Exponential modalities and complementarity

Priyaa Srinivasan
Joint work with Robin Cockett

UNIVERSITY OF CALGARY

Applied Category Theory, July 2021

Exponential modalities

Linear logic treats logical statements as resources (linear types) which cannot be duplicated or destroyed

Exponential modalities

Linear logic treats logical statements as resources (linear types) which cannot be duplicated or destroyed

Linear logic accommodates non-linear types using exponential modalities

Exponential modalities

Linear logic treats logical statements as resources (linear types) which cannot be duplicated or destroyed

Linear logic accommodates non-linear types using exponential modalities

Two exponential modalities in linear logic:
! read as the 'bang' / 'of course' and ? read as the 'why not' / 'whimper'

Exponential modalities

Linear logic treats logical statements as resources (linear types) which cannot be duplicated or destroyed

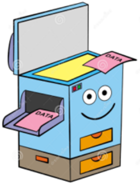
Linear logic accommodates non-linear types using exponential modalities

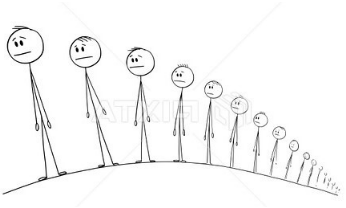
Two exponential modalities in linear logic:
! read as the ‘bang’ / ‘of course’ and ? read as the ‘why not’ / ‘whimper’

For any resource A,

!**A** refers to an infinite supply of the resource A

?**A** represents the notion of infinite demand.





!A can be duplicated and destroyed.

Exponential modalities

Linear logic treats logical statements as resources (linear types) which cannot be duplicated or destroyed

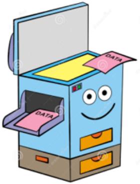
Linear logic accommodates non-linear types using exponential modalities

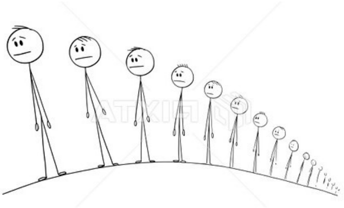
Two exponential modalities in linear logic:
! read as the 'bang' / 'of course' and ? read as the 'why not' / 'whimper'

For any resource A,

!**A** refers to an infinite supply of the resource A

?**A** represents the notion of infinite demand.





$!A$ can be duplicated and destroyed.

! is used a de facto structure to model arbitrary dimensional spaces such as Bosonic Fock spaces in Physics.

complementarity in quantum mechanics

A **quantum observable** refers to a measurable property of quantum system.

A pair of quantum obsevables are **complementary** if measuring one observable increases uncertanity regarding the value of the other.

Example: position and momentum of an electron

Question

Is there a connection between exponential modalities of linear logic and complementary observables of quantum mechanics?

Question

Is there a connection between exponential modalities of linear logic and complementary observables of quantum mechanics?

Theorem:

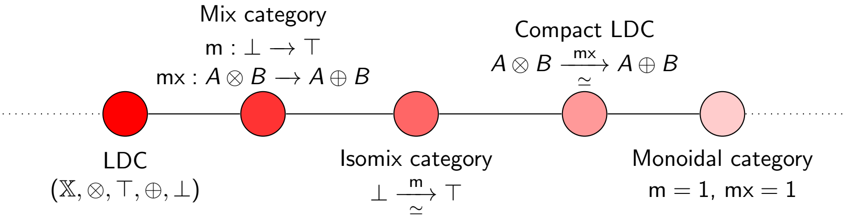
In a $(!, ?)$-$\dagger$-isomix category with free exponentials, every complementary system arises as a splitting of a $\dagger$-binary idempotent on the $\dagger$-linear bialgebra induced on the free exponentials.

Linearly distributive categories (LDC):

$$(\mathbb{X}, \otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes) \qquad (\mathbb{X}, \oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$$

linked by linear distributors: $\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$

Monoidal categories: LDCs in which $\otimes = \oplus$



Categorical semantics of ! and ?

