Noncommutative Network Models

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Outline

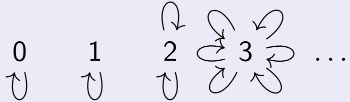
Noncommutative network models, Mathematical Structures in Computer Science, 2020. arXiv:1804.07402

- Network models
- Eckmann–Hilton for network models
- Kneser graphs
- Graph products of monoids
- Free undirected network models

Species

Definition

Let S *denote the* **symmetric groupoid**, *the category consisting of sets* $n = \{1, \ldots, n\}$ *for objects* ($0 = \emptyset$), *and bijections for the morphisms.*



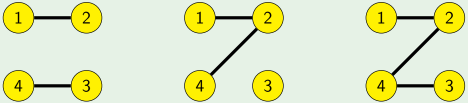
Notice, all morphisms are automorphisms. This is a skeleton of FinBij, the maximal subgroupoid in FinSet.

Definition ([Joy81])

A **combinatorial species** *is a functor* $F \colon \mathsf{S} \to \mathsf{Set}$.

Example

$SG \colon \mathsf{S} \to \mathsf{Set}$ by $SG(n) =$ the set of simple graphs with n nodes.

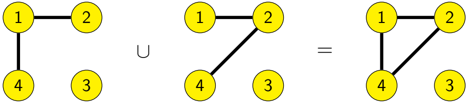


Network models

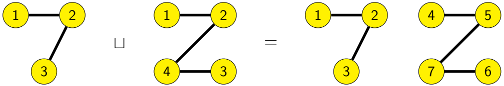
Definition ([BFMP20])

A **network model** *is a symmetric lax monoidal functor of the form* $(F, \sqcup)\colon (\mathsf{S}, +) \to (\mathsf{Mon}, \times)$.

"overlay" $\cup_n\colon F(n) \times F(n) \to F(n)$



and "disjoint union"
$\sqcup_{m,n}\colon F(m) \times F(n) \to F(m+n)$



Example

- simple graphs
- directed edges
- multiple edges
- edge colors
- hypergraphs
- Petri nets

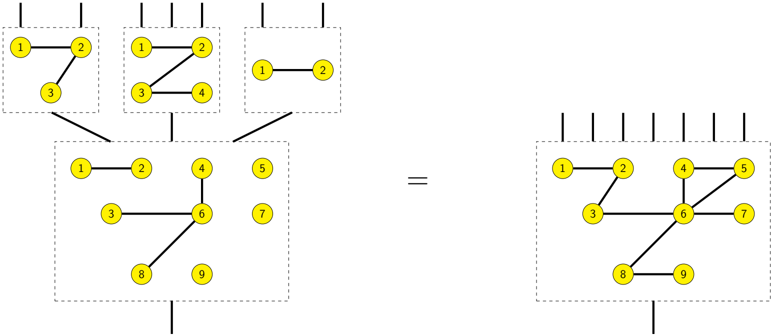
Nonexample

acyclic graphs

Operad from a network model

The original motivation for network models is to construct operads modeling network design:

$$\mathsf{NetMod} \xrightarrow{\int} \mathsf{SMC} \xrightarrow{U} \mathsf{Oprd}$$



Constructing network models

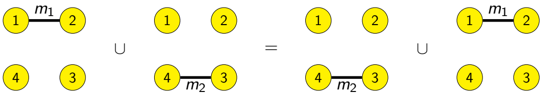
Construction

You can get a network model from any monoid. There's a functor Mon $\to$ NetMod given by $M \mapsto (n \mapsto M^{\binom{n}{2}})$.

Example

- $M = (\mathbb{B}, +)$ recovers the simple graphs network model.
- $M = (\mathbb{N}, +)$ gives graphs with multiple (indistinguishable) edges.
- $M = (2^X, \cup)$ gives graphs with edges labeled in X.

But notice different edge components automatically commute with each other:



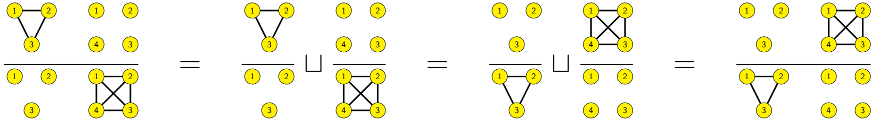
$$(m_1, 0, 0, 0, 0, 0) \cup (0, m_2, 0, 0, 0, 0)$$
$$= (0, m_2, 0, 0, 0, 0) \cup (m_1, 0, 0, 0, 0, 0)$$

Can we define $\Gamma_M \colon \mathsf{S} \to \mathsf{Set}$ by $\Gamma_M(n) = \coprod^{\binom{n}{2}} M$?

