A categorical framework for the expression of composable constraints

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Motivated by recent developments in the foundations and applications of quantum theory in which particular constraints on the behaviours of gates within a circuit-based protocol are guaranteed to be satisfied, we introduce *composable constraints* as a framework for expressing the compatibility of a set of constraints with the structure of the category on which they are imposed. We show that the existence of a composable constraint allows for the construction of a *constrained category*, featuring a calculus of morphisms and their constraints in parallel. A subclass of such constraints is then presented via a general construction from the notion of a *route-category*. We finish by demonstrating the use of the induced constrained category by computing the spread of decoherence through systems in a compact and scalable way.

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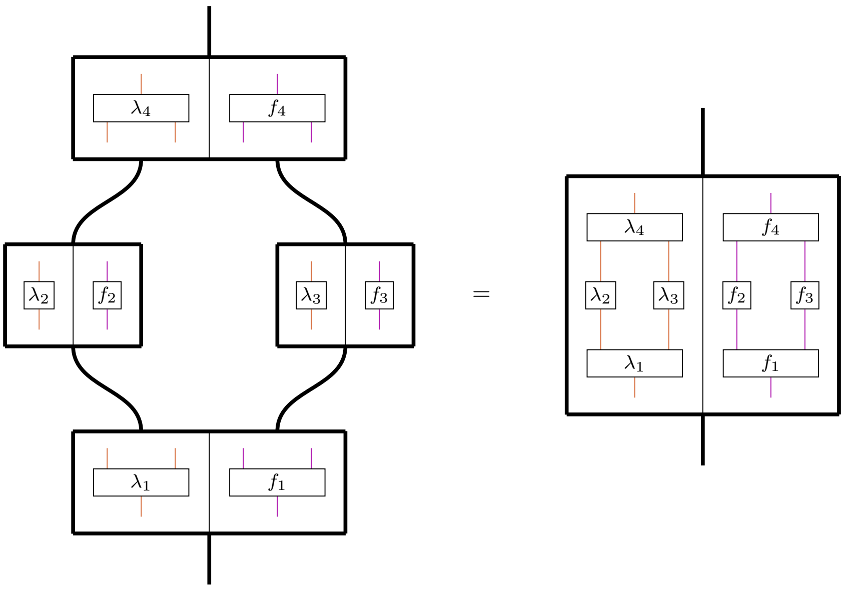
1 Introduction

The aim of this work is to present a general categorical construction on a category $\mathbf{C}$ that 1) captures the possibility to impose *constraints* on morphisms of $\mathbf{C}$, and 2) allows to compose these constraints, in a way that is compatible with the composition of the morphisms of $\mathbf{C}$. By a set of constraints on possible maps $A \xrightarrow{f} B$ in $\mathbf{C}$, we mean a piece of data λ singling out, among the hom-set $\mathbf{C}(A, B)$, a subset $\mathbf{C}_\lambda(A, B)$, whose elements are said to be the maps that *follow* these constraints. Now, given λ and another set of constraints σ on possible maps $B \xrightarrow{g} C$, it is often the case that one can think of a sense in which σ and λ can be composed to form a set of constraints '$\sigma \circ \lambda$' that is followed by $g \circ f$ whenever g follows σ and f follows λ. As we shall see, formally capturing this kind of 'structure of constraints', compatible with the structure of morphisms themselves, helps not only to model scenarios in which constraints have to be taken into account, but also to unlock a handy 'constraint calculus': a calculus only performed on the constraints that morphisms follow, allowing to deduce properties about their compositions while 'bypassing' the handling of the (usually more intricate) data about the morphisms themselves.

The need for a formal theory of constraints has recently arisen in different contexts. First, constraints can appear in the description of physical, communicational or computational scenarios in which some key operations can be freely chosen, yet can only be picked among a subset of the possible operations between their domain and codomain, due to restrictions arising e.g. from physical constraints or from the rules of a game. This is for instance the case for the study of superpositions of channels in quantum theory [1–3] – and more generally for that of the coherent control of gates and channels [4–13] –, a notion whose formal definition is a subtle matter, and for which a recently proposed formalism [14] makes a crucial use of so-called *sectorial constraints* on morphisms. Introducing constraints can also be used as a way of enriching the structure of a given category, in order to make it expressive enough to capture some notions in an elegant and consistent way; this has been the case in the study of so-called *causal decompositions*, i.e. diagrammatic decompositions of unitary channels that are equivalent to these channels' causal structure [15–18] (see in particular Ref. [17]). Indeed, some causal decompositions cannot be written in terms of standard circuits, but only using more elaborate circuits (later called *index-matching circuits* [14]), which relied on constraints and whose exact semantics remained unclear[1].

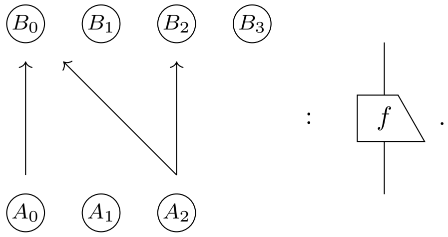
Ref. [14] provided a well-defined formal account of the two examples cited above; the structural and categorical features of this account, however, were toned down in order to make it suitable to working physicists. The framework we will present here aims to be a way more general, and fully structural, account, able to model the inclusion of, and reasoning about, constraints on morphisms in various theoretical contexts. After formalising the notion of a compositional constraint as a lax functor $\mathcal{L}$ with a particular domain, we show that any such lax functor can be used to construct a new *constrained* category $\mathcal{C}_{\mathcal{L}}$ in which morphisms and their constraints are manipulated together as pairs (λ, f) in which λ (in **Con**) is called the *constraint*, and f (in **C**) satisfies the constraint λ – in the sense that $f \in \mathcal{L}(\lambda)$. Defining all operations pairwise, suitably *monoidal* (or † or *compact closed*) properties of such a lax-functor $\mathcal{L}$ lift to properties of the constrained category. In the constrained category, the whole calculus is thus doubled, and performed in parallel in **C** on the one hand, and in **Con** on the other hand, e.g.:, guaranteeing a circuit language for morphisms and their constraints:



We will then present a construction of a class of such compositional constraints based on a generalisation of the constructions of Ref. [14], which will serve, throughout the present paper, as a conceptual guideline and as a source of meaningful instances for the general categorical construction we present. Let us present the outline of these specific constructions, so as to illustrate and motivate our general strategy. The point of Ref. [14] is, in the context of the category **FHilb**

[1] In Appendix A, we also explain why other standard categorical constructions, CP*[**FHilb**] and Karoubi[CPM[**FHilb**]], cannot be used either to model superpositions of paths and causal deocmpositions.

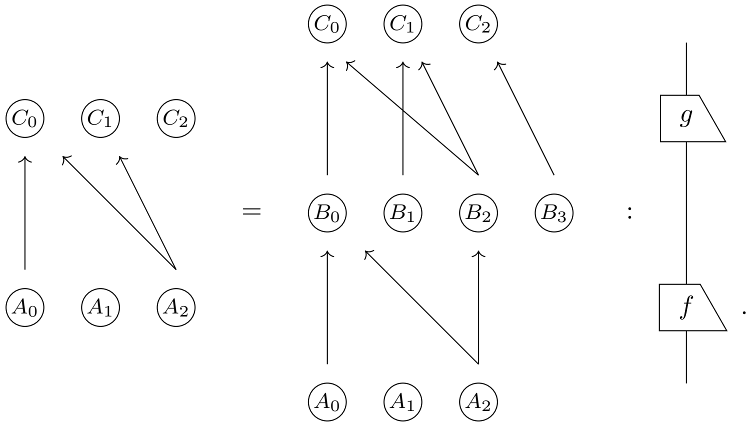
of linear maps on finite-dimensional Hilbert spaces, to build a theory encompassing *sectorial constraints* on these linear maps. Sectorial constraints express the fact that a given linear map is forbidden to relate some sectors (i.e. orthogonal subspaces) of its domain with some sectors of its codomain. For example, the set of arrows in the following figure expresses a set of sectorial constraints on a map $f \in \mathbf{FHilb}(A, B)$, where A and B are partitioned into a direct sum of sectors:



(1)

In this figure, the sectorial constraints correspond to the absence of arrows between some sectors. For instance, the absence of arrows fro A_1 to B_1, B_2 and B_3 means that, for a f following these constraints, one has $f(A_0) \subseteq B_0$; the same goes with the other sectors.

However, as we already mentioned, when we talk about a framework 'including constraints', we do not just mean to allow for the possibility to add these constraints 'by hand' on some morphisms; what we want is a theory in which one can *compose* these constraints in various ways, compatibly with the structure of the original category. An example will make this point clearer. For any two linear maps $A \xrightarrow{f} B \xrightarrow{g} C$ following some sectorial constraints, it is easy to deduce a set of sectorial constraints which $g \circ f$ will necessary follow, as depicted in the following figure:



(2)

In other words, the constraints themselves feature some structure (here, a composition), and, crucially, this structure is compatible with that of the underlying category: if two maps each follow a set of constraints, then their composition follows the composition of these sets of constraints. In fact, sectorial constraints exhibit a whole dagger compact structure – that of finite relations –, completely consistent with the dagger compact structure of **FHilb**. It is this kind of structural compatibility that we want to describe and exploit fully, whenever some notion of constraints features it.

The strategy we will follow here is to capture this structure using *route categories*. Given a symmetric monoidal[2] category $\mathbf{C}$, a dagger compact category $\mathbf{R}$ will be called a *route category*

[2]As is done at several points in this paper, this strategy can be naturally extended to any structure shared, in a

for **C** if there exists a suitably well-behaved (although, as we shall see, not functorial) embedding $\mathcal{E} : \mathbf{R} \to \mathbf{C}$, where the 'good behaviour' of $\mathcal{E}$ means that it allows to think of the morphisms in **R** as describing constraints on **C**'s morphisms[3].

The structure of this paper is as follows. First, we present a very general notion of composable constraint on a category. Then, in order not to lose the reader in the technicalities of our definition of route categories, we introduce its core elements at the conceptual level (Section 3); then, we get the formal work done (Section 4). We present three general classes of route categories, that one can construct for large varieties of categories (Section 5). We explain how, from a category **C** and a route category **R** for it, one can build a routed category featuring a 'double calculus' (Section 6). We study how our constructions interplay with the CPM construction, showing that, if **R** is a route category for **C**, then CPM[**R**] can be considered to be a route category for CPM[**C**] in a canonical way (Section 7). Finally, we show in an example how the structure of routed categories can unlock a handy calculus bypassing the calculus of the target category **C** (Section **??**). We conclude in Section 8.