In a $(!, ?)$-LDC[1]

- ! is a monoidal coalgebra comodality
  - $(!, \delta : ! \Rightarrow !!, \varepsilon : ! \Rightarrow \mathbb{I})$ is a monoidal comonad
  - For each A, $(!A, \Delta_A, e_A)$ is a $\otimes$-cocommutative comonoid
- ? is a comonoidal algebra modality
  - $(?, \mu : ?? \Rightarrow ?, \eta : \mathbb{I} \Rightarrow ?)$ is a comonoidal monad
  - For each A, $(?A, \nabla_A, u_A)$ is a $\oplus$-commutative monoid

[1] Richard Blute, Robin Cockett, and Robert Seely (1996). "! and ? - Storage as tensorial strength."

Categorical semantics of ! and ?

In a $(!, ?)$-LDC[1]

- ! is a monoidal coalgebra comodality
  - $(!, \delta :! \Rightarrow !!, \varepsilon :! \Rightarrow \mathbb{I})$ is a monoidal comonad
  - For each A, $(!A, \Delta_A, e_A)$ is a $\otimes$-cocommutative comonoid

- ? is a comonoidal algebra modality
  - $(?, \mu :?? \Rightarrow ?, \eta : \mathbb{I} \Rightarrow ?)$ is a comonoidal monad
  - For each A, $(?A, \nabla_A, u_A)$ is a $\oplus$-commutative monoid

- $(!, ?)$ is a linear functor

- the pairs (δ, μ), (ε, η), (Δ, ∇) are linear transformations

[1] Richard Blute, Robin Cockett, and Robert Seely (1996). "! and ? - Storage as tensorial strength."

Examples of $(!, ?)$-LDC

Category of sets and **relations**, Rel:
Given a set X, $!X$ is the set of all finite multisets of elements of X.

Category of **finiteness spaces** and finiteness relations, FRel:
Category of finiteness spaces and finiteness matrices over a comm. rig, FMat(R):
Given a finiteness space, $(X, F(X))$, smililar to Rel, $!(X, F(X))$, consists of set of all finite multisets of elements of X with an appropriate finiteness structure.

Category of **Chu spaces** over complex vector spaces with the unit as the dualizing object, $\mathsf{Chus}_I(\mathsf{Vec}_{\mathbb{C}})$[2]

[2] Michael Barr (1991). "Accessible categories and models of linear logic".

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-**monoidal categories** $(\otimes, I)$
compact MLL

†-**LDCs** $(\otimes, \top, \oplus, \bot)$
non-compact MLL

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-monoidal categories $(\otimes, I)$
compact MLL

- For objects, $A^\dagger = A$

†-LDCs $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-monoidal categories $(\otimes, I)$
compact MLL

- for objects, $A^\dagger = A$
- for maps, $f^{\dagger\dagger} = f$

†-LDCs $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general
- $\iota : A \xrightarrow[\simeq]{} A^{\dagger\dagger}$

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-monoidal categories $(\otimes, I)$
compact MLL

- for objects, $A^\dagger = A$
- for maps, $f^{\dagger\dagger} = f$
- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$

†-LDCs $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general
- $\iota : A \xrightarrow[\simeq]{} A^{\dagger\dagger}$
- $\lambda_\otimes : (A^\dagger \otimes B^\dagger) \xrightarrow[\simeq]{} (A \oplus B)^\dagger$;
  $\lambda_\oplus : (A^\dagger \oplus B^\dagger) \xrightarrow[\simeq]{} (A \otimes B)^\dagger$;
  $\lambda_\top : \top \xrightarrow[\simeq]{} \bot^\dagger$; $\lambda_\bot : \bot \xrightarrow[\simeq]{} \top^\dagger$

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-monoidal categories $(\otimes, I)$
compact MLL

