Frobenius-Eilenberg-Moore objects in dagger 2-categories

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A note on Frobenius-Eilenberg-Moore objects in dagger 2-categories
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The formal definition of monads

The *formal definition of monads* due to Benábou (1967).

A *monad* in a 2-category $\mathcal{K}$ is a monoid object $(A, s, \mu, \eta) = (A, s)$ in the category $\mathcal{K}(A, A)$, for some $A \in \mathcal{K}$.

Equivalently: A monad in a 2-category $\mathcal{K}$ is a lax functor $\mathbf{1} \longrightarrow \mathcal{K}$ from the terminal 2-category $\mathbf{1}$ to $\mathcal{K}$.

For each 2-category $\mathcal{K}$, this defines a 2-category

$$\mathrm{Mnd}(\mathcal{K}) = \mathrm{LaxFun}(\mathbf{1}, \mathcal{K})$$

EM-objects

Eilenberg-Moore objects (Street, 1972)

For each monad (A, s) in a 2-category $\mathcal{K}$, there is a 2-functor

$$\mathcal{K}^{\mathrm{op}} \longrightarrow \mathrm{Cat} : X \longmapsto \mathcal{K}(X, A)^{\mathcal{K}(X,s)}$$

If this 2-functor is representable, A^s is the representing object, and is called the *Eilenberg–Moore (EM) object* of the monad (A, s).

That is,

$$\mathcal{K}(X, A^s) \cong \mathcal{K}(X, A)^{\mathcal{K}(X,s)}$$

2-naturally in the arguments.

Example: in 2-category Cat, EM-objects are usual Eilenberg-Moore categories for the monad.

The construction of EM-algebras

A 2-category $\mathcal{K}$ admits the *construction of EM-algebras* when the obvious inclusion 2-functor

$$\mathcal{K} \longrightarrow \mathrm{Mnd}(\mathcal{K}) : X \longmapsto (X, 1)$$

has a right adjoint $\mathrm{EM} : \mathrm{Mnd}(\mathcal{K}) \longrightarrow \mathcal{K}$.

Fact: For a monad (A, s) in $\mathcal{K}$

$$\mathrm{Mnd}(\mathcal{K})((X,1),(A,s)) \cong \mathcal{K}(X,A)^{\mathcal{K}(X,s)}$$

Therefore,

Theorem

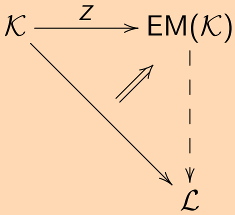
$\mathcal{K}$ admits the construction of EM-algebras if and only if $\mathcal{K}$ has all EM-objects.

Free completion under EM-objects

EM objects are weighted limits (Street, 1976) $\Longrightarrow$ free completion under EM objects.

Theorem (Lack & Street, 2002)

For a 2-category $\mathcal{K}$, there is a 2-category $\mathrm{EM}(\mathcal{K})$ *having EM-objects and fully faithful* $Z : \mathcal{K} \longrightarrow \mathrm{EM}(\mathcal{K})$ *with*



Free completion, cont.

The Eilenberg-Moore completion can also be given an explicit description (Lack & Street, 2002). $\mathsf{EM}(\mathcal{K})$ has:

- objects as monads (A, s) of $\mathcal{K}$
- 1-cells as morphisms of monads $(u, \phi) : (A, s) \longrightarrow (B, t)$
- 2-cells $\rho : (u, \phi) \longrightarrow (v, \psi)$ as 2-cells ρ in $\mathcal{K}$ suitably commuting with a "Kleisli composition".

In general, $\mathsf{EM}(\mathcal{K}) \not\approx \mathsf{Mnd}(\mathcal{K})$

But: $E : \mathsf{Mnd}(\mathcal{K}) \longrightarrow \mathsf{EM}(\mathcal{K})$, which is identity on 0- and 1-cells

Frobenius monads

A monad (X, t, μ, η) in a 2-category $\mathcal{K}$ is a *Frobenius monad* if there is a comonad (X, t, δ, ϵ) such that the *Frobenius law* is satisfied:

$$t\mu \cdot \delta t = \delta \cdot \mu = \mu t \cdot t\delta$$

Example: One-object 2-category $\Sigma(\mathbf{Vect}_k)$ = the suspension and strictification of $\mathbf{Vect}_k$. A Frobenius monad in $\Sigma(\mathbf{Vect}_k)$ is just usual notion of a Frobenius algebra; that is, a k-algebra A with a nondegenerate bilinear form $\sigma : A \times A \longrightarrow k$ that satisfies:

$$\sigma(ab, c) = \sigma(a, bc)$$

Frobenius monads, cont.

