Restricting Power: Pebble-relation comonad in finite model theory

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Let $\sigma$ be a set of relational symbols with positive arities, we can define a category of $\sigma$-structures $\mathcal{R}(\sigma)$:

- **Objects** are $\mathcal{A} = (A, \{R^A\}_{R \in \sigma})$ where $R^A \subseteq A^r$ for $r$-ary relation symbol $R$.

- **Morphisms** $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^A(a_1, \ldots, a_r) \Rightarrow R^B(f(a_1), \ldots, f(a_r))$$

- If there exists a morphism $f : \mathcal{A} \to \mathcal{B}$, we write $\mathcal{A} \to \mathcal{B}$
Category theorists look at structures “as they really are”; i.e. up to isomorphism \( A \cong B \)

Model theorists look at structures with “fuzzy glasses” imposed by a logic \( \mathcal{L} \):

\[
\mathcal{A} \equiv^\mathcal{L} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \models \phi \iff \mathcal{B} \models \phi
\]

\[
\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \equiv^\mathcal{L} \mathcal{B}
\]

Used to study what properties are inexpressible in \( \mathcal{L} \)

To show \( P \) inexpressible in \( \mathcal{L} \), define \( \mathcal{A}, \mathcal{B} \) where \( P(\mathcal{A}) \) and not \( P(\mathcal{B}) \). Must show that \( \mathcal{A} \equiv^\mathcal{L} \mathcal{B} \)
Over finite structures, $\equiv^\text{FOL}$ is the same as $\equiv$

Finite model theorists look at structures with a “fuzzy phoropter” imposed by grading a logic:

- Quantifier rank $\leq n$, $QR_n$
- Restrict number of variables be $\leq k$, $V^k$

$$\phi = \exists x_1 (\exists x_2 (E(x_1, x_2) \wedge \exists x_3 E(x_3, x_2)) \wedge \forall x_4 E(x_1, x_4))$$

$\phi \in QR_3$ and $\phi \in V^4$

To show $P$ inexpressible in $\mathcal{L}$ over the finite, define $A_k, B_k$ for every $k$ where $P(A_k)$ and not $P(B_k)$. Must show that $A_k \equiv^\mathcal{L} B_k$
CSP: Find assignment of variables $\mathcal{A}$ to a domain of values $\mathcal{B}$ satisfying a set of constraints, which can be encoded as relations on $\mathcal{B}$

A CSP can be formulated in $\mathcal{R}(\sigma)$ as deciding if there exists a morphism $h : \mathcal{A} \rightarrow \mathcal{B}$

Non-uniform problem $\text{CSP}(\mathcal{B})$: fixing the set of values $\mathcal{B}$ and varying the variables $\mathcal{A}$.

In general, $\text{CSP}(\mathcal{B})$ is NP-complete

Tractable cases of $\text{CSP}(\mathcal{B})$ can be identified by considering approximations to homomorphism
Approximating homomorphisms

Equivalence in a logic with parameter $k$ approximates isomorphism:

$$A \cong B \Rightarrow A \equiv_{L^k} B$$

Preservation in the existential-positive fragment is an approximation to homomorphism:

$$A \rightarrow B \Rightarrow A \models_{\exists^+L^k} B$$

$$A \models_{\exists^+L^k} B \iff \forall \phi \in \exists^+L^k, A \models \phi \Rightarrow B \models \phi$$

We will consider the existential-positive fragment of $k$-variable logic $\exists^+\forall_k$
For all finite $\mathcal{A}$,

$$\mathcal{A} \equiv \exists^+ \forall^k \mathcal{B} \rightarrow \mathcal{A} \rightarrow \mathcal{B}$$

then $\mathcal{B}$ has $k$-treewidth duality

$\mathcal{B}$ has $k$-treewidth duality $\Rightarrow$ CSP($\mathcal{B}$) $\in$ PTIME

**Proposition**

The following are equivalent:

- $\mathcal{A} \equiv \exists^+ \forall^k \mathcal{B}$
- Duplicator has a winning strategy in a forth $k$-pebble game
- For all finite $\mathcal{C}$ w/ treewidth $< k$, $\mathcal{C} \rightarrow \mathcal{A} \Rightarrow \mathcal{C} \rightarrow \mathcal{B}$
Forth $k$-pebble game

