Constructing Initial Algebras Using Inflationary Iteration

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Accompanying paper: arXiv:2105.03252
Agda formalization: www.cl.cam.ac.uk/users/amp12/agda/coniau

Initial algebras

Given: category $\mathbf{C}$ + endofunctor $F : \mathbf{C} \to \mathbf{C}$,

recall the notion of F-algebra: $F(A) \xrightarrow{\alpha} A$

They are the objects of a category
(with the obvious notion of morphism).

If that category has an initial object, we denote it

$$\boxed{F(\mu F) \xrightarrow{\iota_F} \mu F}$$

Computer Science applications of initial algebras of endofunctors

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The relevant toposes do not satisfy logical principles (LEM,AC) needed for classical constructions of initial algebras, but they do satisfy the Weakly Initial Sets of Covers (WISC) axiom due to Streicher (type theory), Moerdijk, Palmgren & van den Berg (set theory).

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Main result: constructive version of Adámek's classical theorem which is useful for toposes satisfying WISC.

Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α, and hence in particular an initial object 0.

Iterate $F : \mathbf{C} \to \mathbf{C}$ transfinitely, starting at 0

$$0 \xrightarrow{\iota_0} F0 \xrightarrow{\iota_1} F^2 0 \xrightarrow{\iota_2} \cdots \to F^\alpha 0 \xrightarrow{\iota_\alpha} F^{\alpha^+} 0 \to \cdots$$

$$F^\alpha 0 = \begin{cases} 0 & \text{if } \alpha = 0 \\ F(F^\beta 0) & \text{if } \alpha = \beta^+ \text{ is a successor ordinal} \\ \mathrm{colim}_{\beta<\lambda} F^\beta 0 & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases}$$

$$\iota_\alpha = \begin{cases} \text{unique, by initiality of } 0 & \text{if } \alpha = 0 \\ F(\iota_\beta) & \text{if } \alpha = \beta^+ \\ \text{use univ. prop. of } \mathrm{colim}_{\beta<\lambda} & \text{if } \alpha = \lambda \end{cases}$$

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Theorem [Adamek, 1974] If F preserves colimits of shape $(\kappa, <)$ for some limit ordinal κ (so that ι_κ is an isomorphism), then it has initial algebra

$$\mu F = F^\kappa 0 = \mathrm{colim}_{\alpha < \kappa} F^\alpha 0$$

(with algebra structure given by $F(F^\kappa 0) = F^{\kappa^+} 0 \xrightarrow[\cong]{(\iota_\kappa)^{-1}} F^\kappa 0$)

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Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α, and hence in particular an initial object 0.

Law of Excluded Middle (LEM)

$\forall p.\ p \vee \neg p$

is needed for the usual theory of ordinal numbers

…ting at 0

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Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α,

Without some form of choice principle
there won't be many such F

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For the rest of the talk we work (informally) in the internal language of a topos with NNO and universes:

Martin-Löf's type theory + extensional equality

impredicative universe of propositions Ω

universes $\mathbf{Set} = \mathbf{Set}_0 : \mathbf{Set}_1 : \mathbf{Set}_2 : \cdots$
containing Ω & $\mathbb{N}$, closed under Σ, Π-types

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Agda formalization:
www.cl.cam.ac.uk/users/amp12/agda/coniau

is predicative & uses intensional equality (satisfying UIP)

Constructive Adamek - step 1

Avoid zero/successor/limit case distinction in

$$F^0 0 = 0$$
$$F^{\alpha^+} 0 = F(F^\alpha 0)$$
$$F^\lambda 0 = \mathrm{colim}_{\alpha<\lambda} F^\alpha 0$$

by using instead an "inflationary" iteration
(Abel-Pientka, after Sprenger-Dam)

$$\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$$

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$$\mu_i F = \mathrm{colim}_{j<i}\, F(\mu_j F)$$

and replace use of ordinals α by the elements i of any size

Definition. A size is a set κ equipped with a binary relation $<$ which is transitive, directed and *well-founded*

$$\forall S \subseteq \kappa.\ (\forall i.(\forall j < i.\ j \in S) \Rightarrow i \in S) \Rightarrow S = \kappa$$

(sizes play the role of limit ordinals in the constructive theory)

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Lemma. Constructively, assuming $\mathbf{C}$ has small colimits, given any endofunctor $F : \mathbf{C} \to \mathbf{C}$ and size $(\kappa, <)$, there are objects $\mu_i F \in \mathbf{C}$ for each $i \in \kappa$ satisfying

$$\mu_i F = \mathrm{colim}_{j<i} F(\mu_j F)$$

Proof. Just need transitivity and well-foundedness of $<$, but not directedness, to construct $(\mu_i F \mid i \in \kappa)$ by well-founded recursion.