Theorem (Lauda, 2006)

For 1-cells $f : A \longrightarrow B$ and $u : B \longrightarrow A$ in a 2-category $\mathcal{K}$, if $f \dashv u \dashv f$ is an ambidextrous adjunction, then the monad uf generated by the adjunction is a Frobenius monad.

Corollary (Lauda, 2006)

Given a Frobenius monad (X, t) in a 2-category $\mathcal{K}$, in $\mathrm{EM}(\mathcal{K})$ the left adjoint $f^t : X \longrightarrow X^t$ to the forgetful 1-cell $u^t : X^t \longrightarrow X$ is also right adjoint to u^t. Hence, the Frobenius monad (X, t) is generated by an ambidextrous adjunction in $\mathrm{EM}(\mathcal{K})$.

In particular, every Frobenius algebra (and hence every 2D TQFT) is generated by an ambidextrous adjunction in $\mathrm{EM}(\Sigma(\boldsymbol{Vect}_k))$.

Characterising Frobenius algebras

Question: Under appropriate conditions, can we more directly characterize Frobenius objects in a monoidal category? That is, via construction?

- Given a Frobenius monad, can we define an appropriate notion of a "Frobenius-Eilenberg-Moore object"?
- Can we describe FEM-objects as some kind of limit as well as the completion of a 2-category under such FEM-objects like is done for the EM construction?
- Is there an explicit description of this FEM-completion similar to the EM-completion?

Frobenius categories

Theory of accessible categories: A category $\boldsymbol{C}$ is *accessible* if it is equivalent to $\mathrm{Ind}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Theory of locally connected categories: A category $\boldsymbol{C}$ is *locally connected* if it is equivalent to $\mathrm{Fam}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Question: Can we develop the theory of *Frobenius categories*, i.e. A category $\boldsymbol{C}$ is *Frobenius* if it is equivalent to $\mathrm{FEM}(\boldsymbol{S})$ for some category $\boldsymbol{S}$.

Wreaths

A *wreath* $((A, t), (s, \lambda), \sigma, \nu)$ is an object of $\mathsf{EM}(\mathsf{EM}(\mathcal{K}))$.

Examples: The crossed product of Hopf algebras, factorization systems on categories.

EM is an endo-2-functor 2-Cat $\longrightarrow$ 2-Cat, the universal property of the EM construction determines a 2-functor

$$\mathsf{wr}_{\mathcal{K}} : \mathsf{EM}(\mathsf{EM}(\mathcal{K})) \longrightarrow \mathsf{EM}(\mathcal{K})$$

called the *wreath product*, and there is the embedding 2-functor

$$\mathsf{id}_{\mathcal{K}} : \mathcal{K} \longrightarrow \mathsf{EM}(\mathcal{K})$$

sending objects in $\mathcal{K}$ to the identity monad on them. In total $(\mathsf{EM}, \mathsf{wr}, \mathsf{id})$ is a 2-monad.

Frobenius wreaths

A wreath $((A,t),(s,\lambda),\sigma,\nu)$ in a 2-category $\mathcal{K}$ is called *Frobenius* when, considered as a monad in $\mathrm{EM}(\mathcal{K})$, it is a Frobenius monad.

Theorem (Street, 2004)

The wreath product of a Frobenius wreath on a Frobenius monad is Frobenius.

For our proposed FEM construction and its universal property, this result is immediate since:

$$\mathrm{wr}_{\mathcal{D}} : \mathrm{FEM}(\mathrm{FEM}(\mathcal{D})) \longrightarrow \mathrm{FEM}(\mathcal{D})$$

Dagger categories

A *dagger category* $\boldsymbol{D}$ is a category with an involutive functor $\dagger : \boldsymbol{D}^{\mathrm{op}} \longrightarrow \boldsymbol{D}$ which is the identity on objects.

A *dagger functor* between dagger categories is a functor which preserves daggers.

A *monoidal dagger category* is a dagger category that is also a monoidal category, satisfying $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and, whose coherence morphisms are *unitary*.

