMS, is a final $M \otimes -$ coalgebra, and is Bilipschitz equivalent to $\mathbb{SC}$, the Sierpinski Carpet in the plane with the Euclidean metric.

In this work we formulate a set of category theoretic results related to Hutchinson's Theorem. By studying a particular fractal, we connect analytic concepts with category theoretic ones via colimits in metric spaces, short maps approximated by non-short maps, corecursive algebras as an alternative to infinite sum, and the like.

The matter of finding fractal sets as final coalgebras by way of completions of initial algebras is rather delicate - in categories of sets this happens some of the time, but not always, and in metric spaces we get our results up to Bilipschitz equivalence.

Thank you!