2 Compositional Constraints - Formally

In this section we present the concept of a compositional constraint category **Con** that may be imposed on a category $\mathcal{C}$. The notion of a constraint τ encoded as a subset of the morphisms $\mathcal{C}(A,B)$ can be captured at its heart by the notion of a lax functor. From a category $\mathcal{C}$ one may construct a power-set category $\mathcal{P}[\mathcal{C}]$:

Definition 1 (Power-Set Category)**.** *For any category $\mathcal{C}$ the category $\mathcal{P}[\mathcal{C}]$ has objects given by the objects of $\mathcal{C}$ and for morphisms*

$$\mathcal{P}[\mathcal{C}](A,B) := \{S| \quad S \subseteq \mathcal{C}(A,B)\}$$

with identity given by $\{id\}$ and composition given by

$$S \circ T := \{f \circ g| \quad (f,g) \in S \times T\}$$

$\mathcal{P}[\mathcal{C}]$ may furthermore be viewed as a 2-Category with 2-Morphisms given by subset inclusion

$$\mathcal{P}[\mathcal{C}](S,T) := \begin{array}{l} \{\leq\} \textit{ if } S \subseteq T \\ \emptyset \quad \textit{otherwise} \end{array}$$

For any lax functor

$$\mathcal{L} : \mathbf{Con} \longrightarrow \mathcal{P}[\mathcal{C}]$$

the laxity condition implies precisely that $\mathcal{L}(\tau) \circ \mathcal{L}(\lambda) \subseteq \mathcal{L}(\tau \circ \lambda)$ meaning that if $f \in \mathcal{L}(\tau)$ and $g \in \mathcal{L}(\lambda)$ then the composition $f \circ g \in \mathcal{L}(\tau) \circ \mathcal{L}(\lambda) \subseteq \mathcal{L}(\tau \circ \lambda)$, which is exactly the notion we proposed as a "compositional constraint"[4].

compatible way, by **C** and **R**. For instance, in this paper we also consider cases where **C** is a †-SMC or a †-compact category.

[3] Later in the paper, we will introduce a generalisation that allow for the constraints' expressions to be expressed in a third category **V**. What will then be required is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{V}$, and a (not necessarily functorial) embedding $\mathcal{E} : \mathbf{R} \to \mathbf{V}$, interacting suitably.

[4] The authors are extremely grateful for the insight of an anonymous reviewer at ACT 2021 who pointed out the phrasing of compositional constraints via lax-ness

Definition 2 (Compositional Constraint)**.** *A compositional constraint for a category $\mathcal{C}$ is a lax functor $\mathcal{L} : \mathbf{Con} \longrightarrow \mathcal{P}[\mathcal{C}]$ from some any category* **Con** *into the power set category $\mathcal{P}[\mathcal{C}]$*

Our second goal is to give the notion of a circuit theory in which formally the morphisms of $\mathcal{C}$ and the constraints **Con** they satisfy live together in one category, so that the string diagrams of such a category immediately capture a side-by-side calculus of morphisms and their constraints. Firstly we note that for simple (non-monoidal) categories such a construction can easily be written down.

Definition 3 (Category of Constrained Morphisms)**.** *Let $\mathcal{L} : \mathbf{Con} \longrightarrow \mathcal{P}[\mathcal{C}]$ be a compositional constraint, The constrained category $\mathcal{C}_{\mathcal{L}}$ has as objects the objects of* **Con** *and as morphisms*

$$\mathcal{C}_{\mathcal{L}}(a, b) := \{(\tau, f) | f \in \mathcal{L}(\tau : a \to b)\}$$

with composition defined component-wise by $(\tau, f) \circ (\lambda, g) := (\tau \circ \lambda, f \circ g)$ and well-defined since by laxity $f \in \mathcal{L}(\tau)$ and $g \in \mathcal{L}(\lambda) \implies f \circ g \in \mathcal{L}(\tau \circ \lambda)$. The identity is given by (id, id) which is well-defined since $\{id\} \subseteq \mathcal{L}(id)$.

To construct a circuit theory one must take a further step and consider a symmetric-monoidal structure. For any symmetric monoidal category $\mathcal{C}$ the category $\mathcal{P}[\mathcal{C}]$ can be viewed as a monoidal category with $S \otimes T := \{f \otimes g | (f, g) \in S \times T\}$ and with coherence isomorphisms inherited as singletons E.G $\alpha_{\mathcal{P}[\mathcal{C}]} := \{\alpha_{\mathcal{C}}\}$. From this definition of the monoidal product it follows that whenever $S \subseteq S'$ and $T \subseteq T'$ then $S \otimes T \subseteq S' \otimes T'$ it is tedious but simple to check that $\mathcal{P}[\mathcal{C}]$ in-fact forms a symmetric monoidal 2-Category in the sense of being a suitably strict symmetric monoidal bicategory. A monoidal composiitonal constraint is a compositional constraint such that whenever $(f, f') \in \mathcal{L}(\tau) \times \mathcal{L}(\tau')$ then $f \otimes f' \in \mathcal{L}(\tau \otimes \tau')$.

Definition 4 (Monoidal Compositional Constraint)**.** *A monoidal compositional constraint for a monoidal category $\mathcal{C}$ is a lax functor*

$$\mathcal{L} : \mathbf{Con} \longrightarrow \mathcal{P}[\mathcal{C}]$$

equipped with an oplax natural transformation

$$\phi : \mathcal{L}(-) \otimes \mathcal{L}(-) \Rightarrow \mathcal{L}(- \otimes -)$$

a morphism with $\phi_o : I \to \mathcal{L}(I)$ and 2-Morphisms:

$$\begin{tikzcd}
(\mathcal{L}(A) \otimes \mathcal{L}(B)) \otimes \mathcal{L}(C) \arrow[r, "\alpha"] \arrow[d, "\phi_{A,B} \otimes i"'] & \mathcal{L}(A) \otimes (\mathcal{L}(B) \otimes \mathcal{L}(C)) \arrow[d, "i \otimes \phi_{B,C}"] \arrow[ddl, Rightarrow] \\
\mathcal{L}(A \otimes B) \otimes \mathcal{L}(C) \arrow[d, "\phi_{A \otimes B, C}"'] & \mathcal{L}(A) \otimes \mathcal{L}(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
\mathcal{L}((A \otimes B) \otimes C) \arrow[r, "\mathcal{L}(\alpha)"] & \mathcal{L}(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathcal{L}(A) \arrow[r, "\phi_o \otimes i"] & \mathcal{L}(I) \otimes \mathcal{L}(A) \arrow[d, "\phi_{I,A}"] \\
\mathcal{L}(A) \arrow[u, "\lambda"] \arrow[r, "\mathcal{L}(\lambda)"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{L}(I \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}(A) \otimes I \arrow[r, "i \otimes \phi_o"] & \mathcal{L}(A) \otimes \mathcal{L}(I) \arrow[d, "\phi_{A,I}"] \\
\mathcal{L}(A) \arrow[u, "\rho"] \arrow[r, "\mathcal{L}(\rho)"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{L}(A \otimes I)
\end{tikzcd}$$

The reader may recognise the above as a significant weakening of the notion of a homomorphism of monoidal bicategories, the statement of coherence conditions and the phrasing of 2-Morphisms as modifications will be unnecessary for our purposes since all 2-morphisms are unique. We refer to a monoidal compositional constraint as *strong* if the components of the oplax natural transformation $\phi_{A,B} : \mathcal{L}(A) \otimes \mathcal{L}(B) \to \mathcal{L}(A \otimes B)$ and ϕ_o are isomorphisms, taken to its extreme a *strict* monoidal compositional constraint is then one for which $\phi_{A,B}$ and ϕ_o are identities, the strict case can be concisely rephrased by required that the oplax natural transformations ϕ be an *icon*. The strict case is the most easily interpreted, it entails precisely the desired lax-ness of the tensor product

$$\mathcal{L}(\tau) \otimes \mathcal{L}(\lambda) \leq \mathcal{L}(\tau \otimes \lambda)$$

alongside the preservation of coherence isomorphisms up to inclusion $\lambda \leq \mathcal{L}(\lambda)$, $\rho \leq \mathcal{L}(\rho)$, $\alpha \leq \mathcal{L}(\alpha)$ in direct analogy to preservation of the identity by standard lax functors.

It is expected by the authors that the above notion when equipped with the coherence conditions of [] define the most-lax notion of morphism between monoidal bi-categories (crucially not a homomorphism which would impose $\mathcal{L}(\tau) \otimes \mathcal{L}(\lambda) \cong \mathcal{L}(\tau \otimes \lambda)$), this phrasing is left for future development, since the definition given will be adequate for our purposes, essentially because coherence conditions are guaranteed to be satisfied by the uniqueness of 2 morphisms in $\mathcal{P}[\mathcal{C}]$. We now show that for every monoidal compositional constraint one can construct a monoidal category (and so a circuit theory) of morphisms in parallel with their constraints.

Theorem 1 (Monoidal Category of Constrained Morphisms)**.** *Let $\mathcal{L} : \mathbf{Con} \longrightarrow \mathcal{P}[\mathcal{C}]$ be a strong monoidal compositional constraint, The constrained category $\mathcal{C}_{\mathcal{L}}$ is a monoidal category.*

In the strict case it is easy to check that one can define a monoidal category by:

- $(\tau, f) \otimes (\tau', f') := (\tau \otimes \tau', f \otimes f')$
- All coherence isomorphisms defined component wise E.G: $\alpha_{\mathcal{C}_{\mathcal{L}}} := (\alpha_{\mathbf{Con}}, \alpha_{\mathcal{C}})$

crucially the proposed coherence morphisms are well-defined since $\alpha_{\mathcal{C}} \in \{\alpha_{\mathcal{C}}\} = \alpha_{\mathcal{P}[\mathcal{C}]} \subseteq \mathcal{L}(\alpha_{\mathbf{Con}})$ and similarly for left and right unitors. The explicit proof will be given in further development. By including the analogous conditions for additional structures on circuit theories such as

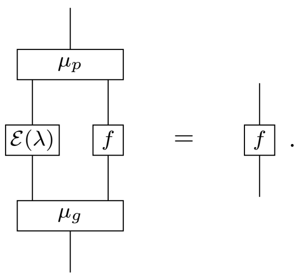
- Symmetric Monoidal: $\beta_{\mathcal{P}[\mathcal{C}]} \subseteq \mathcal{L}(\beta)$
- †-Monoidal: $\mathcal{L} \circ \dagger_{\mathbf{Con}} = \dagger_{\mathcal{P}[\mathcal{C}]} \circ \mathcal{L}$
- Compact Closed: $\cup_{\mathcal{P}[\mathcal{C}]} \subseteq \mathcal{L}(\cup_{\mathbf{Con}})$

one can correspondingly construct symmetric, †, and compact categories of constrained morphisms, meaning for example that whenever one has a theory of compact constraints over a compact category, one may construct a new category in which the constraints may be computed in parallel with the morphisms they constrain *and* the string diagram calculus for the parallelised category also comes with caps, cups, and so a convenient diagrammatic language. We conclude by presenting two basic examples of monoidal compositional constraints in quantum theory, sectorial constraints and signalling constraint, both having constraints which are essentially relations.

3 Route categories – informally

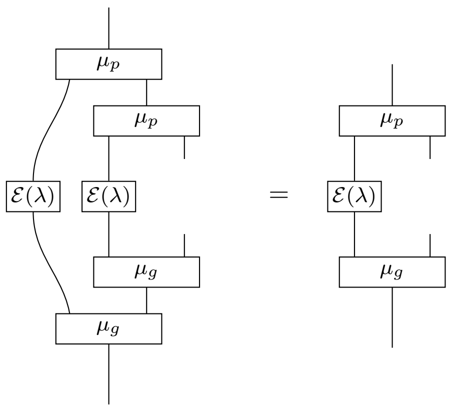
In this section, we provide an introductory description of our strategy to spell out the basic structural requirements endowing a category $\mathbf{R}$ with the interpretation that it encodes constraints for another category $\mathbf{C}$. This section can be seen as a non-technical version of Section 4, allowing the reader to grasp it conceptually before we go into the technicalities.