Examples:

- Any groupoid, with $f^\dagger = f^{-1}$.
- The category **Hilb** of complex Hilbert spaces and bounded linear maps, taking the dagger of $f : A \longrightarrow B$ to be its adjoint, i.e. the unique linear map $f^\dagger : B \longrightarrow A$ satisfying $\langle f(a), b \rangle = \langle a, f^\dagger(b) \rangle$ for all $a \in A$ and $b \in B$.

Dagger 2-categories

A 2-category $\mathcal{D}$ is a *dagger 2-category* when the hom-categories $\mathcal{D}(A, B)$ are dagger categories, and horizontal and vertical composition commute with daggers.

Example: The dagger 2-category DagCat of dagger categories, dagger functors and natural transformations.

A 2-functor is a *dagger 2-functor* when each of its component functors are dagger functors.

Dagger Frobenius monads

A monad (D, t, μ, η) in a dagger 2-category $\mathcal{D}$ is a *dagger Frobenius monad* (Heunen and Karvonen, 2016) if the Frobenius law is satisfied:

$$t\mu \cdot \mu^\dagger t = \mu^\dagger \cdot \mu = \mu t \cdot t\mu^\dagger$$

Example: A *dagger Frobenius monoid* in a monoidal dagger category **D** is a monoid which satisfies the Frobenius law. In fact:

B dagger Frobenius monoid $\iff - \otimes B$ dagger Frobenius monad

FEM-algebras

Frobenius-Eilenberg-Moore algebras (Heunen & Karvonen, 2016)

A *Frobenius-Eilenberg-Moore algebra* for a dagger Frobenius monad (T, μ, η) is an Eilenberg-Moore algebra (D, δ) for T, such that:

$$\mu_D \cdot T(\delta)^\dagger = T(\delta) \cdot \mu_D^\dagger$$

Example: Free algebras for a dagger Frobenius monad are FEM-algebras.

$\text{FEM}(\mathbf{D}, T) \subseteq \mathbf{D}^T$ is the "largest" subcategory of $\mathbf{D}^T$ having a dagger.

Quantum measurements

Example (Heunen & Karvonen, 2016): If B is a dagger Frobenius monoid in **FHilb**, a FEM-algebra (D, δ) for the dagger Frobenius monad

$$T = - \otimes B : \mathbf{FHilb} \longrightarrow \mathbf{FHilb}$$

corresponds precisely to *quantum measurements* on D: orthogonal projections on D that sum to the identity.

FEM-algebras

Lemma

Let T be a dagger Frobenius monad. An EM-algebra (D, δ) is a FEM-algebra if and only if

$$\delta^\dagger : D \longrightarrow T(D)$$

is a homomorphism of EM-algebras $(D, \delta) \longrightarrow (T(D), \mu_D)$.

Proof (one direction): A morphism f is *self-adjoint* if $f^\dagger = f$.

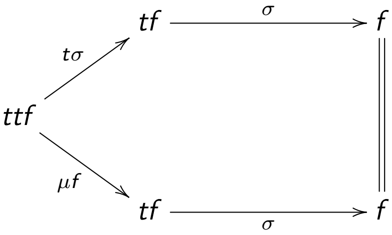
$$\begin{array}{ccc} T(D) & \xrightarrow{T(\delta^\dagger)} & T^2(D) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \mu_D} \\ D & \xrightarrow[\delta^\dagger]{} & T(D) \end{array} \qquad \Longrightarrow \qquad \mu_D \cdot T(\delta^\dagger) = \delta^\dagger \cdot \delta = T(\delta) \cdot \mu_D^\dagger$$

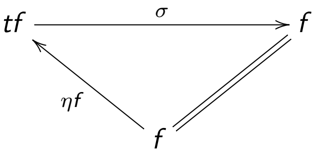
Dagger Frobenius monads

The dagger 2-category $\mathrm{DFMnd}(\mathcal{D})$ should obey a "daggerfied" universal property: for a dagger Frobenius monad (D, t, μ, η) in $\mathcal{D}$

$$\mathrm{DFMnd}(\mathcal{D})((X,1),(D,t)) \cong \mathrm{FEM}(\mathcal{D}(X,D),\mathcal{D}(X,t))$$

That is, $(f : X \longrightarrow D, \sigma : tf \longrightarrow f)$ is a FEM-algebra for $\mathcal{D}(X,t)$ iff:





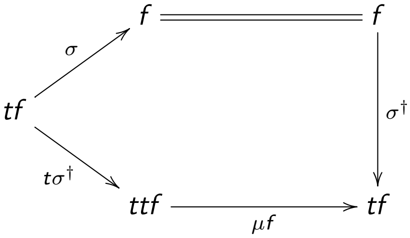
Dagger Frobenius monads, cont.