- for objects, $A^\dagger = A$
- for maps, $f^{\dagger\dagger} = f$
- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- All basic natural isomorphisms are unitary (i.e., $a_\otimes^\dagger = a_\otimes^{-1}$)

†-LDCs $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general
- $\iota : A \xrightarrow[\simeq]{} A^{\dagger\dagger}$
- $\lambda_\otimes : (A^\dagger \otimes B^\dagger) \xrightarrow[\simeq]{} (A \oplus B)^\dagger$;
  $\lambda_\oplus : (A^\dagger \oplus B^\dagger) \xrightarrow[\simeq]{} (A \otimes B)^\dagger$;
  $\lambda_\top : \top \xrightarrow[\simeq]{} \bot^\dagger$; $\lambda_\bot : \bot \xrightarrow[\simeq]{} \top^\dagger$
- (lots of) coherence conditions

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-monoidal categories $(\otimes, I)$
compact MLL

- for objects, $A^\dagger = A$
- for maps, $f^{\dagger\dagger} = f$
- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- All basic natural isomorphisms are unitary (i.e., $a_\otimes^\dagger = a_\otimes^{-1}$)

†-LDCs $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general
- $\iota : A \xrightarrow{\simeq} A^{\dagger\dagger}$
- $\lambda_\otimes : (A^\dagger \otimes B^\dagger) \xrightarrow{\simeq} (A \oplus B)^\dagger$;
  $\lambda_\oplus : (A^\dagger \oplus B^\dagger) \xrightarrow{\simeq} (A \otimes B)^\dagger$;
  $\lambda_\top : \top \xrightarrow{\simeq} \bot^\dagger$; $\lambda_\bot : \bot \xrightarrow{\simeq} \top^\dagger$
- (lots of) coherence conditions

†-isomix categories :- $\mathsf{m} : \bot \xrightarrow{\simeq} \top$

Categorical semantics of multiplicative †-linear logic (†-MLL)

Dagger is a **contravariant** functor

†-**monoidal categories** $(\otimes, I)$
compact MLL

- for objects, $A^\dagger = A$
- for maps, $f^{\dagger\dagger} = f$
- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- All basic natural isomorphisms are unitary (i.e., $a_\otimes^\dagger = a_\otimes^{-1}$)

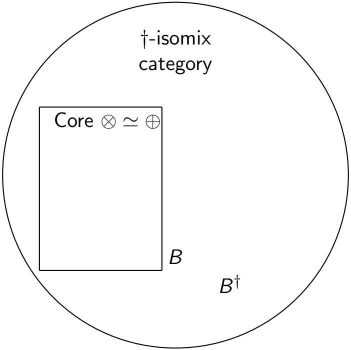
†-**LDCs** $(\otimes, \top, \oplus, \bot)$
non-compact MLL

- $A^\dagger \neq A$ in general
- $\iota : A \xrightarrow{\simeq} A^{\dagger\dagger}$
- $\lambda_\otimes : (A^\dagger \otimes B^\dagger) \xrightarrow{\simeq} (A \oplus B)^\dagger$;
  $\lambda_\oplus : (A^\dagger \oplus B^\dagger) \xrightarrow{\simeq} (A \otimes B)^\dagger$;
  $\lambda_\top : \top \xrightarrow{\simeq} \bot^\dagger$; $\lambda_\bot : \bot \xrightarrow{\simeq} \top^\dagger$
- (lots of) coherence conditions

†-**isomix categories** :- $\mathsf{m} : \bot \xrightarrow{\simeq} \top$

†-monoidal cats: Rel, Hilb   †-isomix cats: FRel, FMat($\mathbb{C}$), Chus$_I$(Vec$_\mathbb{C}$)