- Spoiler and Duplicator each have $k$ pebbles. On each round of $\exists^+ \text{Peb}_k(\mathcal{A}, \mathcal{B})$:
  - Spoiler places his pebble $p \in \mathbf{k}$ on an element $a_i \in \mathcal{A}$
    - If $p$ was already placed, Spoiler moves the pebble.
  - Duplicator places her corresponding pebble $p \in \mathbf{k}$ on $b_i \in \mathcal{B}$

Duplicator wins if

$$\gamma = \{(a, b) \mid p \in \mathbf{k} \text{ w/ } p \text{ pebbling } a \in \mathcal{A}, b \in \mathcal{B} \}$$

is a partial homomorphism

If Duplicator can always produce a winning move for any choice made Spoiler, than Duplicator has a winning strategy
Theorem ([KV90])

*Duplicator* has a winning strategy in $\exists^+ \text{Peb}_k(\mathcal{A}, \mathcal{B})$ iff

$\mathcal{A} \Rightarrow^{\exists^+ \forall^k} \mathcal{B}$

Intuition:

$\mathcal{A} \models \exists x_p \phi(x_p, \bar{y}) \Rightarrow \mathcal{A} \models \phi(a/x_p, \bar{y})$

Spoiler places $p$ on witness $a \in \mathcal{A}$

Suppose Duplicator responds by putting $p$ on $b \in \mathcal{B}$

Partial homomorphism in winning condition $\Rightarrow$

$\mathcal{B} \models \phi(b/x_p, \bar{y}) \Rightarrow \mathcal{B} \models \exists x_p \phi(x_p, \bar{y})$
Intuitively, Spoiler is moving a $k$-sized window around the structure $\mathcal{A}$ during a play.

Duplicator than has to choose a homomorphism from the $k$-sized window into $\mathcal{B}$.

If Duplicator can’t produce such a partial homomorphism than Spoiler wins.

The $k$ sized window is local ‘view’ of the structure.
We can ‘internalize’ $\exists^+ \text{Peb}_k$ game by encoding it as a comonad $\mathbb{P}_k$, for every $k$, over $\mathcal{R}(\sigma)$

Suprisingly: we are also able to define the combinatorial parameter treewidth using coalgebrs of $\mathbb{P}_k$
Given a $\sigma$-structure $A$, we can create $\sigma$-structure on the set of Spoiler moves $\mathbb{P}_k A$ in $\exists^+ \text{Peb}_k(A, \cdot)$, i.e. non-empty sequences of pairs $(p, a)$ where $p \in k = \{1, \ldots, k\}$ and $a \in A$

Let $\varepsilon_A : \mathbb{P}_k A \to A$ be $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto a_n$ and $\pi_A : \mathbb{P}_k A \to k$ be $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto p_n$.

$$R_{\mathbb{P}_k A}(s_1, \ldots, s_r) \iff s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \text{ for } i, j \in r$$

and $\pi_A(s_i)$ does not appear in $\text{suffix}(s_i, s)$

where $s = \max(s_1, \ldots, s_r)$

and $R^A(\varepsilon_A(s_1), \ldots, \varepsilon_A(s_r))$

For $f : \mathbb{P}_k A \to B$ define $f^* : \mathbb{P}_k A \to \mathbb{P}_k B$ recursively:

$$f^*(s[(p, a)]) = f^*(s)[f(s[(p, a)])]$$
Functions $f : \mathbb{P}_k A \to B$ are Duplicator’s strategies in $\exists^+ \text{Peb}(A, B)$

Chose relations so that $\sigma$-morphisms $f : \mathbb{P}_k A \to B$ are Duplicator’s \textbf{winning} strategies.

Coextension $f^* : \mathbb{P}_k A \to \mathbb{P}_k B$ models history preservation of the game

Theorem ([ADW17])

The following are equivalent:

1. \textit{Duplicator has a winning strategy in $\exists^+ \text{Peb}(A, B)$}
2. \textit{There exists a coKleisli morphism $f : \mathbb{P}_k A \to B$}

Can be strengthened to a bijective correspondence using relative comonads and explicit equality in signature
Another characterization of this ‘k-approximate homomorphism relation’

**Proposition**

The following are equivalent:

- \( A \Rightarrow_{\exists^+ \nu^k} B \)
- Duplicator has a winning strategy in \( \exists^+ \text{Peb}_k(A, B) \)
- For all finite \( C \) w/ treewidth < \( k \), \( C \rightarrow A \Rightarrow C \rightarrow B \)
- There exists a Kleisli morphism \( \mathbb{P}_k A \rightarrow B \)
Eilenberg-Moore category of coalgebras

We want to use coalgebras of $\mathbb{P}_k$ to define treewidth.