But also by previous lemma

$$\sigma^\dagger : (f, \sigma) \longrightarrow (\mathcal{D}(X,t)(f), \mathcal{D}(X,\mu)(f)) = (tf, \mu f)$$

is a homomorphism of Eilenberg-Moore algebras for the monad $\mathcal{D}(X,t)$.

$$\sigma^\dagger \cdot \sigma = \mu f \cdot t\sigma^\dagger \qquad \Longleftrightarrow$$



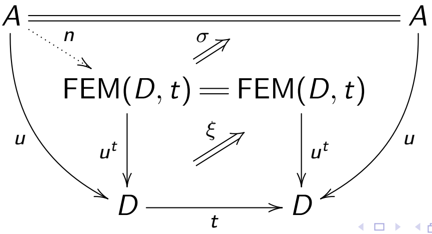
Dagger lax functors

A *dagger lax functor* $F : \mathcal{D} \longrightarrow \mathcal{C}$ between dagger 2-categories is a lax functor satisfying an additional *Frobenius axiom*...

Equivalently: A dagger Frobenius monad in a dagger 2-category $\mathcal{D}$ is a dagger lax functor $\mathbf{1} \longrightarrow \mathcal{D}$ from the terminal 2-category $\mathbf{1}$ to $\mathcal{D}$. So

$$\mathrm{DFMnd}(\mathcal{D}) = \mathrm{DagLaxFun}(\mathbf{1}, \mathcal{D})$$

Dagger lax-natural transformations, dagger lax modifications, dagger lax limits,...



FEM-objects

Frobenius-Eilenberg-Moore objects

For each dagger Frobenius monad (D, t) in a dagger 2-category $\mathcal{D}$, there is a dagger 2-functor

$$\begin{aligned} \mathcal{D}^{\mathsf{op}} &\longrightarrow \mathsf{DagCat} \\ X &\longmapsto \mathsf{FEM}(\mathcal{D}(X, D), \mathcal{D}(X, t)) \end{aligned}$$

If this dagger 2-functor is representable, $\mathsf{FEM}(D, t)$ is the representing object, and is called the *Frobenius-Eilenberg–Moore (FEM) object* of (D, t).

That is,

$$\mathcal{D}(X, \mathsf{FEM}(D, t)) \cong \mathsf{FEM}(\mathcal{D}(X, D), \mathcal{D}(X, t))$$

dagger 2-naturally in the arguments.

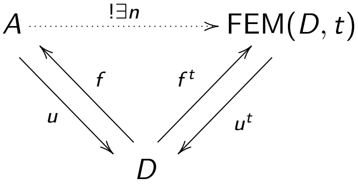
Important properties

Theorem

FEM($\boldsymbol{D}, T$) *is FEM-object for a dagger Frobenius monad* $(\boldsymbol{D}, T)$ *in* DagCat.

Theorem

Suppose (D, t) *in* $\mathcal{D}$ *generated by the adjunction* $f \dashv u : D \longrightarrow A$ *has a FEM-object. Then, there exists a unique* 1*-cell* $n : A \longrightarrow \mathrm{FEM}(D, t)$ *– called the right comparison* 1*-cell – such that* $u^t n = u$ *and* $nf = f^t$.



FK objects

Frobenius-Kleisli objects

A *Frobenius-Kleisli object* for a dagger Frobenius monad (D, t) in $\mathcal{D}$ is dual to $\mathrm{FEM}(D, t)$. Denoted $\mathrm{FK}(D, t)$. In particular

$$\mathcal{D}(\mathrm{FK}(D, t), X) \cong \mathrm{FEM}(\mathcal{D}(D, X), \mathcal{D}(t, X))$$

2-natural in each of the arguments.

Theorem

Each dagger Frobenius monad $T = (T, \mu, \eta)$ on a dagger category $\boldsymbol{D}$ has an FK-object, which is the Kleisli category $\boldsymbol{D}_T$ of the monad T.