Eckmann–Hilton for network models

Disjoint components must commute with each other: Let $a \in F(m)$ and $b \in F(n)$. Then

$$\begin{aligned}(a \sqcup \emptyset) \cup (\emptyset \sqcup b) &= (a \cup \emptyset) \sqcup (\emptyset \cup b) \\ &= (\emptyset \cup a) \sqcup (b \cup \emptyset) \\ &= (\emptyset \sqcup b) \cup (a \sqcup \emptyset)\end{aligned}$$



So what this means is that we want to define Γ_M to be $n \mapsto \coprod_{\binom{n}{2}} M/\sim$ where $\sim$ tells us to impose commutativity for edge components which are disjoint. Let's take a look at the first few levels to see how this relation looks.

$$0 \mapsto 1, \quad 1 \mapsto 1, \quad 2 \mapsto M, \quad 3 \mapsto \coprod^{3} M,$$

$$4 \mapsto \coprod^{6} M/\langle a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,3}b_{2,4} = b_{2,4}a_{1,3}, a_{1,4}b_{2,3} = b_{2,3}a_{1,4}\rangle$$

$$\begin{aligned}5 \mapsto \coprod^{10} M/\langle a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,2}b_{3,5} = b_{3,5}a_{1,2},\\
a_{1,3}b_{2,4} = b_{2,4}a_{1,3}, a_{1,3}b_{2,5} = b_{2,5}a_{1,3}, a_{1,3}b_{4,5} = b_{4,5}a_{1,3},\\
a_{1,4}b_{2,3} = b_{2,3}a_{1,4}, a_{1,4}b_{2,5} = b_{2,5}a_{1,4}, a_{1,4}b_{3,5} = b_{3,5}a_{1,4},\\
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a_{2,3}b_{4,5} = b_{4,5}a_{2,3}, a_{2,4}b_{3,5} = b_{3,5}a_{2,4}, a_{2,5}b_{3,4} = b_{3,4}a_{2,5}\rangle\end{aligned}$$

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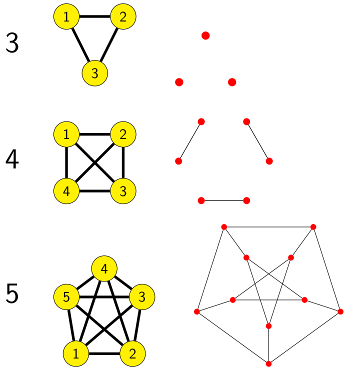
$$4 \mapsto \coprod^{6} M/\langle ☺☺☺☺☺☺ \rangle$$

$$5 \mapsto \coprod^{10} M/\langle ☺☺☺☺☺ \; ☺☺☺☺☺ \; ☺☺☺☺☺ \rangle$$



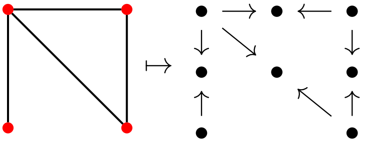
Graph products of monoids

Definition ([Gre90, Vel01])

Let G be a graph with N nodes, and M_i a family of N monoids. The **graph product** *is the monoid*

$$G(M_i) = \coprod M_i / \langle a_k b_\ell = b_\ell a_k \text{ if } (k, \ell) \in G \rangle.$$

Define $IC\colon \mathsf{SimpleGrph} \to \mathsf{Cat}$ by



Let $D\colon IC(G) \to \mathsf{Mon}$ be the diagram

$$\begin{array}{ccccc} M_1 & \longrightarrow & M_1 \times M_2 & \longleftarrow & M_2 \\ \downarrow & \searrow & & & \downarrow \\ M_1 \times M_4 & & M_1 \times M_3 & & M_2 \times M_3 \\ \uparrow & & & \nwarrow & \uparrow \\ M_4 & & & & M_3 \end{array}$$

Proposition (M.)

$G(M_i) \cong \operatorname{colim} D.$

Now for a given monoid M we define a network model $\Gamma_M \colon (\mathsf{S}, +, 0) \to (\mathsf{Mon}, \times, 1)$ by $n \mapsto KG_{n,2}(M)$.

Theorem (M.)

Γ_M *defined above is a network model. Moreover, we have an adjunction*

$$\mathsf{Mon} \underset{ev_2}{\overset{\Gamma}{\underset{\bot}{\rightleftarrows}}} \mathsf{NetMod}^+$$

where NetMod^+ *is the subcategory of* NetMod *of network models with trivial involution (thanks to Mike Shulman for pointing out a mistake in the original).*

Network models with trivial involution are essentially "undirected network models" (thanks Mike Shulman).

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