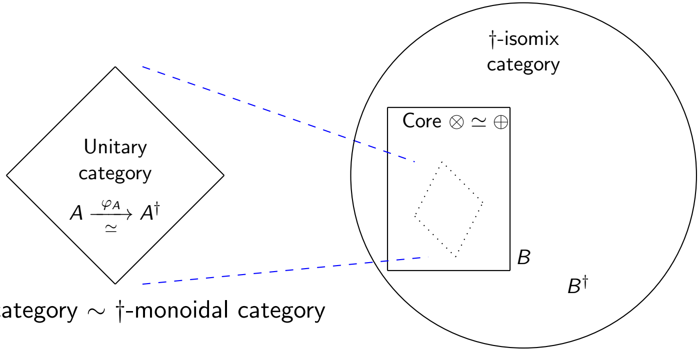
Extracting a †-monoidal category from an isomix †-LDC



Pre-unitary objects: An object A in the **core** with $\alpha : A \xrightarrow{\simeq} A^\dagger$ satisfying

$$\begin{array}{c} A \xrightarrow{\alpha} A^\dagger \xrightarrow{\alpha^{-1\dagger}} A^{\dagger\dagger} \\ \iota : A \to A^{\dagger\dagger} \end{array}$$

Extracting a †-monoidal category from an isomix †-LDC

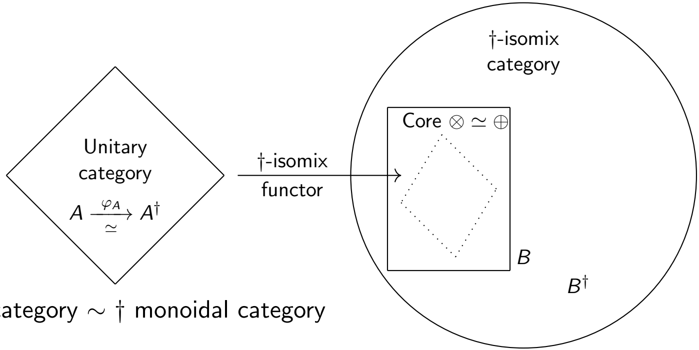


Unitary category $\sim$ †-monoidal category

Pre-unitary objects: An object A in the **core** with $\alpha : A \xrightarrow{\simeq} A^\dagger$ satisfying

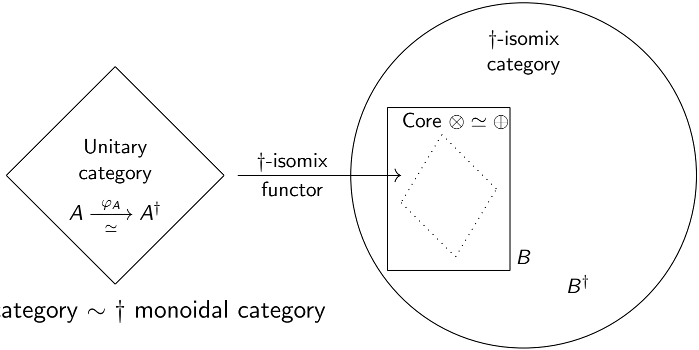
$$A \xrightarrow{\alpha} A^\dagger \xrightarrow{\alpha^{-1\dagger}} A^{\dagger\dagger}, \qquad \iota : A \to A^{\dagger\dagger}$$

Mixed Unitary Categories (MUCs)



Unitary category $\sim$ † monoidal category

Mixed Unitary Categories (MUCs)



Unitary category $\sim$ † monoidal category

A 'canonical' MUC consists of the unitary category of pre-unitary objects embedded into the †-isomix category.

Examples of MUCs

- Every †-monoidal category is a MUC

Examples of MUCs

- Every †-monoidal category is a MUC
- FinRel $\hookrightarrow$ FRel: Finite relations embedded into finiteness relations

Examples of MUCs

- Every †-monoidal category is a MUC
- FinRel $\hookrightarrow$ FRel: Finite relations embedded into finiteness relations
- Mat($\mathbb{C}$) $\hookrightarrow$ FMat(R): Complex finite dimensional matrices embedded into finiteness matrices over a commutative rig R