Free cocompletions

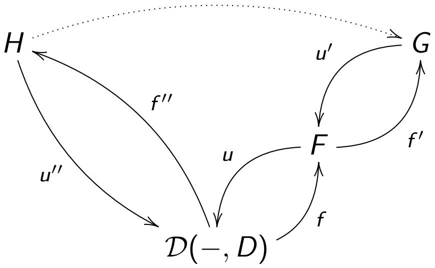
Kelly (2005) provides very general theory of cocompletions. Hard (impossible?) to transfer to the dagger context (e.g. Karvonen, 2019)

Build closure $\overline{\mathcal{K}}$ via transfinite process: take $[\mathcal{K}^{\mathsf{op}}, \mathsf{Cat}]$ and start with representables. At each stage, add colimits of the previous stage.

Plan: imitate this for FK-objects without general theory.

Free cocompletions, cont.

Transfinite process ends in after one step. **Proof**: In $[\mathcal{D}^{\mathrm{op}}, \mathrm{DagCat}]$



$\mathrm{FK}(\mathcal{D})$ is replete, full dagger-sub-2-category of $[\mathcal{D}^{\mathrm{op}}, \mathrm{DagCat}]$ of objects resulting from the single step.

Explicit definition

We want $\mathsf{FEM}(\mathcal{D}) = \mathsf{KL}(\mathcal{D}^{\mathsf{op}})^{\mathsf{op}}$. So we define $\mathsf{FEM}(\mathcal{D})$ as:

- 0-cells are dagger Frobenius monads in $\mathcal{D}$
- 1-cells are the same as 1-cells in $\mathsf{DFMnd}(\mathcal{D})$
- A 2-cell $(f, \sigma) \longrightarrow (g, \gamma) : (D, t) \longrightarrow (C, s)$ is a 2-cell $\alpha : f \longrightarrow gt$ in $\mathcal{D}$ suitably commuting with a "Kleisli composition".

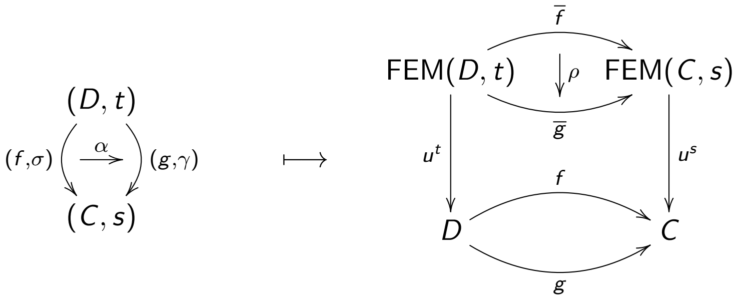
There is an embedding $I : \mathcal{D} \longrightarrow \mathsf{FEM}(\mathcal{D})$, $D \longmapsto (D, 1)$.

Explicit definition, cont.

Theorem

When a dagger 2-category $\mathcal{C}$ has FEM-objects, there is an equivalence of categories $\mathrm{FEM}(\mathcal{C}) \longrightarrow \mathcal{C}$*.*

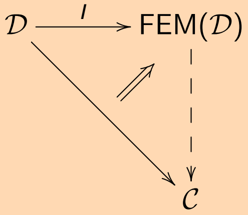
Proof: By bijection of mates under the adjunction $f^t \dashv u^t$ in $\mathcal{D}$



Universal property of FEM construction

Theorem

For a dagger 2-category $\mathcal{D}$, and $\mathcal{C}$ a dagger 2-category with FEM-objects, each dagger 2-functor extends to a FEM-object preserving 2-functor



That is,

$$[\mathsf{FEM}(\mathcal{D}), \mathcal{C}]_{\mathsf{FEM}} \approx [\mathcal{D}, \mathcal{C}]$$

Examples

Calculate $\mathrm{FEM}(\Sigma(\mathbf{FHilb}))$:

<u>0-cells:</u> (Heunen & Vicary, 2019)
Let $\boldsymbol{G}$ be a finite groupoid, and G its set of objects. The assignments

$$1 \longmapsto \sum_{A \in G} \mathrm{id}_A \qquad f \otimes g \longmapsto \begin{cases} f \cdot g & \text{if } f \cdot g \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

define a dagger Frobenius monoid in **FHilb**. Any dagger Frobenius monoid in **FHilb** is of this form.

<u>1-cells:</u> Any isometry $f : A \longrightarrow B$ between 0-cells preserving (co)multiplication and the unit. More generally, seem to be related to the *unitary transformations of fibre functors* of D. Verdon.

References

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- "Frobenius algebras and ambidextrous adjunctions" - A. Lauda (2006)
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