Lovász-Type Theorems and Game Comonads

Tomáš Jakl (join work with Anuj Dawar and Luca Reggio)

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Lovász-Type Theorems

Theorem (Lovász, 1967)

For finite relational structures A and B,

$$A \cong B \iff \hom(C, A) \cong \hom(C, B) \quad \forall \textit{finite } C$$

Yoneda lemma implies $A \cong B \iff \hom(-, A) \cong \hom(-, B)$

Generalisations

Theorem (Pultr, 1973)

Every finitely well-powered, locally finite categories with (extremal epi, mono) factorization system is **combinatorial**, *i.e.*

$$A \cong B \iff \hom(-, A) \text{ “}\cong\text{” } \hom(-, B)$$

unnatural isomorphism

Other categorical reformulations: (Isbell, 1991), (Lovász, 1972).

Notable refinements

Theorem (Dvořák, 2009)

For finite relational structures A and B,

$$A \equiv_{\mathrm{FO}^k(\#)} B \iff \hom(C, A) \cong \hom(C, B)$$

$$\forall \textit{finite } C \textit{ with tree-width} < k$$

Theorem (Grohe, 2020)

For finite relational structures A and B,

$$A \equiv_{\mathrm{FO}_k(\#)} B \iff \hom(C, A) \cong \hom(C, B)$$

$$\forall \textit{finite } C \textit{ with tree-depth} \leq k$$

$$\mathrm{FO}(\#) \;=\; \mathrm{FO} \;+\; \exists^{\geq n} x \;+\; \exists^{\leq n} x$$

Comonads

Kleisli–Manes definition

Comonad $(\mathbb{C}, \varepsilon, \overline{(-)})$ on category $\mathcal{A}$:

- $\mathbb{C}\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{A})$
- counit $\varepsilon_A\colon \mathbb{C}A \to A, \qquad \forall A \in \mathrm{obj}(\mathcal{A}),$
- lifting $\overline{f}\colon \mathbb{C}A \to \mathbb{C}B, \qquad \forall f\colon \mathbb{C}A \to B,$
- laws:

$$\overline{\varepsilon_A} = \mathrm{id}_{\mathbb{C}A}, \qquad \varepsilon_B \circ \overline{f} = f, \qquad \overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$$

Example: Lists/words comonad on Sets

$\mathrm{List}_k(A) = \{\, [a_1, \ldots, a_m] \mid a_i \in A,\ 1 \leq m \leq k\}$

- $\varepsilon([a_1, \ldots, a_m]) = a_m$.
- $f\colon \mathrm{List}_k(A) \to B$ lifts to $\overline{f}\colon\ [a_1, \ldots, a_m] \ \mapsto\ [b_1, \ldots, b_m]$ where $b_i = f([a_1, \ldots, a_i])$

Comonad Coalgebras

$\alpha\colon A \to \mathbb{C}A$ is a coalgebra iff

$$\begin{array}{ccc} A & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\mathsf{id}} & \\ \mathbb{C}A & \xrightarrow[\varepsilon_A]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & \mathbb{C}A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta_A = \overline{\mathsf{id}}} \\ \mathbb{C}A & \xrightarrow[\mathbb{C}\alpha]{} & \mathbb{C}^2 A \end{array}$$

$$\begin{array}{c} \mathtt{CoAlg}(\mathbb{C}) \\ U \downarrow \;\dashv\; \uparrow F \\ \mathcal{A} \end{array}$$

Example: For a fixed set A,

$$\text{Coalgebras } \alpha\colon A \to \mathrm{List}_k(A)$$

$$\updownarrow$$

$$\text{forest orders } (\leq) \subseteq A \times A$$
$$\text{of depth } \leq k$$

Ehrenfeucht-Fraissé comonad(s)

Fix a relational signature σ and category $\mathcal{R}(\sigma)$.

Given $A \in \mathcal{R}(\sigma)$, define $\mathbb{E}_k(A)$ on the set $\mathrm{List}_k(A)$ with

$$(w_1, \ldots, w_u) \in R^{\mathbb{E}_k(A)} \quad \text{if} \quad w_s \sqsubseteq w_t \text{ or } w_s \sqsupseteq w_t \quad (\forall s, t)$$
$$\text{and} \quad (\varepsilon_A(w_1), \ldots, \varepsilon_A(w_u)) \in R^A$$

Sequence of subcomonads $\mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \mathbb{E}_3 \hookrightarrow \ldots$

Theorem (Abramsky, Shah, 2018)

A coalgebra $A \to \mathbb{E}_k(A)$ exists iff tree-depth of A is $\leq k$.