First, it is important to make a distinction between the morphisms in a route category $\mathbf{R}$, and the expression of the constraints they represent for morphisms in $\mathbf{C}$, which will necessarily have to be expressed within $\mathbf{C}$. Let us focus on the latter. For morphisms $A \to B$, it is natural to express a set of constraints λ (in $\mathbf{R}$) through a map $Z_A \overset{\mathcal{E}(\lambda)}{\to} Z_B$ between two auxiliary objects of $\mathbf{C}$, together with two maps $A \overset{\mu_g}{\to} Z_A \otimes A$ and $Z_B \otimes B \overset{\mu_p}{\to} B$, and say that a map f follows λ if



(3)

We use this condition because it nicely matches the standard practice of defining constraints through projectors: here, we want the higher-order map formed by the μ_g, $\mathcal{E}(\lambda)$ and μ_p to act essentially as a (higher-order) projector[5]. This leads to the following requirement on the structure we can use, corresponding to the idempotency of a projector:

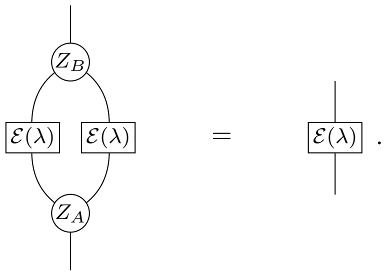


(4)

In addition, other requirements can be spelled out, to ensure that this representation of constraints interplays suitably with the dagger symmetric monoidal structure of maps in $\mathbf{C}$. The categorical structure which naturally satisfies all these requirements is that of updates structures over special semigroups (SAs). More precisely – we will spell out these notions in detail later in the paper –, Z_A and Z_B will each have the structure of a SA, and μ_g and μ_p will respectively be the action and coaction of update structures on $A \otimes Z_A$ and $B \otimes Z_B$. In lots of interesting cases, such as that of sectorial constraints on **FHilb**, these can be taken to be stronger structures:

[5] Of course, whenever $\mathbf{C}$ is compact, we could also represent the set of constraints directly as a projector on the object $A^* \otimes B$, whose states are in one-to-one correspondence with morphisms $A \to B$. However, such a picture is unsuitable if one wants to endow constraints with their own compositional structure, as it spoils the distinction between their domain and codomain; this is why we rather frame them as maps $Z_A \to Z_B$.

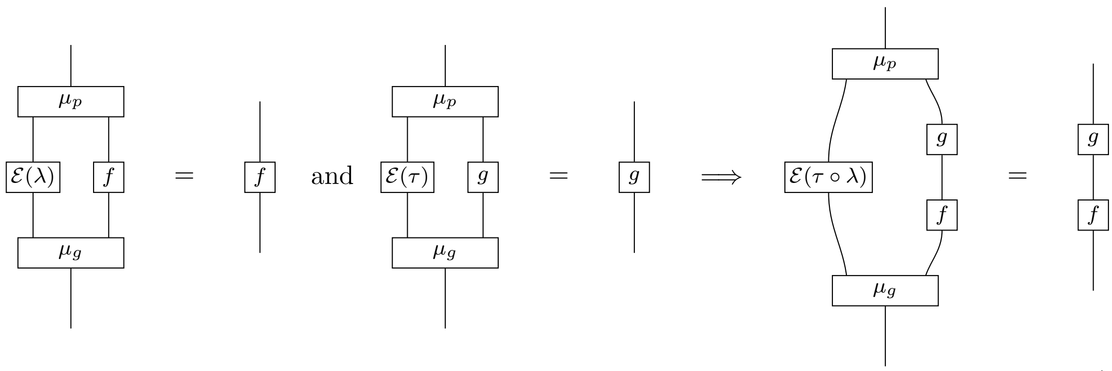
dagger modules over dagger special commutative Frobenius algebras (†-SCFAs)[6] [19–23]. In this context, (4) will in particular translate to the condition that the $\mathcal{E}(\lambda)$ be so-called element-wise idempotents, i.e.



(5)

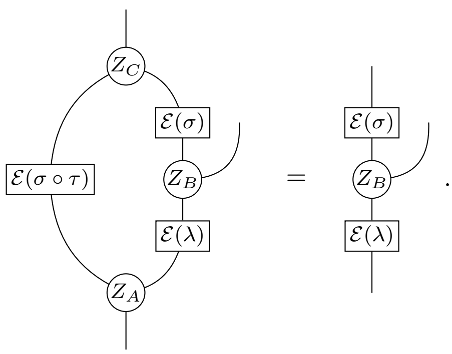
There is an important point, however: the composition of routes, given by the composition in $\mathbf{R}$, does not in general correspond to the composition in $\mathbf{C}$ of the expressions $\mathcal{E}(\lambda)$ of their constraints, i.e. one does not have $\mathcal{E}(\tau \circ \lambda) = \mathcal{E}(\tau) \circ \mathcal{E}(\lambda)$. The transformation $\mathcal{E}$ from $\mathbf{R}$ to $\mathbf{C}$ will thus have the peculiar feature that it preserves all the structure of a dagger compact category (identities, monoidal products, units and counits, adjoints...), *except* its composition. We will coin the notion of a *faketor* in order to frame this behaviour.

Even though $\mathcal{E}$ will not preserve composition per se, there will still be a sense in which the compositions of the two categories will be compatible. This sense corresponds to a loosening condition: we want $\mathcal{E}(\tau \circ \lambda)$ to impose looser constraints than $\mathcal{E}(\tau)$ and $\mathcal{E}(\lambda)$ taken together, so that "f follows λ and g follows τ" implies "$g \circ f$ follows $\tau \circ \lambda$".



(6)

Using the update structure, this loosening condition can be translated into the following one, which we shall call the *loosening condition*:



(7)

[6] Note, however, that the $\mathcal{E}(\lambda)$ will *not* necessarily be homomorphisms of Frobenius algebras.

The loosening condition (7) actually implies the element-wise idempotency (5). Any faketor of dagger compact categories (or of †-SMCs) satisfying (7) will thus meet our needs, and be called a route faketor, the specification of which makes **R** a route category for **C**. Given a route faketor, we can build a routed category with all the suitable structure; this is done in Section 6.

If we go back to our introductory example of sectorial constraints in **FHilb**, these elements will all find a natural meaning. In this case, one has $\mathbf{C} = \mathbf{FHilb}$ and the route category **R** is (equivalent to) the category **FRel** of finite sets and relations; we will use †-SCFAs, corresponding to preferred bases of a Hilbert space, the †-module over them will correspond to orthogonal partitions of Hilbert spaces, and $\mathcal{E}$ will map a relation to the linear map whose matrix (in the preferred bases determined by the †-SCFAs) only contain 1's and 0's, determined by the relation. As we will show, this particular manner of using **FRel** as a route category can actually be applied not only to **FHilb**, but to any †-compact category enriched in commutative monoids – i.e. admitting sums and zeroes on its hom-sets. We expand on this example and its generalisation to any †-SMC enriched in commutative monoids in Section 5.1.

In addition, we will provide a second example of a route category, that of *matching routes*, which can be built for *any* †-compact category, and whose route faketor maps to all the †-SCFAs of this category. This construction also originates from the study of sectorial constraints in **FHilb** [14], in which it appeared as a subcase of interest of the previous construction, giving rise to so-called *index-matching diagrams*, which are used to write causal decompositions [17].

4 Route categories – formally

Having shown that from any monoidal composiitonal constraint over a category $\mathcal{C}$ one can construct a new monoidal constrained category, and so a circuit theory of constraints, we now present a construction for a class of such constraints generalising the motivating example of the consideration of **fRel** as a theory of constraints for **fHilb**. This generalisation will include as special cases the lax-functorial embedding of boolean matrices into complex matrices and then notion of a functor into a Cartesian monoidal category.

4.1 Route faketors

As explained in the previous section, a crucial component of our constructions will be the mapping $\mathcal{E}$ going from a route category **R**, in which the route morphisms live and get composed, to the target category **C**, in which the images of these route morphisms by $\mathcal{E}$ serve to denote constraints on morphisms[7]. $\mathcal{E}$ will have the peculiar feature that it preserves all of the structure of **R** (identities, monoidal products, units and counits, adjoints...), *except its composition.* We will coin the notion of a *faketor* – or a symmetric monoidal faketor, a dagger compact faketor, etc., depending on the structure it preserves – to describe this behaviour. A *route faketor* will then be a dagger compact faketor which additionally maps to a well-defined †-SCFA of **C** and satisfies the loosening condition

[7] Strictly speaking, $\mathcal{E}$ will not exactly map to **C**, but to its special semigroup splitting $\mathbf{Sg}[\mathbf{C}]$; we write $\mathcal{E} : \mathbf{R} \to \mathbf{C}$ as a slight abuse of notation.

(7) with respect to the latter[8].

An intuitive way of understanding what we are building is to look at our main example, that of the structure of sectorial constraints in **FHilb** [14]. In this example, the structure of the route category **R** is essentially that of the category of finite relations **FRel**. $\mathcal{E}$ can then be loosely described as the mapping which, to a relation represented by a boolean matrix λ, associates the complex matrix $\mathcal{E}(\lambda)$ which is "the same matrix", in the sense that the boolean scalars 0 and 1 of λ are replaced, in $\mathcal{E}(\lambda)$, by the complex scalars 0 and 1, respectively. It is easy to see that $\mathcal{E}$ does not preserve composition; yet it preserves all the rest of the dagger compact structure of **FRel** into that of **FHilb**. We will thus in particular provide a sense in which **Rel** lives inside, can be used inside, and can be built from **FHilb** through the use of faketors.

Definition 5 (Faketor)**.** *A faketor* $\mathcal{E} : \mathbf{R} \longrightarrow \mathbf{C}$ *is a map sending each object* A *of* $\mathbf{R}$ *to an object* $\mathcal{E}(A)$ *on* $\mathbf{C}$*, and a map sending each morphism* $f : A \to A'$ *to a morphism* $\mathcal{E}(f) : \mathcal{E}(A) \to \mathcal{E}(A')$ *such that* $\mathcal{E}(1_{\mathbf{R}}) = 1_{\mathbf{C}}$*.*

In general, we will use the term 'structure-faketor' in place of 'structure-functor' to signify that a given mapping satisfies all of the defining constraints of the functor, barr that of composition preservation. For instance, in order to capture the functorial nature of just the parallel composition structure of the two categories, we introduce the notion of being a monoidal faketor.

Definition 6 (Strong Monoidal Faketor)**.** *A Strong Monoidal Faketor* $(\mathcal{E}, \theta, \phi) : \mathbf{R} \longrightarrow \mathbf{C}$ *between Symmetric Monoidal Categories is*

- *A faketor* $\mathcal{E} : \mathbf{R} \longrightarrow \mathbf{C}$
- *An isomorphism* $\phi : I_{\mathbf{C}} \to \mathcal{E}(I_{\mathbf{R}})$
- *A family of isomorphisms* $\theta_{A,A'} : \mathcal{E}(A) \otimes \mathcal{E}(A') \to \mathcal{E}(A \otimes A')$

which together satisfy all of the standard coherence conditions for a Strong Monoidal Functor [24] including the naturality square for θ*.* $(\mathcal{E}, \theta, \phi)$ *is furthermore a* †*-Strong Monoidal Faketor between* †*-Symmetric Monoidal Categories if* θ *and* ϕ *are unitary.*

Similarly, we can talk about faketors that preserve a †-compact structure.