Coalgebras are morphisms $\alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A}$ satisfying the equations:

\[
\epsilon_\mathcal{A} \circ \alpha = \text{id}_\mathcal{A} \quad \mathbb{C}_k \alpha \circ \alpha = \delta_\mathcal{A} \circ \alpha
\]

with $\delta_\mathcal{A} = \text{id}_{\mathbb{P}_k\mathcal{A}} : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathbb{P}_k\mathcal{A}$.

We can define the Eilenberg-Moore category $\mathcal{EM}(\mathbb{P}_k)$:

- Objects are coalgebras $(\mathcal{A}, \alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A})$.
- Morphisms are commuting squares:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\alpha} & \mathbb{P}_k\mathcal{A} \\
\downarrow f & & \downarrow \mathbb{P}_k f \\
\mathcal{B} & \xrightarrow{\beta} & \mathbb{P}_k\mathcal{B}
\end{array}
\]
For every structure $\mathcal{A}$, define the Gaifman graph $\mathcal{G}(\mathcal{A})$ w/ vertices $\mathcal{A}$ and

$$a \bowtie a' \in \mathcal{G}(\mathcal{A}) \iff a = a' \text{ or } a, a' \text{ appear in some tuple of } R^\mathcal{A}$$

Intuition: Treewidth $\text{tw}(\mathcal{A})$ measures how far $\mathcal{G}(\mathcal{A})$ is from being a tree

Often implicit in dynamic programming algorithms, i.e $k$-consistency algorithms

Formally: Treewidth is the minimum width of a tree-decomposition of $\mathcal{G}(\mathcal{A})$
Definition
A tree decomposition of $\mathcal{A}$ of width $k$ is a triple $(T, \leq_T, \lambda : T \to \mathcal{P}A)$:

- Every $a \in \mathcal{A}$ is in some node of $T$
- All the nodes containing $a \in \mathcal{A}$ form a subtree
- For every $a \sim a' \in \mathcal{G}(\mathcal{A})$, $\{a, a'\} \subseteq \lambda(x)$
- $k = \max\{|\lambda(x)|\}_{x \in T} - 1$
Figure: Tree decomposition of width 3 for $G(A)$
Figure: Tree decomposition of width 3 for $G(A)$
Figure: Tree decomposition of width 3 for $\mathcal{G}(\mathcal{A})$
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Figure: Tree decomposition of width 3 for $\mathcal{G}(A)$
We can define a category of $k$-pebble forest covers $\mathcal{F}(\sigma)^k$, where objects $(\mathcal{A}, \leq, p : \mathcal{A} \rightarrow \mathbb{K})$ satisfying:

- All elements below $a \in \mathcal{A}$ in $\leq$ form a chain
- If $a \sim a' \in \mathcal{G}(\mathcal{A})$, $a \leq a'$ or $a' \leq a$
- If $a \sim a'$ and $a \leq a'$, then for all $b$ with $a < b \leq a'$, $p(a) \neq p(b)$

Morphisms are functions that preserve immediate successors in the order $\leq$ and the pebbling function.
\( \mathbb{P}_k \) arises from the comonadic adjunction \( U^k \dashv F^k \) where 
\( U^k : \mathcal{F}(\sigma)^k \to \mathcal{R}(\sigma), \ F^k A = (\mathbb{P}_k A, \sqsubseteq, \pi_A) \)

**Theorem ([AM20])**

The category of coalgebras \( \mathcal{EM}(\mathbb{P}_k) \) is isomorphic to \( \mathcal{F}(\sigma)^k \)
Theorem ([ADW17, AS18])

The following are equivalent:

1. $A$ has a tree decomposition of width $< k$
2. $A$ has a $k$-pebble forest cover, i.e. coalgebra $A \to \mathbb{P}_k A$

Let $\kappa^C(A)$ be the least $k$ such that there exists coalgebra $A \to \mathcal{C}_k A$

Corollary ([ADW17])

$\kappa^\mathbb{P}(A) = tw(A) + 1$
What about pathwidth?

We say a tree decomposition \((T, \leq, \lambda)\) of \(A\) is a *path decomposition* if \(\leq\) is a linear order.

Pathwidth \(pw(A)\) is the minimum width of a path decomposition of \(A\).

Closely linked to CSPs in \(\text{NLOGSPACE}\) analogous to treewidth’s relationship to \(\text{PTIME}\).

