A Canonical Algebra of Open Transition Systems

Elena Di Lavore, Alessandro Gianola, Mario Román, Nicoletta Sabadini
and Paweł Sobociński

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Part 1: Spans of Graphs
Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.
Span (Graph), algebra of open transition systems

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How to model this situation?
**Span (Graph)**: 

- Compositional, stateful transition systems.
- Synchronization by composition.
- Transition systems are encoded as graphs.
- Boundaries may be single-vertex graphs, $\text{Span} (\text{Graph})_\ast$. 
**Span(Graph)**, algebra of open transition systems

**Span(Graph):**

- Compositional, stateful transition systems.
- Synchronization by composition.
- Transition systems are encoded as graphs.
- Boundaries may be single-vertex graphs, $\text{Span(Graph)}_*$. 

Ad hoc?
Part 2: Stateful Morphisms
The \( \text{St}(\cdot) \) construction

**Definition.** For \((C, \otimes, I)\) symmetric monoidal,

\[
\text{St}(C)(A,B) := \{(S, \psi) \mid S \in \text{ob}C, \quad \psi: S \otimes A \to S \otimes B \}/\sim,
\]

quotiented by the equivalence relation

\[
\left( \begin{array}{c}
\begin{array}{c}
S \\
\downarrow \\
\psi \\
\downarrow \\
S
\end{array}
\end{array} \right) \sim
\left( \begin{array}{c}
\begin{array}{c}
T \\
\downarrow \\
\psi \\
\downarrow \\
T
\end{array}
\end{array} \right)
\]

where \( \psi: S \cong T \) is any isomorphism.

---

Diagrammatic algebra: from linear to concurrent systems. Bonchi, Holland, et al.
Memoryful geometry of interaction. Hoshino, Muroya, Hasuo.
The \textit{St}() construction

Composition is given by:

\[
\left( \begin{array}{c}
S \\
A \\
B
\end{array} \right) \psi
\left( \begin{array}{c}
S \\
T \\
B \\
C
\end{array} \right) = \left( \begin{array}{c}
S \otimes T \\
A \\
B \\
C
\end{array} \right).
\]

Tensoring is given by:

\[
\left( \begin{array}{c}
S \\
A \\
B
\end{array} \right) \otimes \left( \begin{array}{c}
S' \\
A' \\
B'
\end{array} \right) = \left( \begin{array}{c}
S \otimes S' \\
A \\
B \\
A' \\
B'
\end{array} \right).
\]

Universal property?

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
The \( \text{St}(\cdot) \) construction

\[
\text{St}(\text{Set}_x) : \quad S \times A \rightarrow S \times B \quad \text{Mealy transition system}
\]

\[
\text{St}(\text{Set}_+) : \quad S + A \rightarrow S + B \quad \text{Elgot transition system}
\]

\[
\text{St}(\text{Rel}_x) : \quad S \times A \rightarrow P(S \times B) \quad \text{Non-deterministic transition system}
\]
The $\text{St}(\cdot)$ construction

$\text{St}(\text{Set}_x)$: $S \times A \rightarrow S \times B$  Mealy transition system

$\text{St}(\text{Set}_+)$: $S + A \rightarrow S + B$  Elgot transition system

$\text{St}(\text{Rel}_x)$: $S \times A \rightarrow P(S \times B)$  Non-deterministic transition system

$\text{St}(\text{Span}((\text{Set})))$: $S \times A \leftrightarrow E \rightarrow S \times B$
The St(•) construction

\[ \text{St}(\text{Set}_x) : S \times A \rightarrow S \times B \]  Mealy transition system

\[ \text{St}(\text{Set}_+) : S + A \rightarrow S + B \]  Elgot transition system

\[ \text{St}(\text{Rel}_x) : S \times A \rightarrow \mathcal{P}(S \times B) \]  Non-deterministic transition system

\[ \text{St}(\text{Span}(\text{Set})) : A \leftarrow E \rightarrow B \]
The $\text{St}(\cdot)$ construction