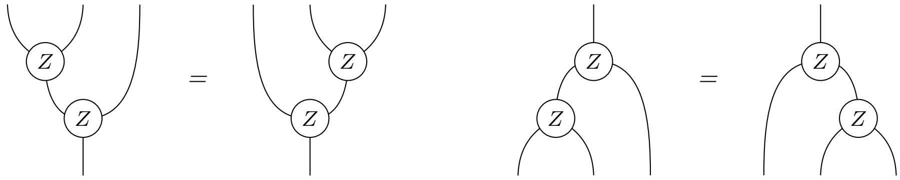
Definition 7 (†-Compact Faketor)**.** *A* †*-Compact Faketor* $\mathcal{E}$ *between* †*-Compact categories is a* †*-Strong Monoidal Faketor such that*

- $\theta_{A^*,A} \circ \mathcal{E}(\cup) \circ \phi = \cup$

Now that we have defined dagger compact faketors, the next step on our way to route faketors is to define where they go. Indeed, they don't exactly map to **C**, but rather to an equivalent category **Sg**[**C**] in which special semigroups over the objects of **C** are hardcoded into those objects. In the sense in which we embed **FRel** into **FHilb** this captures the notion that each relation will be encoded into a particular basis. Let us first define special semigroups.

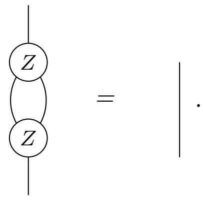
[8]In Section 4.2, we will also show that in some (though not all) cases, the structure of route faketors can be more neatly described as corresponding to a 2-category with poset enrichment.

Definition 8. *A special semigroup* $(A, Z_g : A \otimes A \to A, Z_c : A \to A \otimes A)$ *is a (co)-associative (co)-magma pair over an object* A



(8)

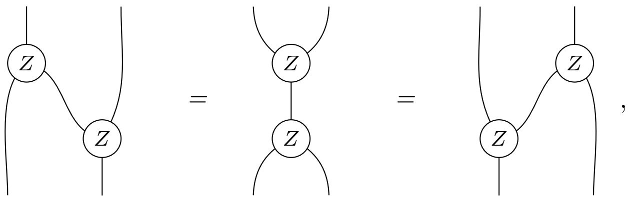
which is furthermore special in the sense that



(9)

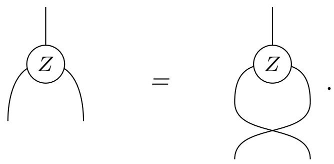
The structure of a special (c is sufficiently general to capture some quite distinct behaviours. An example is that of dagger special commutative (semi-)algebras.

Example 1 (†-SCFsA)**.** *A †-Special Commutative Frobenius semi-Algebra (†-SCFsA) is a magma* Z *on an object* A *in a †-Symmetric Monoidal Category such that* $(A, Z, Z^\dagger)$ *defines a special semigroup which furthermore satisfies the frobenius laws,*



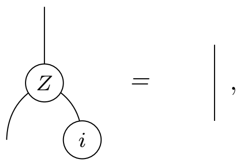
(10)

and such that Z *is commutative, i.e.*



(11)

If m *also has a unit, in the sense that there exists some state* i *satisfying*

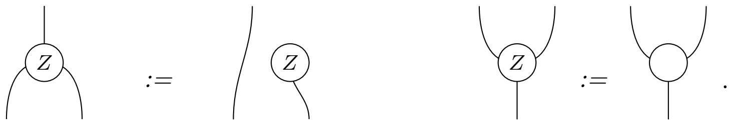


(12)

then the word semi is dropped and Z *is referred to as a †-Special Commutative Frobenius Algebra †-SCFA.*

For instance, the †-SCF(s)As with [21] and without [23] units can be used to characterise orthonormal bases in **FHilb** and **Hilb** respectively. Another example, in which the algebras are merely special, can be found in the case of cartesian monoidal categories.

Example 2 (Delete-Copy Algebra)**.** *The right-delete magma and copy co-magma which come for free in any cartesian monoidal category [24] can be used to define the delete-copy algebra:*



(13)

When the meaning is clear we will use the symbol Z interchangeably for the magma, co-magma, and the object on which they are defined. We will require a relabelling of the category $\mathbf{C}$ by the special semigroups of $\mathbf{C}$, generalising the notion of a hyper-graph category [**?**, 25–30]. The re-labelled category $\mathbf{Sg}[\mathbf{C}]$ is equivalent to a full subcategory of $\mathbf{C}$, and only serves to hardcode the special semigroup structure into its objects.

Definition 9 ($\mathbf{Sg}[\mathbf{C}]$)**.** *The special semigroup-splitting* $(\mathbf{Sg}[\mathbf{C}], \otimes_{\mathbf{Sg}[\mathbf{C}]}, I_{\mathbf{Sg}[\mathbf{C}]})$ *of a symmetric monoidal category* $(\mathbf{C}, \otimes, I)$ *is the symmetric monoidal category such that*

- *The objects of* $\mathbf{Sg}[\mathbf{C}]$ *are the special semigroups of* $\mathbf{C}$
- $\mathbf{Sg}[\mathbf{C}](Z, Z') = \mathbf{C}(Z, Z')$
- $\circ_{\mathbf{Sg}[\mathbf{C}]} = \circ_{\mathbf{C}}$
- $\otimes_{\mathbf{Sg}[\mathbf{C}]}$ *is the standard tensor product of special semigroups inherited from* $\otimes$*.*
- $I_{\mathbf{Sg}[\mathbf{C}]}$ *is the unique frobenius algebra defined by the unitor of* $\mathbf{C}$

Furthermore the special semigroup splitting of a $\dagger$*-compact category* $(\mathbf{C}, \otimes, I, \dagger, \cup)$ *is the* $\dagger$*-Compact category* $(\mathbf{Sg}[\mathbf{C}], \otimes_{\mathbf{Sg}[\mathbf{C}]}, I_{\mathbf{Sg}[\mathbf{C}]}, \dagger_{\mathbf{Sg}[\mathbf{C}]}, \cup_{\mathbf{Sg}[\mathbf{C}]})$ *such that* $\cup_{\mathbf{Sg}[\mathbf{C}]} = \cup_{\mathbf{C}}$*.*

The inheritance of a compact structure in the above sense requires that the existence of a special semigroup on an object A garuntees the existence of a special semigroup on the dual object A^*, such a special semigroup on an object A^* can indeed be constructed from a special semigroup on A by applying cups and caps to all wires converting the magma of A into a co-magma on A^* and vice-versa. For any $\dagger$-Symmetric Monoidal Category $\mathbf{C}$ we furthermore denote the sub-category defined by including only $\dagger$-SCFsAs of $\mathbf{C}$ by $\mathbf{FsA}[\mathbf{C}]$ and the sub-category including only the $\dagger$-SCFAs of $\mathbf{C}$ by $\mathbf{FA}[\mathbf{C}]$. A well behaved mapping of a category into the Special-SemiGroup splitting of another category is one in which the structure morphisms are homomorphisms with with respect to the labelling of objects by their algebras

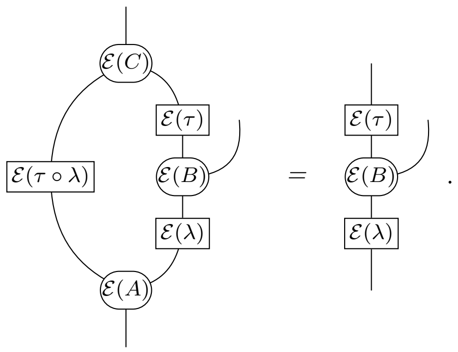
Definition 10. *A special semigroup homomorphism* $h : (A, Z_g, Z_c) \to (A', Z'_g, Z'_c)$ *is a morphism* $h : A \to A'$ *such that* h *is both a magma homomorphism* $h : Z_g \to Z'_g$*:*
and simmillaraly a co-magma homomorphism $h : Z_c \to Z'_c$*.*

encoding that the structural morphisms of a monoidal functor be special semigroup homomorphisms essentially generalises the notion of a hyper-graph functor.

Definition 11. *An Sg-*$\dagger$*-Strong-Monoidal-Faketor from* $\mathbf{R}$ *to* $\mathbf{Sg}[\mathbf{C}]$ *is a* $\dagger$*-Strong-Monoidal-Faketor* $(\mathcal{E}, \theta, \phi) : \mathbf{R} \longrightarrow \mathbf{Sg}[\mathbf{C}]$ *such that each* $\theta : Z \to Z'$ *and* $\phi : X \to X'$ *are special-semigroup homomorphisms from* Z *to* Z' *and from* X *to* X' *respectively.*

Finally, as the faketor we will manipulate needs to express sets of constraints, we will introduce a *loosening condition*, which will ensure that the set of constraints encoded by $\mathcal{E}(\sigma \circ \lambda)$ is *at most as tight* as that expressed by $\mathcal{E}(\sigma)$ and $\mathcal{E}(\lambda)$ taken together. This will conclude our definition of route faketors.

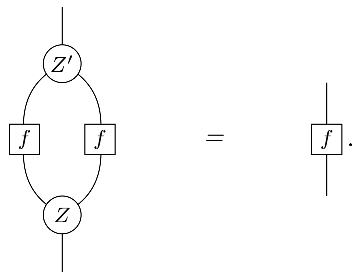
Definition 12 (route faketor)**.** *A route faketor $\mathcal{E} : \mathbf{R} \rightsquigarrow \mathbf{C}$ between †-symmetric monoidal (or †-compact) categories is an Sg-Frobenius-†-symmetric monoidal (or †-compact) faketor $\mathcal{E} : \mathbf{R} \rightsquigarrow \mathbf{Sa}[\mathbf{C}]$ such that the following loosening condition is satisfied for any τ and λ:*



(14)

In the motivating case of relations in **FHilb**, the image of any relation is a boolean matrix in a particular basis, in general we will not need the entire Sg splitting, instead just the closure of the boolean matrices, that is - the element-wise idempotents.

Definition 13. *A morphism $f : Z \to Z'$ in $\mathbf{Sg}[\mathbf{C}]$ is an element-wise idempotent from Z to Z' if*



(15)

Corollary 1 (Routes are element-wise idempotents)**.** *For any morphism $\tau \in \mathbf{R}(A, B)$ the image $\mathcal{E}(\tau)$ is an element-wise idempotent from $\mathcal{E}(A)$ to $\mathcal{E}(B)$.*

Proof. Follows by insertion of $\sigma = 1_A = 1_{Z_A}$ into eq (2), followed by specialness. $\square$

4.2 Relation Between Route-Faketors and Standard Categorical Notions of Functors

We expect that the notion of a faketor might be perceived as unwieldy and not structural enough. In this section we present two restricted classes of examples in which route-faketors reduce to standard categorical notions of functors. We show that in some cases, such as that of sectorial constraints on **FHilb**, the existence of route faketors can be recast in a more mainstream way, in terms of a 2-categorical structure. We furthermore show that in the context of cartesian monoidal categories, route functors who images are copy-delete algebras are exactly functors. In this sense the notion of a route functor actually puts a certian class of (non-functorial) oplax functors and a certian class of functors under the same umberella.

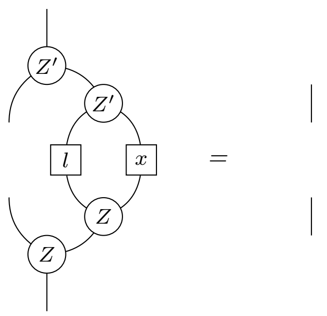
4.2.1 Route-Faketors as Oplax Functors

A subcategory of the special semigroup splitting of **FHilb** is given by the element-wise idempotents, concretely given by keeping only those objects that represent bases and and keeping only matrices generated by boolean matrices in those bases. This subcateegory can be equipped with 2-Morphisms, which capture a partial order on the matrices, in terms of those components which are non-zero. We now generalise the above story by examining the categorical properties of the closure $\mathbf{EW}[\mathbf{C}]$ of the element-wise idempotents in the subcategory $\mathbf{FsA}[\mathbf{C}]$, which inherits the $\cup$ and $\dagger$ of $\mathbf{FsA}[\mathbf{C}]$.