Examples of MUCs

- Every †-monoidal category is a MUC
- FinRel $\hookrightarrow$ FRel: Finite relations embedded into finiteness relations
- Mat($\mathbb{C}$) $\hookrightarrow$ FMat(R): Complex finite dimensional matrices embedded into finiteness matrices over a commutative rig R
- FHilb $\hookrightarrow$ $\mathsf{Chus}_I(\mathsf{Vec}(\mathbb{C}))$: Finite-dimensional Hilbert spaces embedded into Chu spaces over complex vector spaces

! and ? in †-linear logic

In a $(!, ?)$-**dagger**-LDC is a $(!, ?)$-LDC and a dagger LDC such that:

- $(!, ?)$ is a **dagger** linear functor

$$(!A)^\dagger \simeq ?(A^\dagger) \qquad !(A^\dagger) \simeq (?A)^\dagger$$

- The pairs (δ, μ), (ε, η), (Δ, ∇) are **dagger** linear transformations

Examples: FRel, FMat(R), (conjecture) Chus$_I$(Vec($\mathbb{C}$))

A rough plan

Step 1: Measurements in MUCs

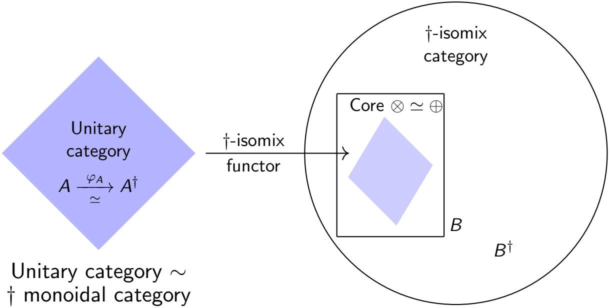
Step 2: complementary systems in MUCs

Step 3: Prove the connection between exponential modalities and complementary observables

Step 1: Measurement in MUCs

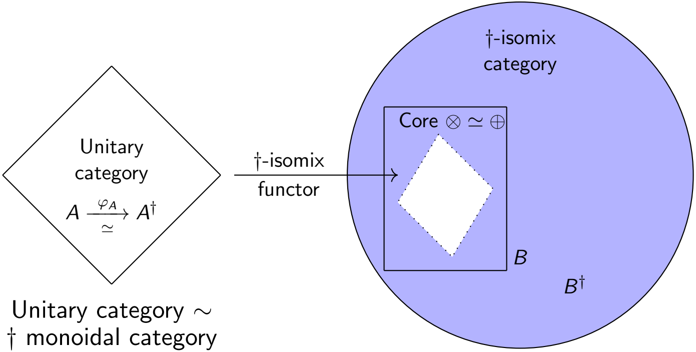
In a †-monoidal category, a **demolition measurement**[3] on an object A is retract from A to a special commutative †-Frobenius algebra (an abstract quantum observable), E.

$$A \underset{r^\dagger}{\overset{r}{\rightleftarrows}} E \quad \text{such that } r^\dagger r = 1_E$$



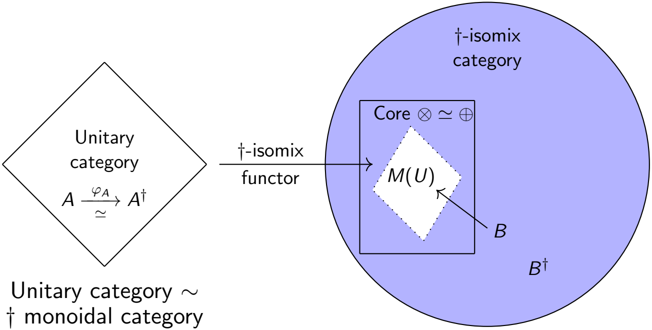
Unitary category ∼
† monoidal category

[3] Bob Coecke and Dusko Pavlovic (2006). "Quantum measurements without sums"