$\text{St}(\text{Set}_x) : S \times A \rightarrow S \times B$  
$\text{St}(\text{Set}_+) : S + A \rightarrow S + B$  
$\text{St}(\text{Rel}_x) : S \times A \rightarrow \mathcal{P}(S \times B)$  
$\text{St} (\text{Span} (\text{Set}))$:

Non-deterministic transition system

Mealy transition system

Elgot transition system
The $\text{St}()$ construction

$\text{St}(\text{Set}_x) : \quad S \times A \to S \times B$  \quad \text{Mealy transition system}

$\text{St}(\text{Set}_+) : \quad S + A \to S + B$  \quad \text{Elgot transition system}

$\text{St}(\text{Rel}_x) : \quad S \times A \to P(S \times B)$  \quad \text{Non-deterministic transition system}

$\text{St}(\text{Span}(\text{Set})) : \quad \begin{array}{ccc}
A & \leftarrow & E \rightarrow & B \\
\downarrow & & \downarrow & \downarrow \\
1 & \leftarrow & S \rightarrow & 1
\end{array}$  \quad \text{Span(Graph)}_x
The $\text{St}(\cdot)$ construction

$\text{St} (\text{Set}_x) : S \times A \rightarrow S \times B$ \hspace{1cm} Mealy transition system

$\text{St} (\text{Set}_+ ) : S + A \rightarrow S + B$ \hspace{1cm} Elgot transition system

$\text{St} (\text{Rel}_x) : S \times A \rightarrow P(S \times B)$ \hspace{1cm} Non-deterministic transition system

$\text{St} (\text{Span} (\text{Set})) : A \leftarrow E \rightarrow B$

$\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow$

$1 \leftarrow S \rightarrow 1$

$\text{Span} (\text{Graph})_*$

**Theorem.** There is a monoidal isomorphism:

$\text{St} (\text{Span} (\text{Set})) \cong \text{Span} (\text{Graph})_*$

\text{stateful synchronization: spans of graphs}
Part 3: Feedback
Categories with feedback

Symmetric monoidal category with an operator
\[ fbk_s : \text{hom}\,(S \otimes A, S \otimes B) \to \text{hom}\,(A,B), \]
such that:

1. \[ u; fbk_s(f); v = fbk_s((u \otimes \text{id}); f; (v \otimes \text{id})) \]
2. \[ fbk_s(1) = f \]
3. \[ fbk_s(fb_k(I)(f)) = fbk_{S \otimes 1}(f) \]
4. \[ fbk_s(f) \circ g = fbk_s(f \circ g) \]
5. \[ fbk_k((f; (h \otimes 1)); f) = fbk_k((h \otimes 1); f) \]

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
Categories with feedback

Symmetric monoidal category with an operator \( \text{fbks} : \text{hom}(S \otimes A, S \otimes B) \to \text{hom}(A, B) \), such that:

1. \( u \circ f \circ v = u \circ f \circ v \)
2. \( f \circ f = f \)
3. \( f \circ f = f \)
4. \( f = g \)
5. \( f \circ \psi = \psi \circ f \) (\( \psi \text{ iso} \))

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
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1. \( u \circ f \circ v = u \circ f \circ v \)
2. \( f = f \)
3. \( f = f \)
4. \( f \circ g = f \circ g \)
5. \( f \circ \varepsilon = \varepsilon \circ f \)

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Symmetric monoidal category with an operator \( \text{fbks} : \text{hom}(S \otimes A, S \otimes B) \to \text{hom}(A, B) \), such that:

1. \( u \circ f \circ v = u \circ f \circ v \)
2. \( f \circ f = f \)
3. \( f \circ f = f \)
4. \( f \circ g = g \circ f \)
5. \( \emptyset \circ f = f \circ \emptyset \) (\( \emptyset \) iso)

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
Categories with feedback

Symmetric monoidal category with an operator $\text{fbks}: \text{hom}(S \otimes A, S \otimes B) \to \text{hom}(A, B)$, such that:

1. $u \circ f \circ v = u \circ f \circ v$
2. $f \circ f = f$
3. $f \circ f = f$
4. $f \circ g = g$
5. $f \circ \vartheta \circ f = \vartheta$ (\$iso\$)