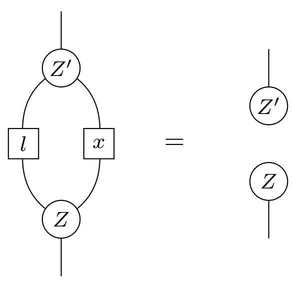
Definition 14. *The category* $\mathbf{EW}[\mathbf{C}]$ *of element-wise idempotents on* $\mathbf{C}$ *is the subcategory of* $\mathbf{FsA}[\mathbf{C}]$ *generated by for each* $Z, Z' \in o(\mathbf{FsA}[\mathbf{C}])$*, the element-wise idempotents* $f \in \mathbf{FsA}[\mathbf{C}](Z, Z')$

Similarly the sub-category of $\mathbf{FA}[\mathbf{C}]$ generated by element-wise idempotents is denoted $\mathbf{EW_u}[\mathbf{C}]$ (in which the u stands for unital). We finish by noting some additional properties of the special case of route faketors from **Rel** to **FHilb** which generalise to 2-categorical properties of a general class of route faketors. First we introduce a generalisation of the notion of being able to re-scale the elements of a matrix to a matrix of `ones`.

Definition 15. *A morphism* l *is invertible wrt* Z, Z' *if there exists a morphism such that*



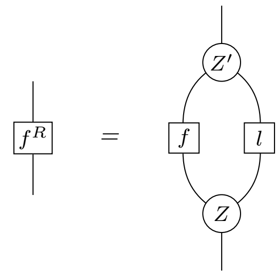
The above diagram with open holes is not formal, it should be interpreted by extending all downwards pointing legs to the bottom of the page and all upwards pointing legs to the top of the page, the informal expression is given for readability, it makes clear that x rescales l to produce a matrix which behaves like a matrix of `ones` in the sense that it trivially rescales any matrix placed into the slot to itself. Note that in the special case of a pair of †-SCFAs this condition is equivalent to



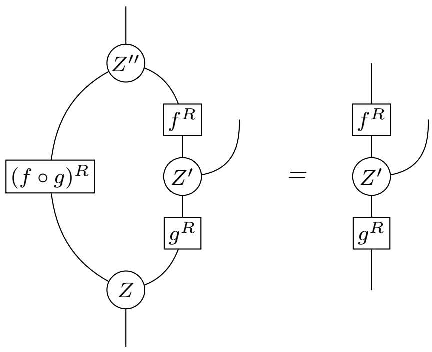
Next we introduce some basic additional conditions on element-wise idempotents of a category $\mathbf{C}$,

Definition 16. *The category* $\mathbf{EW}(\mathbf{C})$ *has loosened re-normalisation if*

- *For every morphism $f : Z \to Z'$ there exists a unique element-wise idempotent $f^R : Z \to Z'$ such that there is an invertible morphism $l \in \mathbf{C}$ satisfying*



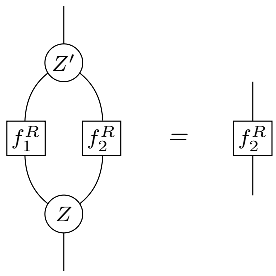
- *For every pair of morphisms f, g such that $f \circ g$ is well typed, $(f \circ g)^R$ satisfies*



We now have the terminology in place to explore the 2-categorical structure of $\mathbf{EW}[\mathbf{C}]$.

Lemma 1. *Any category $\mathbf{EW}[\mathbf{C}]$ with loosened-re-normalisation can be viewed as a 2-category by poset enrichment of each $\mathbf{EW}[\mathbf{C}](Z, Z')$ via the relation*

$f \leq f' \iff$



Proof. Given in appendix □

In the case of †-SCFAs route functors become oplax functors [24].

Lemma 2. *let $\mathbf{EW_u}[\mathbf{C}]$ have loosened re-normalisation, then any route faketor $\mathcal{E} : \mathbf{R} \longrightarrow \mathbf{Sg}[\mathbf{C}]$ defines an oplax functor $\mathcal{E} : \mathbf{R} \longrightarrow \mathbf{EW_u}[\mathbf{C}]$.*

Proof. All that is required is to check that $\mathcal{E}(f \circ g) \leq \mathcal{E}(f) \circ \mathcal{E}(g)$ which follows immediately by the loosening condition for route functors. □

Two versions of the loosening condition have appeared in capturing two separate notions of constraint-like composition

- For constraints of one category to be imposed on another, we shall find that the loosening condition on route faketors is the crucial component.

- For the guarantee that the images of constraints have a partial order structure, a notion of coarse graining of constraints which is consistent with composition, the loosening condition is again crucial.

It is for these reasons that we propose the loosening condition is the key ingredient that one should expect to introduce when working with compositional constraints.

4.2.2 Route-Faketors as Functors into Cartesian Monoidal Categories

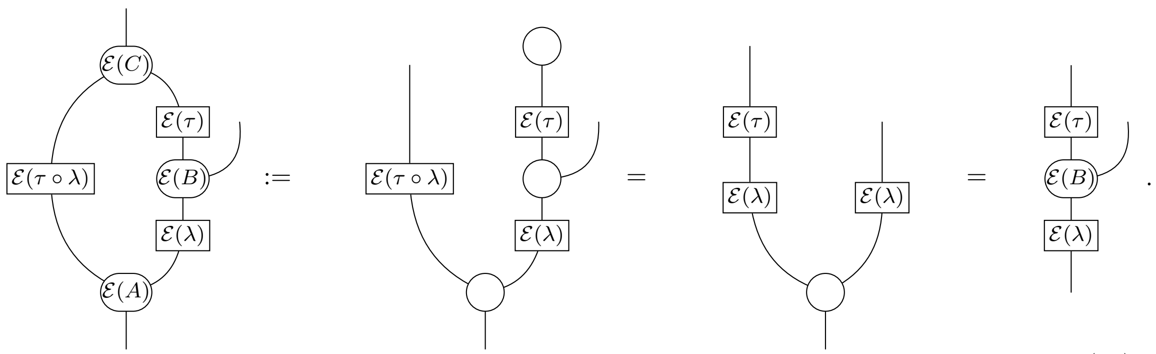
For a cartesian monoidal category $\mathbf{C}$ any strong monoidal functor $\bar{\mathcal{E}} : \mathbf{R} \to \mathbf{C}$ defines a route faketor.

Theorem 2 (route faketor Induced by Functor)**.** *Let $\bar{\mathcal{E}} : \mathbf{R} \to \mathbf{C}$ be a strong monoidal functor into a Cartesian monoidal category $\mathbf{C}$; for each choice of a copy $c_A : A \to A \otimes A$ and delete $d_A : A \to I$ the induced assignment $\mathcal{E}$ given on objects by*

$$\mathcal{E}(A) := (c_A, d_A)$$

and on morphisms by $\mathcal{E}(f) := \bar{\mathcal{E}}(f)$ defines a route faketor $\mathcal{E} : \mathbf{R} \rightharpoonup \mathbf{C}$.

Proof.



(16)

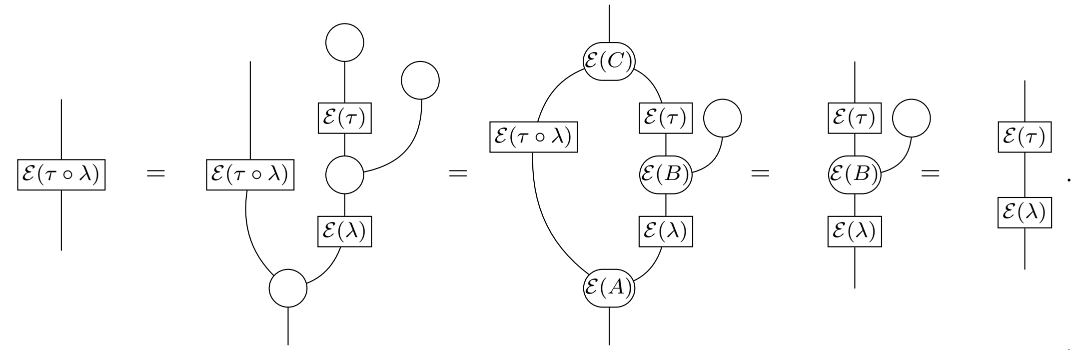
□

We will find that route faketors among those induced by functors in this way essentially specify the behaviour of morphisms on $\otimes$-subsystems. We also note that every route faketor $\mathcal{E} : \mathbf{R} \to \mathbf{C}$ into a Cartesian monoidal category $\mathbf{C}$ who's image consists only of copy-delete algebras is in fact a functor. In this sense one can interpret a route faketor as a generalisation of the notion of a functor into a cartesian monoidal category. We will define by $\mathbf{cd}$ the set of copy-delete algebras on $\mathbf{C}$.

Theorem 3. *Every route faketor $\mathcal{E} : \mathbf{R} \rightharpoonup \mathbf{C}$ into a cartesian monoidal category $\mathbf{C}$ defines a functor $\bar{\mathcal{E}} : \mathbf{R}_{\mathcal{E}^{-1}(\mathbf{cd})} \to \mathbf{C}$.*

Proof. For each object A of $\mathbf{R}_{\mathcal{E}^{-1}(\mathbf{cd})}$ then $\mathcal{E}(A) \in \mathbf{cd}$, define $\bar{\mathcal{E}}(A)$ to be the object over which the copy-delete algebra $\mathcal{E}(A)$ is defined. On morphisms define $\bar{\mathcal{E}}(A) := \mathcal{E}(\tau)$, for each $\lambda \in$

$\mathbf{R}_{\mathcal{E}^{-1}(\mathbf{cd})}(A, B)$ and $\tau \in \mathbf{R}_{\mathcal{E}^{-1}(\mathbf{cd})}(B, C)$ it follows that:



(17)

□

The notion of route-faketor hence reduces, in two rather different settings, to two standard categorical notions of functor between categories.

5 Relational and Index matching route categories

We will now present two particular cases of route categories. Our first example, in which the route category essentially corresponds to relations, can be built for any †-SMC enriched in commutative monoids. Our second example, in which the category essentially corresponds to finite corelations, can be built for any dagger compact category. Our third example, in which the routes faketor is in fact a functor, can be defined for the case of copy/delete structures in cartesian monoidal categories.

5.1 Relational route categories

Our first example of a route faketor $\mathcal{E} : \mathbf{R} \to \mathbf{C}$ is one in which the domain category $\mathbf{R}$ is (essentially) $\mathbf{bRel}$, the category of bounded relations between finite sets.

Definition 17 ($\mathbf{bRel}$)**.** *The category of bounded relations* $\mathbf{bRel}$ *is the* †*-sub-symmetric monoidal category of* $\mathbf{Rel}$ *such that for each* $R \in \mathbf{bRel}(X, Y)$ *there exists a bound* B *which* R *respects in the sense that for each* $x \in X$ *there are less than* B *elements* y *such that* xRy *and similarly for each* y *there are less than* B *elements* x *such that* xRy*. The sub-category of bounded relations between countable sets is denoted* $\mathbf{bRel_c}$*.*

This intuitively corresponds to the case where the objects of $\mathbf{C}$ can be partitioned into "sectors", and in which a route expresses the constraints that a map cannot connect some given sectors of its domain and codomain. In this example, $\mathcal{E}$ will not map to all of the objects in $\mathbf{Sg}[\mathbf{C}]$, but only to †-SCFAs characterise orthonormal bases [21], a result which generalises to a special class of †-SCFsAs in $\mathbf{Hilb}$ [23]. For concreteness, we start by describing this construction in the case of $\mathbf{FHilb}$: this corresponds to the study of sectorial constraints in finite-dimensional quantum theory [14].