Is there an analogous comonad to \(\mathbb{P}_k\), but for pathwidth?
Given a $\sigma$-structure $A$, we can create $\sigma$-structure $\text{PR}_kA$ on the set of pairs $([(p_1, a_1), \ldots, (p_n, a_n)], i)$ with $i \in n$

$\varepsilon_A : \text{PR}_kA \to A$ be $([(p_1, a_1), \ldots, (p_n, a_n)], i) \mapsto a_i$

$\pi_A : \text{PR}_kA \to k$ be $([(p_1, a_1), \ldots, (p_n, a_n)], i) \mapsto p_i$.

For $i < j$, $s(i, j]$ is the subsequence of $s$ starting at $i + 1$ and ending at $j$ (inclusive)

$$R^{\text{PR}_kA}((s, i_1), \ldots, (s, i_r)) \Leftrightarrow \pi_A(s, i_j) \text{ does not appear in } s(i_j, m]$$

where $m = \max(i_1, \ldots, i_j)$

and $R^A(\varepsilon_A(s, i_1), \ldots, \varepsilon_A(s, i_r))$

Let $s = [(p_1, a_1), \ldots, (p_n, a_n)] \in \text{PR}_kA$ and $f : \text{PR}_kA \to B$

$$f^*(s, i) = [(p_1, f(s, 1)), \ldots, (p_n, f(s, n))], i)$$
Pathwidth and $\mathbb{PR}_k$

We can define a subcategory $\mathcal{LF}(\sigma)^k$ of the $k$-pebble forest covers $\mathcal{F}(\sigma)^k$ where the forests are linear forests.

$\mathbb{PR}_k$ arises from the comonadic adjunction $U^k \dashv L^k$ where $U^k : \mathcal{LF}(\sigma)^k \to \mathcal{R}(\sigma)$, $L^k A = (\mathbb{PR}_k A, \leq^*, \pi_A)$

$$(t, i) \leq^* (t', j) \iff t = t' \text{ and } i \leq j$$

**Theorem ([AM20])**

The category of coalgebras $\mathcal{EM}(\mathbb{PR}_k)$ is isomorphic to $\mathcal{LF}(\sigma)^k$
Theorem

The following are equivalent:

1. $A$ has a path decomposition of width $< k$
2. $A$ has a $k$-pebble linear forest cover, i.e. coalgebra $A \rightarrow \mathbb{PR}_k A$

Corollary

$\kappa^{\mathbb{PR}}(A) = pw(A) + 1$
What about the logic?

Definition ([Dal05])

Restricted conjunction fragment $\exists^+ \mathcal{N}_k \subseteq \exists^+ \mathcal{V}_k$ where conjunctions $\bigwedge \Psi$ have that $\Psi$:

- At most one formula in $\Psi$ containing quantifiers has a free variable.

Theorem ([Dal05])

The following are equivalent:

- $A \Rightarrow^{\exists^+ \mathcal{N}_k} B$
- Duplicator has a winning strategy in a $k$ pebble relation game $\exists^+ \text{PebR}_k(A, B)$
- For all $C$ w/ pathwidth $< k$, $C \rightarrow A \Rightarrow C \rightarrow B$
The $k$ pebble-relation game is cumbersome to state formally

- Spoiler chooses a at most $k$ sized window on the structure $\mathcal{A}$ (as in the $k$-pebble game)
- Duplicator responds with a set of homomorphisms from that window into $\mathcal{B}$ (non-determinism)
- Response set must extend some of the partial homomorphisms of her previous move
- Spoiler wins if Duplicator can only respond with the empty set
An easier game

We can interpret elements of $\mathbb{PR}_k A$ as Spoiler plays, in some new game.

This produces a simpler equivalent game: preannounced or all-in-one $k$-pebble game.
The pre-announced $k$-pebble game $\exists^+ \text{Peb}_k(A, B)$ is played in one round:

- **Spoiler** chooses a list of $k$-pebble placements on $A$:
  \[ s = [(p_1, a_1), \ldots, (p_n, a_n)] \]

- **Duplicator** chooses a compatible list of $k$-pebble placements on $B$:
  \[ t = [(p_1, b_1), \ldots, (p_n, b_n)] \]

Duplicator wins if for every index $i \in \mathbb{n}$, the pairs of pebble placements in $s(0, i]$ and $t(0, i]$ form a partial homomorphism.

