Norms on Categories

Daniel Luckhardt joint with Matt Insall

Motivation

- categories with large class of morphisms,
- convenient and systematic metrization for equivalence classes of spaces,
- generalization of Cantor-Schröder-Bernstein theorem

Axioms

A **seminorm** on a category $\underline{C} = (\underline{C}_0, \underline{C}_1, ;)$ is a map $\|-\|\colon \underline{C}_1 \to [0, \infty]$ such that

(N1) $\|\mathrm{id}_X\| = 0$ for all $X \in \underline{C}_0$;

(N2) $\|f \,;\, g\| \leq \|f\| + \|g\|$ (triangle inequality).

X, Y are **norm isomorphic** if $\exists f\colon X \to Y, g\colon Y \to X$ inverse to each other with $\|f\| = \|g\| = 0$

A **norm** is a seminorm such that for all $X, Y \in \underline{C}_0$

(N3) if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ with $\|f\| = \|g\| = 0$, then X, Y are norm isomorphic;

(N4) if for all $\varepsilon > 0$,

$$\exists f \in \underline{C}_1 : X \xrightarrow{f,\, \|f\| \leq \varepsilon} Y,$$

then

$$\exists f \in \underline{C}_1 : X \xrightarrow{f,\, \|f\| = 0} Y.$$

Principle

A seminorm becomes a norm on a full subcategory of "compact" objects.

Examples

Set $\|f\|_{\mathrm{set}} \coloneqq \log \sup_{x \in X} \# f^*(\{f(x)\})$, where $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preimage,

Graph Seminorm as above. Becomes a norm when restricting to finite graphs.

$\mathrm{NVect}^*_{\mathbb{R}}$ The category of normed vector spaces over the reals and linear maps.

$$\|A\|_{\mathrm{op}} \coloneqq \log \sup_{v \in V}{}^{1} \frac{\|v\|_V}{\|Av\|_W}$$

If $\|A\| = 0$, then A is expansive. We obtain a norm by restricting to $\mathrm{Hilb}\,\mathrm{NVect}^*_{\mathbb{R}}$, the Banach spaces with Hilbert space structure.

Top $\|f\|_{\mathrm{top}} \coloneqq \|f\|_{\mathrm{comp}} + \|f\|_{\mathrm{dim}}$ where $\|f\|_{\mathrm{comp}}, \|f\|_{\mathrm{dim}}$ resp., measures the number of components, the dimension resp., of preimages of subsets. Norm on compact metrizable spaces.

Metr Metric spaces and multivalued maps. Seminorm $\|f\|_{\mathrm{dil}} \coloneqq \sup\{\, |x\,y| - |f(x)\,f(y)| \mid x, y \in M \,\} \cup \{0\}$. Norm for compact spaces. Relation to Gromov-Hausdorff dist.:

$$((\mathrm{Met}_{\mathrm{cpt}})_0/\!\sim, \mathrm{d}_{\mathrm{GH}}) \xrightarrow[\text{inv. Cauchy cont.}]{\text{2-Lipschitz}} ((\mathrm{Met}_{\mathrm{cpt}})_0/\!\sim, \mathrm{d}^{+}_{\mathrm{dil}})$$

Outlook

Look at Wasserstein distance and Prokhorov metrics.
Prove Theorems:

- Freudenthal-Hurewicz thm.
- Kantorovich-Rubinstein thm.

Use ind-completion $\underline{\text{ind-}C}$ to treat "non-compact" objects: Fix a directed set $I = (I, \leq)$ and an order preserving function $F\colon I \to [0, 1]$, thought of as the distribution of a probability measure. Define

$$f(i) \coloneqq \inf \left\{ \|g\| \;\middle|\; \begin{array}{l} \iota_{ij}(g) = \mathrm{pr}_i\, f, \\ g \in \underline{C}[X_i, Y_j] \end{array} \right\}$$

for $(X_i)_{i \in I}, (Y_j)_{j \in I} \in (\underline{\text{ind-}C})_0$ and $f \in \text{ind-}C[(X_i)_{i \in I}, (X_j)_{j \in J}] = \lim_{i \in I} \operatorname{colim}_{j \in J} \underline{C}[X_i, Y_j]$.

Finally, define the **Choquet integral**

$$\int f(i)\, \mathrm{d}F \coloneqq \int 1 - F(\sup\{\, i \mid f(i) \leq t \,\})\, \mathrm{d}t.$$

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