Theorem 4 (Route Faketor)**.** *There is a route faketor* $\mathcal{E} : \mathbf{FRel} \to \mathbf{FHilb}$*.*

Proof. To each set $X \in o(\mathbf{Rel})$ a †-SCFA $\mathcal{E}(X)$ on the object $\mathbf{C}^{|X|}$ is assigned, furthermore a bijection κ between X and the copyable states of $\mathcal{E}(X)$ is defined. To each relation $\tau : X \to Y$ a linear map is defined by $\mathcal{E}(\tau) = \sum_{ab} \tau_a^b \left|\kappa(b)\right\rangle \left\langle\kappa(a)\right|$ where $\tau_a^b = 1 \iff {}_a\tau_b$ and otherwise $\tau_a^b = 0 \iff \neg_a\tau_b$. The route functor condition amounts to the requirement that for every a, b, c

$$\left\langle\kappa(c)\right| \mathcal{E}(\tau \circ \lambda) \left|\kappa(a)\right\rangle \left\langle\kappa(c)\right| \mathcal{E}(\tau) \left|\kappa(b)\right\rangle \left\langle\kappa(b)\right| \mathcal{E}(\lambda) \left|\kappa(a)\right\rangle = \left\langle\kappa(c)\right| \mathcal{E}(\tau) \left|\kappa(b)\right\rangle \left\langle\kappa(b)\right| \mathcal{E}(\lambda) \left|\kappa(a)\right\rangle$$

which is equivalent to

$$\tau_a^c \tau_a^b \tau_b^c = \tau_a^b \tau_b^c$$

which in turn is satisfied since $\tau_a^c \neq 1 \implies \tau_a^b \tau_b^c = 0$. □

We now show that this example can be extended to infinite-dimensional Hilbert spaces, using bounded relations.

Theorem 5. *There is a route faketor $\mathcal{E} : \mathbf{bRel}_c \to \mathbf{Hilb}$.*

Proof. On objects $\mathcal{E}$ is defined in the same way for finite sets as above, for any countable set X define $\mathcal{E}(X)$ to be a †-SCFsA whos copyable states are an orthonormal basis of l^2 (the seperable Hilbert space of square summable functions). Such a †-SCFsA always exists [23]. To each bounded relation define $\mathcal{E}(\tau)$ to be the continuos linear extension of the following assignment

$$\mathcal{E}(\tau) \left|\kappa(x)\right\rangle := \sum_{y|_x\tau_y} \left|\kappa(y)\right\rangle$$

which is bounded since τ is a bounded relation. The route functor condition reduced to the same form as in the previous theorem, and is satisfied for the same reason. □

Now that we have an intuition of what to do, let us generalise this construction; we find that it can be achieved whenever $\mathbf{C}$ is enriched in commutative monoids.

We denote the set of †-SCFsAs whose copyable states form an orthonormal set as $Z_{\perp\mathbf{C}}$, and we denote the set of copyable states of a frobenius algebra Z by $C(Z)$ and finally we denote

$$C_{\perp\mathbf{C}} := \{C(Z) \mid Z \in Z_{\perp\mathbf{C}}\}.$$

The function $C : Z_{\perp\mathbf{C}} \to C_{\perp\mathbf{C}}$ is defined by $C : Z \mapsto C(Z)$. In anticipation of a critical property of categories enriched in commutative monoids we define the following, where a bounded family of scalars $\tau : X \times Y \to \{0_{\mathbf{C}}, 1_{\mathbf{C}}\}$ is a function such that the relation defined by $xRy \iff \tau(x, y) = 1$ is a bounded relation.

Definition 18 (Component-full)**.** *A subset $S \subseteq C_{\perp\mathbf{C}}$ is component-full if for all $S_A, S_B \in S$ and for every bounded family of scalars $\tau : S_A \times S_B \to \{0_{\mathbf{C}}, 1_{\mathbf{C}}\}$ in $\mathbf{C}$ there exists a morphism $\tau \in \mathbf{C}(o(S_A), o(S_B))$ such that*

$$\left\langle b\right| \tau \left|a\right\rangle = \tau(a, b)$$

where $o(S_A)$ is the object on which that states of S_A exist.

A component-full set S of orthonormal sets is such that one can make any (possibly infinite - with bounded-size rows and columns) matrix of scalars with respect to any pair of orthonormal bases of S. Enrichment in commutative monoids is enough to ensure that the set of finite orthonormal sets makes component-full sets.

Definition 19 (Enrichment in Commutative Monoids)**.** *A category* $\mathbf{C}$ *is enriched in commutative monoids if for each homset* $\mathbf{C}(A,B)$ *there exists a map* $+_{A,B} : \mathbf{C}(A,B) \times \mathbf{C}(A,B) \to \mathbf{C}(A,B)$ *which is commutative, associative, has a unit* u_{AB}*, and is compatible with composition in the sense that*

- $u_{BB} \circ f = u_{A,B} = f \circ u_{A,A}$
- $(f + f') \circ g = f \circ g + f' \circ g$
- $f \circ (g + g') = f \circ g + f \circ g'$

The existence of a summing operation allows one to build up any matrix using scalars and orthonormal elements.

Lemma 3. *Let* $\mathbf{C}$ *be enriched in commutative monoids, the subset* $C_{F\perp\mathbf{C}} \subseteq C_{\perp\mathbf{C}}$ *of finite cardinality orthonormal sets is component-full*

Proof. For any pair of finite cardinality orthonormal sets $\{a\}$,$\{b\} \in C_{F\perp\mathbf{C}}$ and set of scalars $\tau : \{a\} \times \{b\} \to \{0,1\}$ define

$$\tau := \sum_{ab} \tau(a,b) a \circ b^{\dagger}$$

□

We denote by $\times S$ the set of cartesian products of sets in S and by $\mathbf{D}_S$ the full subcategory of $\mathbf{D}$ such that $\mathcal{O}(\mathbf{D}_S) = S$.

Lemma 4 ($\mathbf{fRel}_{\times S}$ Is †-Symmetric Monoidal)**.** *Let* $\mathbf{C}$ *be a †-Symmetric Monoidal Category with a* $\mathbf{0}$ *object, for any component-full* $S \subseteq C_{\perp\mathbf{C}}$ *such that* $\mathcal{E}(Z_\lambda) \in S$ *the category* $\mathbf{fRel}_{\times S}$ *is a †-Symmetric Monoidal Category.*

Proof. Z_λ is the trivial †-SCFA defined by the unitor λ. Clearly 1 is a copyable state of Z_λ, the orthonormality condition then implies that for any other copyable state m then $m = m \bullet 1 = 0$. It follows that the only normalised copyable state of Z_λ is 1 and so $C(\lambda) = \{1\}$ has unit cardinality meaning it may be used as a tensor unit in $\mathbf{fRel}$. The full subcategory $\mathbf{fRel}_{\times S}$ of $\mathbf{fRel}$ defined by restriction to objects in S must then be a symmetric monoidal subcategory. □

Now that we have a suitable route category $\mathbf{fRel}_{\times S}$ we are ready to define a route faketor into $\mathbf{C}$.

Theorem 6. *Let* $\mathbf{C}$ *be a dagger-SMC with a* $\mathbf{0}$ *object. For every component-full subset* $S \subseteq C_{\perp\mathbf{C}}$ *such that* $\mathcal{E}(Z_\lambda) \in S$ *there exists a route faketor*

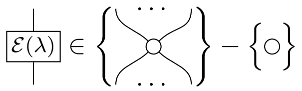
$$\mathcal{E} : \mathbf{fRel}_{\times S} \longrightarrow \mathbf{Sg}[\mathbf{C}]$$

Proof. Given in Appendix D. □

5.2 Matching Route Categories

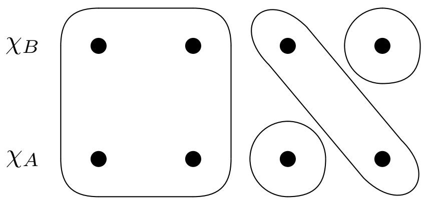
Another example of a route category is that of *matching* routes; this example can be defined for any dagger compact category $\mathbf{C}$, and yields a route category $\mathbf{Match}[\mathbf{C}]$ which contains (objects corresponding to) all †-SCFMs of $\mathbf{C}$.

The idea is to restrict oneself to $\mathcal{E}(\lambda)$'s which can be built out of the sole spiders between Frobenius algebras - that is, out of the multiplication, comultiplication, unit and counits of theses algebras. The $\mathcal{E}(\lambda)$'s will have to be restricted to be somewhat "normalised", in order to satisfy the loosening condition (17). This normalisation condition will take a particularly simple form: it will correspond to the absence of "legless spiders".



Accordingly, the composition of $\mathbf{Match}[\mathbf{C}]$ will have to differ from that in $\mathbf{C}$ in that, when seen in $\mathbf{C}$, it includes writing off any zero-legged spider produced. As we shall see, this is sufficient to yield a route category satisfying the loosening condition (17), out of any dagger compact category. Because of this absence of legless spiders, the dagger special commutative Frobenius algebras of $\mathbf{C}$ will, in $\mathbf{Match}[\mathbf{C}]$, correspond to dagger *extra*special commutative Frobenius algebras, where the "extra" means that their legless spiders are equal to the unit scalar. This entails that $\mathbf{Match}[\mathbf{C}]$ can be characterised as essentially consisting of *corelations*, which are the proper tool to describe the algebraic structure of dagger extraspecial commutative Frobenius algebras.

For an introduction to corelations, their structure, their use, and their connections to extraspecial commutative Frobenius algebras, we refer the reader to the excellent presentation of Ref. [31]. In short, a corelation between two finite sets $\mathcal{X}_A$ and $\mathcal{X}_B$ is a partition of their disjoint union $\mathcal{X}_A \sqcup \mathcal{X}_B$; it can be seen as a collection of non-empty and non-overlapping bubbles covering all of $\mathcal{X}_A \sqcup \mathcal{X}_B$,



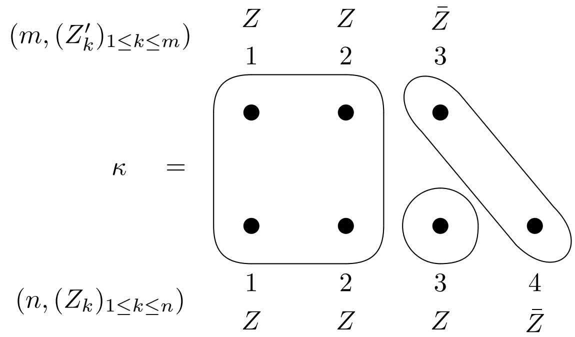
with the composition with a corelation $\mathcal{X}_B \to \mathcal{X}_C$ given by bubble merging. (Finite) corelations owe their name to the fact that they are dual objects to (finite) relations: where relations can be obtained as the category of isomorphism classes of jointly monic spans in the category of finite sets and functions, *co*relations correspond to isomorphism classes of jointly epic *co*spans in the same setting. Finite corelations form a dagger compact category $\mathbf{FCoRel}$, with monoidal product given by the disjoint union. In [31], it is shown that finite corelations are (equivalent to) the PROP for extraspecial commutative Frobenius algebras.

To define $\mathbf{Match}[\mathbf{C}]$ formally, we characterise its objects as finite sequences of †-SCFAs of $\mathbf{C}$, and its morphisms as corelations between the sets of indices of such sequences, such that, if two indices are in the same equivalence class, then their corresponding Frobenius algebras are the same. The corelations tell us which of the Frobenius algebras will be connected by a same spider.