Stewart’s all-in-one existential $k$-pebble game [Ste07]
Proposition

The following are equivalent:

1. $A \Rightarrow \exists^+ N^k B$
2. Duplicator has a winning strategy in $\exists^+ \text{PebR}_k(A, B)$
3. For all finite $C$ w/ pathwidth $< k$, $C \rightarrow A \Rightarrow C \rightarrow B$
4. There exists $f : \text{PR}_k A \rightarrow B$
5. Duplicator has a winning strategy in $\exists^+ \text{PPeb}_k(A, B)$
Definition
A structure $\mathcal{B}$ has the $\mathbb{C}_k$-lifting property if for every structure $\mathcal{A}$:

$$\mathbb{C}_k \mathcal{A} \rightarrow \mathcal{B} \Rightarrow \mathcal{A} \rightarrow \mathcal{B}$$

$\mathcal{B}$ has $k$-treewidth duality iff $\mathcal{B}$ has the $\mathbb{P}_k$-lifting property.

$\mathcal{B}$ has $k$-pathwidth duality iff $\mathcal{B}$ has the $\mathbb{PR}_k$-lifting property.

$\mathcal{B}$ has $k$-treewidth duality for some $k \Rightarrow \text{CSP}(\mathcal{B}) \in \text{P}[\text{DKV02}]$
(converse does not hold [Ats08])

$\mathcal{B}$ has $k$-pathwidth duality for some $k \Rightarrow \text{CSP}(\mathcal{B}) \in \text{NL}[\text{Dal05}]$
(converse open, but hard)
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<tr>
<th>$\mathbb{C}_k$</th>
<th>Logic</th>
<th>$\kappa^\mathbb{C}$</th>
<th>$\rightarrow^\mathbb{C}_k$</th>
<th>$\leftrightarrow^\mathbb{C}_k$</th>
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<tr>
<td>$\mathbb{E}_k$ [AS18]</td>
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<td>tree-depth</td>
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Theorem

1. $\mathcal{A} \xrightarrow{\mathbb{C}} \mathcal{B} \iff \mathcal{A} \equiv \exists^+ \mathcal{L}_k \mathcal{B} \iff \text{Duplicator wins } \exists^+ \mathcal{G}_k(\mathcal{A}, \mathcal{B})$

2. $\mathcal{A} \xleftarrow{\mathbb{C}} \mathcal{B} \iff \mathcal{A} \equiv \mathcal{L}_k \mathcal{B} \iff \text{Duplicator wins } \mathcal{G}_k(\mathcal{A}, \mathcal{B})$

3. $\mathcal{A} \equiv \mathbb{C} \mathcal{B} \iff \mathcal{A} \equiv \mathcal{L}_k(\#) \mathcal{B} \iff \text{Duplicator wins } \# \mathcal{G}_k(\mathcal{A}, \mathcal{B})$

The $\xrightarrow{\mathbb{C}}$ and $\equiv_k$ arise from $\mathcal{K}(\mathbb{C}_k)$

The $\xleftarrow{\mathbb{C}}$ arises from a notion of open map bisimulation in the category of coalgebras over $\mathbb{C}_k$
All structures finite

Theorem ([Lov67])
\[ A \cong B \iff \text{Hom}(C, A) \cong \text{Hom}(C, B) \text{ for } C \]

Theorem ([Gro20])
\[ A \equiv^{QR_n(\#)} B \iff \text{Hom}(C, A) \cong \text{Hom}(C, B) \text{ for } C \text{ w/ } \text{td}(C) \leq n \]

Theorem ([Dvo09])
\[ A \equiv^{V^k(\#)} B \iff \text{Hom}(C, A) \cong \text{Hom}(C, B) \text{ for } C \text{ w/ } \text{tw}(C) < k, \]

Theorem ([DJR21])
\[ A \equiv^{L_k(\#)} B \iff \text{Hom}(C, A) \cong \text{Hom}(C, B) \text{ for } \mathbb{C}_k\text{-coalgebras } C \]
Spoiler-Duplicator game comonads unify and generalize the use of resource measures in finite model theory.

These comonads are robustly defined, i.e. via a model-comparison game or a forest cover/decomposition.

$\mathbb{PR}_k$ extends this framework to link pathwidth and a restricted conjunction fragment of $k$-variable logic $\exists^+ \mathcal{N}_k$.

Provides interesting avenues towards applying category theory to complexity theory:

$\mathcal{B}$ has the $\mathbb{PR}_k$-lifting property for some $k \Rightarrow \text{CSP}(\mathcal{B}) \in \text{NL}$


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