Unitary category $\sim$
† monoidal category

Compaction

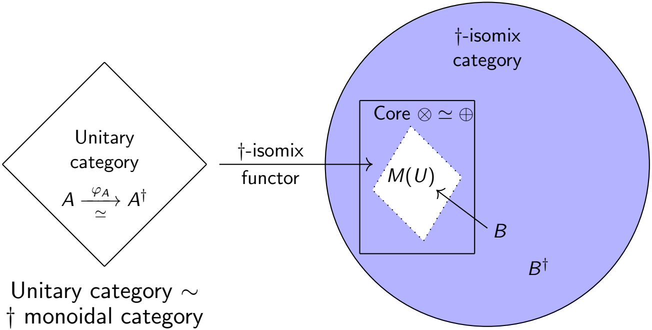


$r : B \longrightarrow M(U)$

$s : M(U) \longrightarrow B$

$sr = 1_{M(U)}$

Compaction



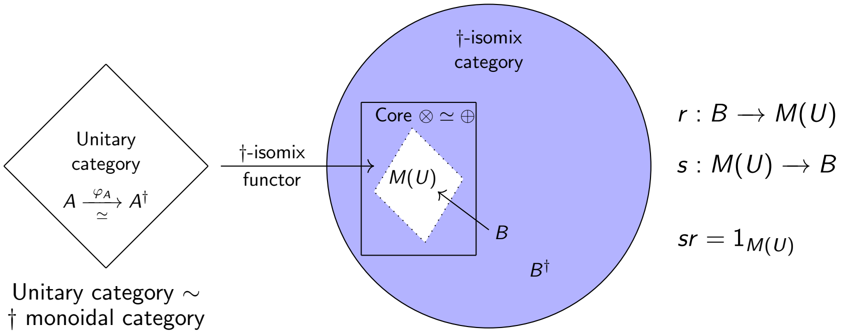
$r : B \longrightarrow M(U)$

$s : M(U) \longrightarrow B$

$sr = 1_{M(U)}$

A **compaction** in a MUC, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a retraction to an object in the unitary core $r : B \longrightarrow M(U)$.

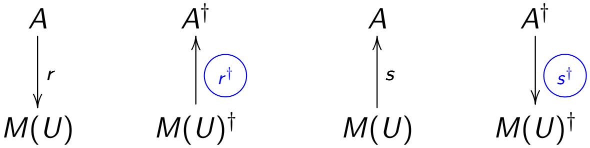
Compaction



A **compaction** in a MUC, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a retraction to an object in the unitary core $r : B \longrightarrow M(U)$.

MUC measurement = Compaction followed by Demolition

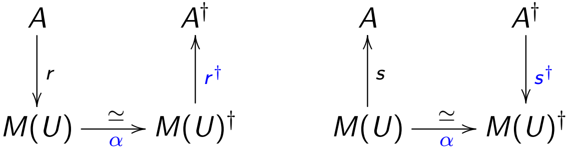
Binary idempotents



Binary idempotents

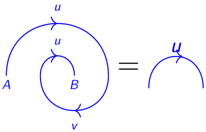
$$
\begin{array}{ccc}
A & & A^\dagger \\
\downarrow{\scriptstyle r} & & \uparrow{\scriptstyle r^\dagger} \\
M(U) & \xrightarrow[\alpha]{\simeq} & M(U)^\dagger
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & & A^\dagger \\
\uparrow{\scriptstyle s} & & \downarrow{\scriptstyle s^\dagger} \\
M(U) & \xrightarrow[\alpha]{\simeq} & M(U)^\dagger
\end{array}
$$

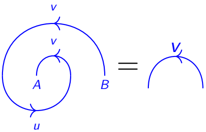
Binary idempotents



Binary idempotent (any category): $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