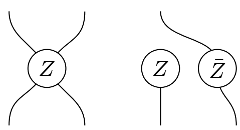
Definition 20. *Given a dagger compact category* $\mathbf{C}$, $\mathbf{Match}[\mathbf{C}]$ *is the dagger compact category defined in the following way:*

- *The objects are of the form* $(n, (Z_k)_{1\leq k\leq n})$ *where* $n \in \mathbb{N}$ *and the* Z_k*'s are †-SCFAs in* $\mathbf{C}$*;*
- *Morphisms from* $(n, (Z_k)_{1\leq k\leq n})$ *to* $(m, (Z'_k)_{1\leq k\leq m})$ *are corelations* κ *from* $[\![1, n]\!]$ *to* $[\![1, m]\!]$ *such that, if* $k, l \in [\![1, n]\!] \sqcup [\![1, m]\!]$ *are in a same equivalence class of* κ*, then one has* $Z_k = Z'_l$
- *the dagger compact structure is the standard one on corelations (see [31]).*

As an example, a morphism in $\mathbf{Match}[\mathbf{C}]$ could be expressed in the following way (corelations can only connect numbers whose corresponding frobenius algebras match):



The route faketor will send each bubble of instances of a frobenius algebra to the spider made from that frobenius algebra which connects all members of the bubble. $\mathbf{C}$

$\mathcal{E}(\kappa)$ =


.

Theorem 7. *For any dagger compact category* $\mathbf{C}$*, there is an f-faketor of dagger compact categories* $\mathcal{E}$ *from* $\mathbf{Match}[\mathbf{C}]$ *to* $\mathbf{C}$*, which:*

- *to an object* $(n, (Z_k)_{1\leq k\leq n})$ *of* $\mathbf{Match}[\mathbf{C}]$*, associates their tensor product in* $\mathbf{Frob}[\mathbf{V}]$*,* $\bigotimes_{1\leq k\leq n} Z_k$*;*
- *to a morphism* $\kappa : (n, (Z_k)_{1\leq k\leq n})$*, associates the morphism in* $\mathbf{Frob}[\mathbf{V}]$ *given by writing down a spider for each of the equivalence classes in* κ*, connecting all the representatives of this equivalence class.*

Furthermore, this functor makes $\mathbf{Match}[\mathbf{C}]$ *a route category for* $\mathbf{C}$*, i.e. it satisfies the loosening condition (17).*

Proof. First, $\mathcal{E}$ is consistently defined because of the requirement that the corelations in $\mathbf{Match}[\mathbf{C}]$ only match indices corresponding to the same †-SCFA. It is straightforward to prove that $\mathcal{E}$ satisfies all the requirements to be a Frobenius-†-compact faketor.

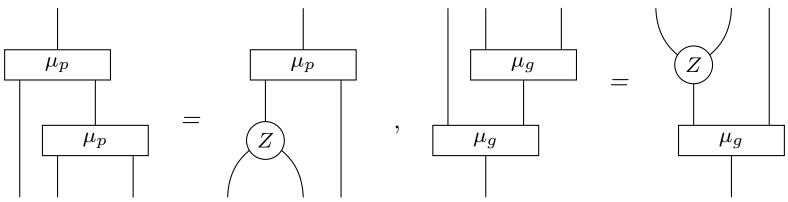
Let us prove that it is satisfies the loosening condition (17). If we take two morphisms $(n, (Z_k)_{1\leq k\leq n}) \xrightarrow{\kappa} (n', (Z'_{k'})_{1\leq k'\leq n'}) \xrightarrow{\kappa'} (n'', (Z''_{k''})_{1\leq k''\leq n''})$ in $\mathbf{Match}[\mathbf{C}]$, we have, from the definition of the composition of corelations, that two elements of $(Z_k)_{1\leq k\leq n} \sqcup (Z''_{k''})_{1\leq k''\leq n''})$ are connected by $\kappa' \circ \kappa$ if and only if they can be connected via a path made of iterations of the connections of κ and κ'. This is exactly the same rule as that of spider fusion; therefore, in $\mathbf{C}$, $\mathcal{E}(\kappa' \circ \kappa)$ connects two Frobenius algebras via spiders if and only if $\mathcal{E}(\kappa') \circ \mathcal{E}(\kappa)$ connects them. In addition, these connections between Frobenius algebras of its domain and codomain fully specify $\mathcal{E}(\kappa' \circ \kappa)$, as it is only made of non-legless spiders. Thus, if we look at the left-hand side diagram in (17), the spiders displayed by the left-hand arm of the diagram are fully redundant and can be absorbed in those displayed in the right-hand arm. This leads to this diagram being equal to the right-hand side of (17). □

A typical application is the study of index-matching routes in **FHilb** [14]. Here we see that this specific example can in fact be extended to any dagger compact category.

6 Routed categories

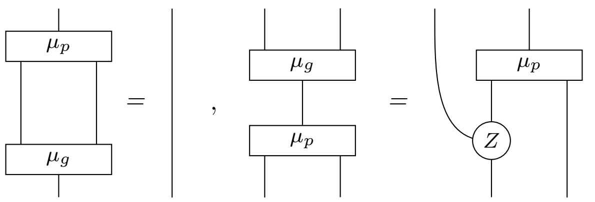
Now that we have a notion of route categories, we are in a position to define routed categories. The core idea is simple: once we have a route faketor $\mathbf{R} \xrightarrow{\mathcal{E}} \mathbf{C}$ between a category $\mathbf{R}$ in which constraints are expressed and a category $\mathbf{C}$ in which constraints are intended to be implemented, what remains is the structure capturing of implementation of constraints. From a route faketor $\mathbf{R} \xrightarrow{\mathcal{E}} \mathbf{C}$ we can build a routed category $\mathbf{Routed}[\mathcal{E}]$, whose morphisms are pairs of a route morphism in $\mathbf{R}$ and a morphism in $\mathbf{C}$ which follows it, and we define all operations on morphisms pairwise. First, to define the objects, we will need *update structures* [32,33]; which generalise the notion of a partition of an object of $\mathbf{C}$ via the specification of a †-module for it over a †-SCFA. [22].

Definition 21 (Update structures)**.** *An* ***update structure*** $\mu : Z \rightsquigarrow A$ *from a special semigroup* Z *to an object* A *is a tuple* (μ_P, μ_G) *such that* μ_P (μ_G) *is a (co)-module over the (co)-algebra of* Z*:*



(18)

and such that the following two additional laws named `PutGet` *and* `GetPut` *hold.*



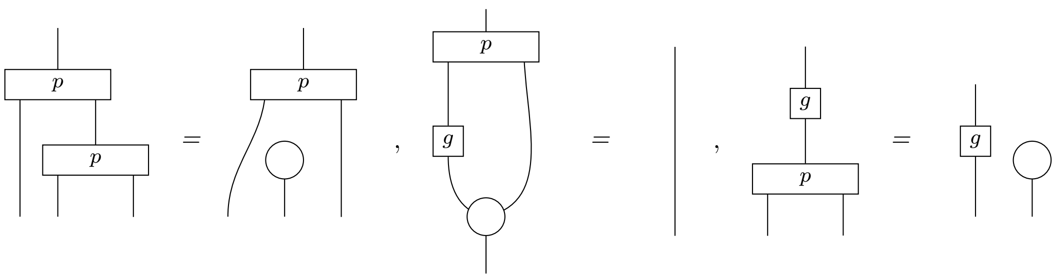
(19)

Two distinct classes of update structure are identified in [32], orthogonal partitions and lenses. Inuitively lens-like updates represent replacements [34–37] whilst partition-like updates represent projections into subspaces.

Definition 22 (Partition)**.** *A partition is an update structure* $\mu : Z \rightsquigarrow A$ *such that* Z *is a †-SCFsA and* $\mu_g = \mu_p^\dagger$

Partitions generalise projector valued spectra on **FHilb** as introduced in [22]; by not making reference to a unit for the frobenius algebra, partitions allow for the expression of constraints on infinite dimensional Hilbert spaces. It is easy to check that partitions over †-SCFAs are exactly projector valued spectra [32]. Indeed in **FHilb**, unital †-SCFAs Z_M correspond to Hilbert spaces with a preferred basis (with the co-multiplication being the copying operation in this basis), and †-modules of an object A over Z_M correspond to partitions of A into orthogonal sectors, with the states of the preferred basis of Z_M serving to label these sectors. The second type of update structure are Lens-like updates, those for which the special semigroup is a copy-delete algebra,

Definition 23. *A vwb-lens $L : V \rightsquigarrow S$ in a cartesian monoidal category is a tuple $(p : V \times S \to S, g : S \to V)$ such that the following equations are satisfied.*



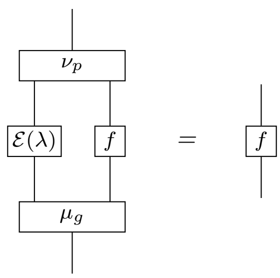
Every vwb-lens defines an update structure $U : Z \rightsquigarrow S$ by taking Z to be the copy delete algebra and for the module and co-module taking $u_p = p$ and $u_g = (g \otimes id) \circ c$.

Categories in which each morphism is supplemented by a constraint, a piece of data that is imposed upon it can be constructed using two basic ingredients, route-functors and update structures.

Theorem 8. *Let $\mathbf{R} \overset{\mathcal{E}}{\to} \mathbf{C}$ be a route faketor, then one can define a constraint category $\mathbf{Con}[\mathcal{E}]$ such that:*

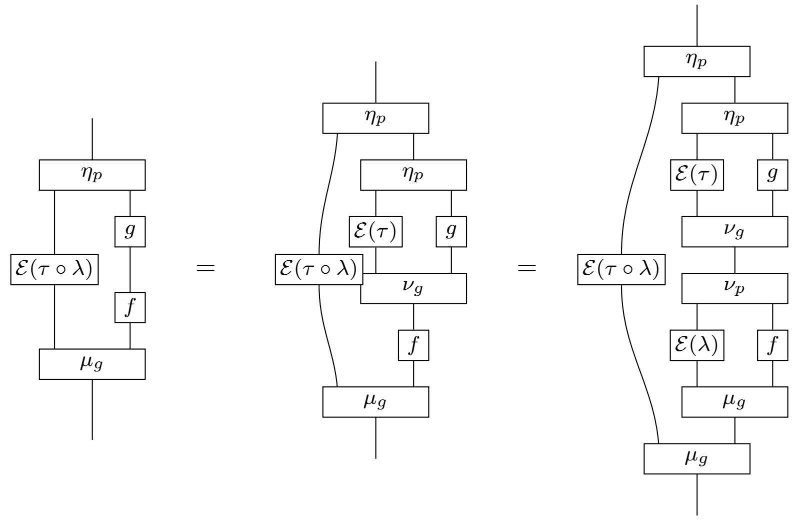
- *Objects of $\mathbf{Con}[\mathcal{E}]$ are tuples (A, B, μ) such that $(A, B) \in \mathcal{O}(\mathbf{R}) \times \mathcal{O}(\mathbf{C})$ and $\mu : \mathcal{E}(A) \rightsquigarrow B$ is an update structure in $\mathbf{C}$;*
- *Morphisms are given by $\mathbf{Con}[\mathcal{E}](A, A') := \mathcal{R}(A, A')$ and composition inherited from $\mathcal{R}$*