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
Categories with feedback

Symmetric monoidal category with an operator

\[ \text{fbks} : \text{hom}(S \otimes A, S \otimes B) \rightarrow \text{hom}(A, B), \]

such that:

1. \[ u \circ f \circ v = u \circ f \circ v \]

2. \[ f = f \]

3. \[ f = f \]

4. \[ f \circ g = f \circ g \]

5. \[ f \circ \eta = \eta \circ f \text{\ (\& iso)} \]

Feedback, trace, and fixed-point semantics.  Katis, Sabadini, Walters.
Categories with feedback

Differences with traced monoidal categories?

i. $\neq$

(Weak Sliding)

ii. $\neq$

(Yanking)
Categories with feedback

Differences with traced monoidal categories?

i. \( f \circ \chi \circ \chi \) equals \( \chi \circ f \) (\( \chi \text{ iso} \))

ii. etc. (\( \text{WEAKSLIDING} \))

etc. (\( \text{YANKING} \))
Categories with feedback

Differences with traced monoidal categories?

i. \[ f \circ \alpha \circ (\alpha \text{ iso}) = \alpha \circ f \] (Weak Sliding)

ii. \[ \exists \] (Yanking)
Categories with feedback

Differences with traced monoidal categories?

i. \[(\gamma \text{ iso}) \quad\]

ii. \[\Box \quad\]

(Weak Sliding)

(Yanking)

- Feedback is weaker than trace (and balanced trace).
- Feedback and guarded trace coincide in compact closed categories.
- Feedback has a different type than delayed trace.
Categories with feedback

Multiple applications of feedback can be reduced into a single one. All of the axioms are needed for this result.
Categories with feedback

Multiple applications of feedback can be reduced into a single one. All of the axioms are needed for this result.

We can use this to show that $\text{St}(C)$ is the free category with feedback.
Categories with feedback

**Proposition.** Let \( \mathcal{C} \) be a symmetric monoidal category. \( S^\mathcal{T}(\mathcal{C}) \) has a feedback structure given by

\[
\text{fbk}_T \left( S \begin{array}{c} S \\ S \end{array} = \begin{array}{c} \text{f} \\ \text{f} \end{array} \begin{array}{c} S \\ S \end{array} \right) = \left( S \otimes T \begin{array}{c} S \\ S \end{array} = \begin{array}{c} \text{f} \\ \text{f} \end{array} \begin{array}{c} S \\ S \end{array} \right).
\]

**Theorem.** Let \( \mathcal{C} \) be a symmetric monoidal category. The symmetric monoidal category \( S^\mathcal{T}(\mathcal{C}) \) is the free category with feedback over \( \mathcal{C} \), meaning that

\[
S^\mathcal{T}(\mathcal{C}) \xrightarrow{\exists! \tilde{F}} \text{D}
\]

Feedback-preserving:

\[
\tilde{F}(\text{fbk}_S(f)) = \text{fbk}_{\mathcal{T}S}(\tilde{F}(f)).
\]

Category with feedback.

Symmetric monoidal functor.

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Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.
Categories with feedback

THEOREM. The following is an isomorphism of categories.

\[ \text{SPAN}(\text{GRAPH})_* \cong \text{ST}(\text{SPAN}(\text{SET})) \]

\( \text{SPAN}(\text{GRAPH})_* \) is the free category with feedback over \( \text{SPAN}(\text{SET}) \).
Categories with feedback

**THEOREM.** The following is an isomorphism of categories.

\[
\text{SPAN}(\text{GRAPH})_* \cong \text{ST}(\text{SPAN} (\text{SET}))
\]

\text{SPAN}(\text{GRAPH})_* \text{ is the free category with feedback over } \text{SPAN} (\text{SET}).