$uvu = u$





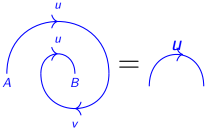
$vuv = u$

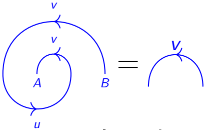
Binary idempotents

$$\begin{array}{ccc} A & & A^\dagger \\ \downarrow r & & \uparrow r^\dagger \\ M(U) & \xrightarrow[\alpha]{\simeq} & M(U)^\dagger \end{array} \qquad \begin{array}{ccc} A & & A^\dagger \\ \uparrow s & & \downarrow s^\dagger \\ M(U) & \xrightarrow[\alpha]{\simeq} & M(U)^\dagger \end{array}$$

Binary idempotent (any category): $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

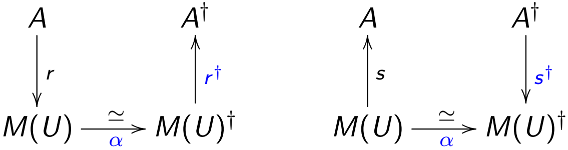
$uvu = u$





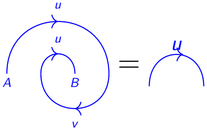
$vuv = u$

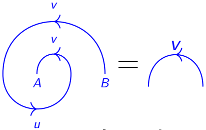
If $e_A := uv$ splits through E, and $e_B := vu$ splits through F, then $E \simeq F$



Binary idempotent (any category): $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

$uvu = u$





$vuv = u$

If $e_A := uv$ splits through E, and $e_B := vu$ splits through F, then $E \simeq F$

†-**binary idempotent**: **(†-LDC)** $A \underset{v}{\overset{u}{\rightleftarrows}} A^\dagger$ such that $\iota u^\dagger = u \quad v\iota = v^\dagger$

Observation: $(e_A)^\dagger = v^\dagger u^\dagger = v\iota u^\dagger = vu = e_{A^\dagger}$

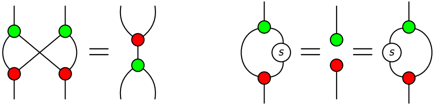
Compaction = splitting coring †-binary idempotents

Theorem:

In a †-isomix category, $r : A \longrightarrow U$ is a compaction if and only if U is given by splitting a †-binary idempotent[4] on A.

[4] The idempotent has to be coring, that is, split through the core.

In a †-monoidal category, two †-Frobenius algebras (*A*, , ), (*A*, , ), on an object are complementary[5] if they interact to produce two Hopf algebras.



[5] Bob Coecke and Ross Duncan (2008). "Interacting quantum observables"

Linear monoids

Linear monoids in LDCs are a general version of Frobenius algebras.

In a symmetric LDC, **a linear monoid**, $A \overset{\circ}{\multimap\!\!\!\Vdash} B$, contains a:

- a monoid $(A, \curlyvee : A \otimes A \longrightarrow A,\ \text{⚲} : \top \longrightarrow A)$

- a dual for A, $(\eta, \varepsilon) : A \dashv\!\!\vdash B$

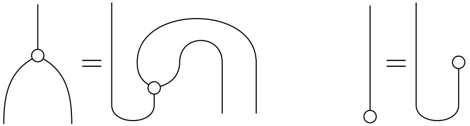
Linear monoids

Linear monoids in LDCs are a general version of Frobenius algebras.

In a symmetric LDC, **a linear monoid**, $A \overset{\circ}{\dashv\!\!\vdash} B$, contains a:

- a monoid $(A, \curlyvee : A \otimes A \to A, \, \top \to A)$
- a dual for A, $(\eta, \varepsilon) : A \dashv\!\!\vdash B$

together producing a comonoid $(B, \curlywedge : B \to B \oplus B, \, : B \to \bot)$



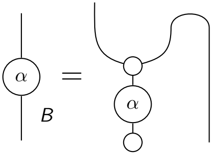
A **self** linear monoid is a linear monoid, $A \overset{\circ}{\dashv\!\!\vdash} B$, with $A \simeq B$

Linear monoids with an extra property are Frobenius

An object which is a Frobenius algebra is always a self-dual, however for linear monoids, the monoid and the comonoid are on distinct but dual objects