Expressing logic fragments

- Define $\sigma^{\mathrm{I}} = \sigma \cup \{I\}$ and $\mathcal{R}(\sigma^{\mathrm{I}}) \circlearrowleft \mathbb{E}_k^{\mathrm{I}}$ as before.
- $\mathcal{R}(\sigma) \xrightarrow{J} \mathcal{R}(\sigma^{\mathrm{I}}) \xrightarrow{F} \texttt{CoAlg}(\mathbb{E}_k^{\mathrm{I}})$ where

$$J\colon A \mapsto (A, I^A) \quad \text{and} \quad I^A = \{(a,a) \mid a \in A\}$$

Theorem (Abramsky, Shah, 2018)

$$A \equiv_{\mathrm{FO}_k(\#)} B \quad \iff \quad FJ(A) \cong FJ(B) \ \textit{in} \ \texttt{CoAlg}(\mathbb{E}_k^{\mathrm{I}})$$

Remark:

$A \equiv_{\mathrm{FO}_k} B$ captured by span of "open pathwise embeddings"

$$FJ(A) \leftarrow R \rightarrow FJ(B) \quad \text{in} \quad \texttt{CoAlg}(\mathbb{E}_k^{\mathrm{I}})$$

The framework

“Theorem” (Dawar, Jakl, Reggio, 2021)

Assuming

1. $\mathbb{C}$ *classifies* $\equiv_{\mathcal{L}}$ *and a class of finite structures* Δ, *(regardless of signature)*
2. $\mathbb{C}$ *restricts to finite structures, and*
3. *for every* (A, I) *in* Δ, *also* $A/I \in \Delta$.

combinatorial core
—————
equality elimination

Then,

$$A \equiv_{\mathcal{L}} B \iff \mathrm{hom}(C, A) \cong \mathrm{hom}(C, B) \quad \forall C \in \Delta$$

Applies to $\mathbb{E}_k$, $\mathbb{P}_{k,n}$ (... also $\mathbb{P}_k$), $\mathbb{M}_k$

Candidate comonads: $\mathbb{H}_{n,k}$ (Ó Conghaile, Dawar, 2020), $\mathbb{G}^{\mathfrak{g}}$ (Abramsky, Marsden, 2021), $\mathbb{PR}_k$ (Shah, 2021)

Ingredients

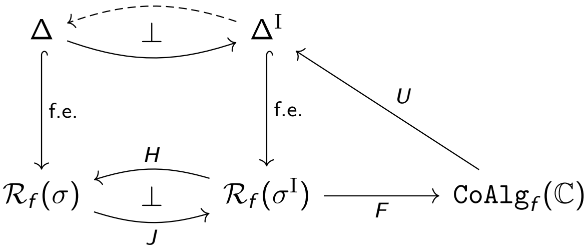
Theorem

Any locally finite category with pushouts and a weak factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} \subseteq \mathrm{Epis}$ and $\mathcal{M} \subseteq \mathrm{Monos}$ is combinatorial.

Because $U\colon \mathtt{CoAlg}(\mathbb{C}) \to \mathcal{R}(\sigma)$ creates colimits and isomorphisms:

Corollary

For any comonad $\mathbb{C}$ on $\mathcal{R}(\sigma)$, the category of finite coalgebras $\mathtt{CoAlg}_f(\mathbb{C})$ is combinatorial.



Theorem

Any locally finite category with pushouts and a weak factorisation system $(\mathcal{E}, \mathcal{M})$ *with* $\mathcal{E} \subseteq \mathrm{Epis}$ *and* $\mathcal{M} \subseteq \mathrm{Monos}$ *is combinatorial.*

Proof sketch:

- Given $F, G \colon \mathcal{A}^{\mathrm{op}} \to \mathsf{FinSet}$ which send $\mathcal{E}$-pushout squares to pullbacks. Then F "$\cong$" G implies $\widehat{F}$ "$\cong$" $\widehat{G}$ where
$$\widehat{F}(z) \;=\; F(z) \;\setminus\; \bigcup\{im(E(f)) \mid f \colon z \to z' \in \mathcal{E} \setminus \mathcal{M}\}.$$
- If $E = \mathrm{hom}(-, x)$ then $\widehat{E}(z) = \mathcal{M}(z, x)$.
- $\mathrm{hom}(-, x)$ "$\cong$" $\mathrm{hom}(-, y)$
  $\implies \mathcal{M}(-, x)$ "$\cong$" $\mathcal{M}(-, y)$
  $\implies x \cong y$

Thank you!