**Example:**

\[
\begin{array}{c}
\text{Set} \longrightarrow \text{Span} \quad \text{lifts to} \quad \text{MealyTS} \longrightarrow \text{SPAN} (\text{GRAPH})_*.
\end{array}
\]

\[
f : A \rightarrow B
\]

\[
\begin{array}{l}
\text{id} \quad A \quad f
\end{array}
\]

\[
\begin{array}{l}
f(a,0) = a \\
f(a,1) = b \\
f(b,0) = b \\
f(b,1) = a
\end{array}
\]

\[
\begin{array}{c}
A \quad \Rightarrow \quad B
\end{array}
\]

\[
\begin{array}{c}
\text{MealyTS}
\end{array}
\]

\[
\begin{array}{c}
A \quad \Rightarrow \quad B
\end{array}
\]

\[
\begin{array}{c}
\text{SPAN} (\text{GRAPH})_*
\end{array}
\]
Categories with feedback

THEOREM. The following is an isomorphism of categories.

$$\text{SPAN} (\text{GRAPH})_* \cong \text{ST} (\text{SPAN}(\text{SET}))$$

$\text{SPAN} (\text{GRAPH})_*$ is the free category with feedback over $\text{SPAN} (\text{SET})$.

Example:

$$\text{Set} \rightarrow \text{Span} \text{ lifts to } \text{MealyTS} \rightarrow \text{Span} (\text{Graph})_*.$$
Part 4:
Generalizing $St(\bullet)$
Generalizing Feedback

Feedback describes a particular flow of information.

\[ \text{input} \rightarrow \text{output} \quad \text{normal flow} \]

\[ \text{input} \rightarrow \text{output} \quad \text{flow with feedback} \]
Generalizing Feedback

Feedback describes a particular flow of information.

\[
\begin{align*}
\text{input} & \quad \rightarrow \quad \text{output} \\
\text{normal flow} & \quad \downarrow \\
\text{input} & \quad \rightarrow \quad \text{output} \\
\text{flow with feedback}
\end{align*}
\]

These are monads in the bicategory $\text{PROF}$ of profunctors:

\[
\begin{align*}
\text{hom}(I, O) & \quad \int_{\text{sec}} \quad \text{hom}(S \otimes I, S \otimes O) \\
\text{monads correspond to a new assignment of morphisms to a category.}
\end{align*}
\]

Open Diagrams via Coend Calculus, Mario Román, ACT’20
Generalizing Feedback

What is the most general form of feedback?

\[ \text{St}_D(A, B) := \int_{D \in D} \text{hom}(D \triangleright A, D \triangleright B) \]

The normal form theorem holds for any pair of monoidal actions \((0, 0)\). These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.
What is the most general form of feedback?

\[ S_{c}(A,B) = \int_{\text{Sec}} \hom(S \otimes A, S \otimes B) \]

The normal form theorem holds for any pair of monoidal actions \((e, e)\). These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.
Generalizing Feedback

What is the most general form of feedback?

\[
\text{St}_{\text{Core } C} (A, B) := \int_{S \in \text{Core } C} \text{hom}(S \otimes A, S \otimes B)
\]

The normal form theorem holds for any pair of monoidal actions \( \ast, \circ \). These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.
What is the most general form of feedback?

\[ \text{St}_{\mathcal{C}/\mathcal{I}}(A, B) := \int_{S, s \in \mathcal{C}/\mathcal{I}} \hom(S \otimes A, S \otimes B) \]

The normal form theorem holds for any pair of monoidal actions \((\circ, \circ)\). These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.
Generalizing Feedback

What is the most general form of feedback?

\[ \text{St}_{c \diamond} (A, B) := \int \text{Sec} \hom (c \diamond A, c \diamond B) \]

The normal form theorem holds for any pair of monoidal actions \((\odot, \odot)\). These generalize:

- Right/left traced categories.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.
Conclusion

- \( \text{Span} (\text{Graph})_* \cong \text{St} (\text{Span} (\text{Set})) \).
- \( \text{St}(\cdot) \) commonly appears across the literature.
- \( \text{St}(\cdot) \) is the free category with feedback.
- Categories with feedback are a weakening of traces.
- Categories with feedback have a normal form theorem.
- \( \text{St}(\cdot) \) can be generalized to variants of feedback.
- Relate to the coalgebraic approach.
References.


Romain. Open diagrams via Coend Calculus. (ACT’20).