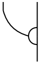
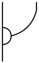
Proposition:

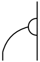
In a monoidal category, a Frobenius algebra is precisely a self linear monoid $A \overset{\circ}{\multimap} B$, ($\alpha : A \xrightarrow{\alpha} B$) satisfying the equation:



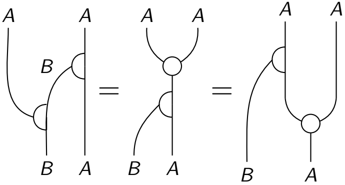
Alternate characterization of linear monoids

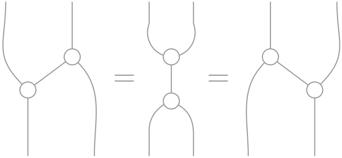
A linear monoid, $A \overset{\circ}{\multimap} B$, consists of a $\otimes$-monoid, $(A,$  $,$  $)$, and a $\oplus$-comonoid, $(B,$  $,$  $)$ and:

- monoid actions:  $: A \otimes B \longrightarrow B$;  $: B \otimes A \longrightarrow A$

- comonoid coactions:  $: B \longrightarrow A \oplus B$;  $: B \longrightarrow A \oplus B$

satisfying certain equations. The Frobenius equation is given as follows:





Linear bialgebras

Linear monoid

a $\otimes$-monoid and a dual:

$(A,\ \curlyvee : A \otimes A \longrightarrow A,\ \dot{\top} : \top \longrightarrow A)$

$(\eta, \varepsilon) : A \dashv\!\!\vdash B$

Linear comonoid

a $\otimes$-comonoid and a dual:

$(A,\ \curlywedge : A \longrightarrow A \otimes A,\ \dot{\bot} : A \longrightarrow \bot)$

$(\eta, \varepsilon) : A \dashv\!\!\vdash B$

Linear bialgebras

- a linear monoid $(A, \curlyvee, \dot{\top})$; $(\eta, \varepsilon) : A \dashv\!\!\vdash B$

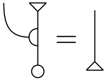
- a linear comonoid $(A, \curlywedge, \dot{\bot})$; $(\eta', \varepsilon') : A \dashv\!\!\vdash B$

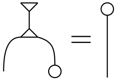
such that $(A, \curlyvee, \dot{\top}, \curlywedge, \dot{\bot})$ is a $\otimes$-bialgebra; $(B, \curlyvee, \top, \curlywedge, \bot)$ is a $\oplus$-bialgebra

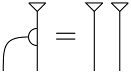
A **self-linear bialgebra** is a linear bialgebra where $A \simeq B$

complementary systems

A **complementary system** in an isomix category a self-linear bialagebra, A (not necessarily in the core), such that:

[comp.1] 

[comp.2] 

[comp.3] 

Lemma: If A is a complementary system, then A is a $\otimes$-Hopf and $\oplus$-Hopf.

Main result: connection with exponential modalities

Theorem:

In a $(!, ?)$-isomix category with **free** exponential modalities, every complementary system arises as a splitting of a binary idempotent on the linear bialgebra induced on the free exponentials.

The structures and results discussed extend directly to $\dagger$-linear bilagebras in $\dagger$-isomix categories with free exponential modalities due to the $\dagger$-linearity of $(!, ?)$, (η, ε), (Δ, ∇), and $(\perp, \top)$.

Future work

Examples in physics to be explored: Modeling quantum Harmonic Oscillators using exponentials[6]

Acknowledgement
Thank you Jean-Simon Lemay for many useful discussions on the exponential modalities and examples!

Pre-prints

Robin Cockett, and Priyaa Srinivasan. Exponential modalities and complementarity. **arXiv:2103.05191** (2021).

Robin Cockett, Cole Comfort, and Priyaa Srinivasan. Dagger linear logic for categorical quantum mechanics. **arXiv:1809.00275** (2018).

[6] Jamie Vicary (2008). Categorical quantum harmonic osciallator