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Lemma. Constructively, assuming $\mathbf{C}$ has small colimits, given any endofunctor $F : \mathbf{C} \to \mathbf{C}$ and size $(\kappa, <)$, there are objects $\mu_i F \in \mathbf{C}$ for each $i \in \kappa$ satisfying

$$\mu_i F = \mathrm{colim}_{j<i} F(\mu_j F)$$

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of size $(\kappa, <)$, then it has initial algebra $\mu F = \mathrm{colim}_{i \in \kappa} \mu_i F$.

Proof uses directedness of $<$.

Constructive Adamek - step 2

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of some size κ, then it has initial algebra given by taking the colimit of the κ-indexed inflationary iteration of F.

are there (m)any F for which there is such a κ?

Classically, given F one tries to find a "big enough" κ and then prove cocontinuity using AC.

"big enough" = has upper bounds for a given infinite set

Recall that a size is a set κ with a transitive, directed and well-founded binary relation $<$

Given $\Sigma = (A : \mathbf{Set}, B : \mathbf{Set}^A)$ say that a size $(\kappa, <)$ is Σ-filtered if

for all $a \in A$, every $B\,a$-indexed family $(f\,b \in \kappa \mid b \in B\,a)$ has a $<$-upper bound in κ.

Theorem. There is a function assigning a Σ-filtered size $(\kappa_\Sigma, <)$ to each Σ.

Proof κ_Σ is a suitable type of well-founded trees and $<$ is Paul Taylor's "plump" order for such trees.

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Definition. A functor $F : \mathbf{C} \to \mathbf{D}$ between cocomplete categories is sized if it preserves colimits of Σ-filtered sizes, for some Σ.

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- identity, composition, constant functors

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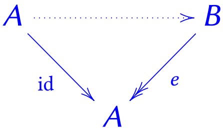
- identity, composition, constant functors
- assuming [WISC]: small colimits, limits, parameterised initial algebras

WISC

Assuming AC, for all $A \in \mathbf{Set}$



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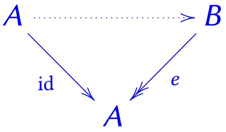

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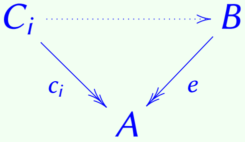
WISC axiom [Streicher; van den Berg, Moerdijk, Palmgren]
weakens AC to merely assume that for each $A \in \mathbf{Set}_n$ there is a Set of surjections ("Covers")
$\left\{ C_i \xrightarrow[c_i]{} A \mid i \in I \right\}$ in $\mathbf{Set}_n$ which is Weakly Initial for covers in $\mathbf{Set}_{n+1}$:



WISC

ZFC **Set** satisfies WISC

If any elementary topos $\mathcal{E}$ satisfies WISC, so do toposes of (pre)sheaves and realizability toposes built from $\mathcal{E}$
[B. van den Berg & I. Moerdijk, J. Math. Logic, 2014]

But there are toposes not satisfying WISC
[D.M. Roberts, Studia Logica, 2015]

Theorem. In any elementary topos $\mathcal{E}$ with NNO and universes satisfying WISC, if $(F_d : \mathbf{Set}_n \to \mathbf{Set}_n \mid d \in \mathbf{D})$ is a diagram of sized functors for some $\mathbf{D}$ in $\mathbf{Set}_n$, then its limit and colimit $\lim_d F_d, \operatorname{colim}_d F_d : \mathbf{Set}_n \to \mathbf{Set}_n$ are also sized.

For proof see accompanying paper: arXiv:2105.03252

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E.g. as a corollary we get that in any topos with NNO and universes satisfying WISC, we can construct initial algebras for Gylterud's symmetric containers

$$F_{\mathbf{G},B}(X) \triangleq \operatorname{colim}_{g \in \mathbf{G}} X^{B\,g}$$

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For further applications of the method (if not the theorems) see M.P. Fiore, AMP & S.C. Steenkamp, *Quotients, Inductive Types and Quotient Inductive Types*, arXiv:2101.02994

Conclusions/Questions

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  An alternative constructive approach to initial algebras: Adámek, Milius & Moss, *An Initial Algebra Theorem without Iteration*, arXiv:2104.09837

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Thank you for your attention!