One can then defined a compositional constraint $\mathcal{L}[\mathcal{E}] : \mathbf{Con}[\mathcal{E}] \longrightarrow \mathcal{C}$ by taking $\mathcal{L}((A, B, \mu)) := B$ $\mathcal{L}(\lambda : (A, B, \mu) \to (A', B', \nu)) \subseteq \mathcal{C}(B, B')$ to be the set of all f such that λ is a route for f, i.e the set of all morphisms f such that:



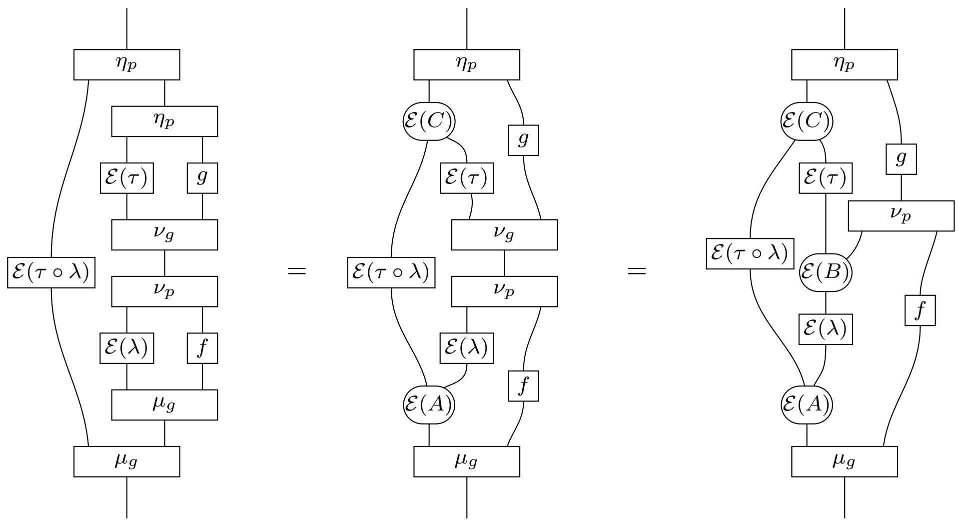
(20)

Proof. To prove lax-ness we must show that whenever λ, τ are routes for f, g then $\tau \circ \lambda$ is a route for $g \circ f$. First consider the insertion of the routing conditions for f and g



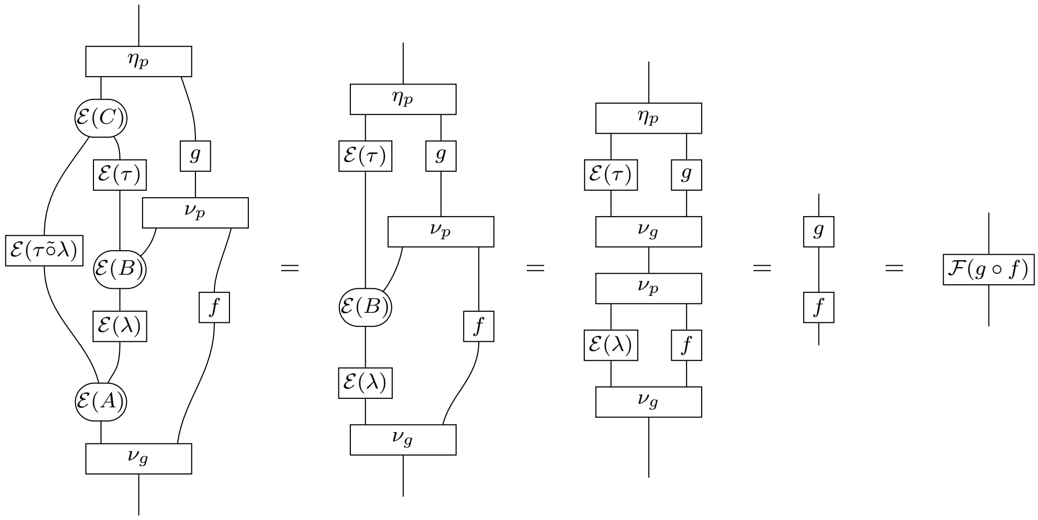
(21)

Then we use two defining properties of partitions, recording information from a system twice is equivalent to recording once and copying, imposing a condition on a system and then recording it from the system is equivalent to making a copy of the condition prior to imposing it on the system.



(22)

Finally the fact that $\mathcal{E}$ is a route faketor can be used to remove the condition $\mathcal{E}(\sigma \circ \lambda)$, after which the partition identities used previously can be reversed.

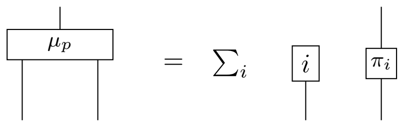


(23)

That $id_{\mathbf{R}}$ is a route for $id_{\mathbf{C}}$ follows from the `GetPut` law for update structures. $\square$

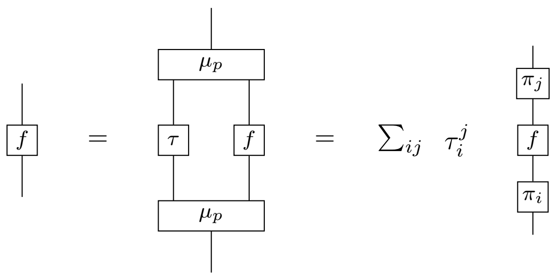
6.1 Examples

The motivating example for this work was the construction of routed quantum circuits [14], in which the route functor $\mathcal{E} : \mathbf{FRel} \to \mathbf{FHilb}$ embeds finite relations as constraints in $\mathbf{FHilb}$ and the category of morphisms to be constrained is also $\mathbf{FHilb}$, that is: $\mathbf{C} = \mathbf{fHilb}$. In [14] a theory of completely positive routed circuits was furthermore constructed. The update structures used are the projector valued spectra, which are sums of projectors indexed by a basis.



(24)

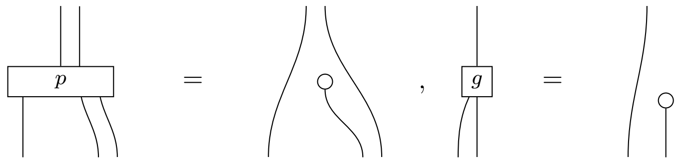
The routing condition then reads:



(25)

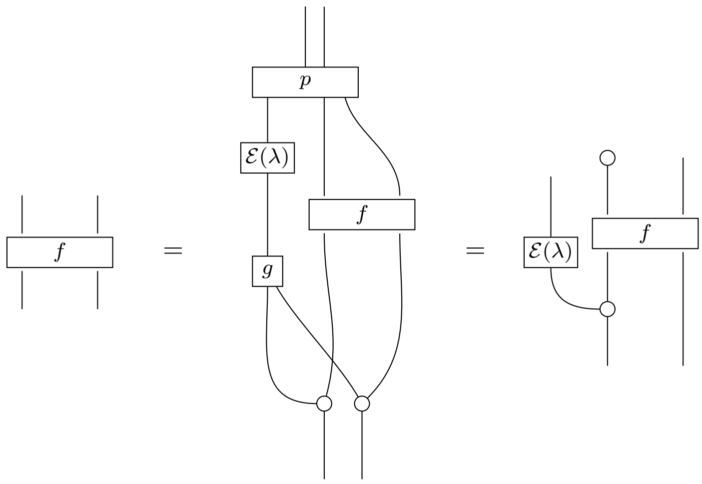
The route functor $\mathcal{E} : \mathbf{bRel}_c \to \mathbf{Hilb}$ can be used to express conditions with respect to infinite partitions of infinite dimensional quantum systems. First for any Hilbert space $\mathbf{H}_1$ a basis B can be chosen and furthermore a partition into subsets can be chosen, that is, a countable partition S_i for B can be chosen, that is, a family of sets S_i such that $\bigcup_i S_i = B$ and $S_i \cap S_j \neq \implies i = j$. Given a separable Hilbert space $\mathbf{H}_2$ equipped with a particular basis E one can fix a bijection $e : S \cong E$. Given such a choice a countable spectrum of projectors into the partition can be defined by their action on members of the basis $\mu_G(b) := b \otimes e(b)$ where $e(b) := e(S_i)$ for the i such that $b \in S_i$. The linear extension of μ_G is bounded and so lifts to a bounded linear operator on $\mathbf{H}_1 \to \mathbf{H}_2 \otimes \mathbf{H}_1$. The update structure equations between μ_G and $\mu_G^\dagger$ are easy to check on the above bases.

Our final example which does not suit an interpretation in terms of families of projectors is the notion of a route faketor $\mathcal{E}$ induced by a strong monoidal functor $\bar{\mathcal{E}}$ into a Cartesian monoidal category $\mathbf{C}$, and the notion of an object (A, B, μ) of $\mathbf{Con}[\mathcal{E}]$ such that μ is a vwb-lens. For example the vwb-lens defined by:



(26)

Essentially defines the replacement of the action of f on a particular subsystem with the action of λ. The morphism (λ, f) encodes the constraint that the effect of f on its right hand input is completely determined by λ,



(27)

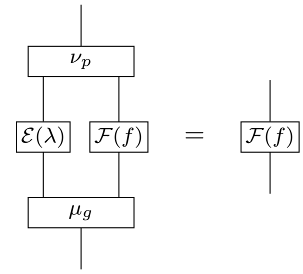
this in turn entails that the morphism f forbids signalling from any other system into the privileged system of the put p. Whilst replacement and projective constraints are very different in nature, we have found that they may be put under the same umbrella, via the key notions of a route faketor and an update structure.

6.2 Generalisation

To address the fact that there are many constructions for building physical theories from raw-material categories, but that the realisation of constraints for physical theories may only be expressible within the raw material category, we introduce a generalisation in which there is a separation between the category $\mathbf{C}$ in which a constraint is implemented, and the category $\mathbf{V}$ of morphisms which are interpreted as being constrained. This is simply captured by a strong monoidal (or compact, or $\dagger$-compact, etc., depending on the structure at hand) functor $\mathcal{F} : \mathbf{V} \to \mathbf{C}$.

$$
\begin{tikzcd}
& \mathbf{C} & \\
\mathbf{R} \arrow[ur, "\mathcal{E}"] & & \mathbf{V} \arrow[ul, "\mathcal{F}"']
\end{tikzcd}
$$

it is easy to check that the routed construction can be generalised, the objects of $\mathbf{Con}[\mathcal{E}, \mathcal{F}]$ are tuples (A, B, μ) in which $\mu : \mathcal{E}(A) \rightsquigarrow \mathcal{F}(B)$ is an update structure from $\mathcal{E}(A)$ to $\mathcal{F}(B)$. The compositional constraint $\mathcal{L}[\mathcal{E}]$ is defined by taking $\mathcal{L}[\mathcal{E},\mathcal{F}](\lambda)$ to be the set of all f such that



(28)

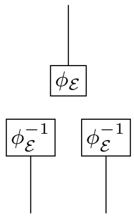
As a result one can immediately define routed categories based from the many physically inspired categorical constructions on **FHilb**, for example:

- There is a Strong †-Compact Functor $\mathcal{F} : \text{CPM}[\mathbf{C}] \to \mathbf{C}$
- There is a Strong Monoidal Functor $\mathcal{F} : \mathbf{Caus}[\mathbf{C}] \to \mathbf{C}$ from the higher order causal category [38] of processes built from a raw material category $\mathbf{C}$.

In particular the use of a functor from $\mathbf{Caus}[\mathbf{FHilb}]$ into **FHilb** is essential for imposing constraints on the causal higher order processes over quantum theory since non-trivial projectors and †-SCFAs are typically non-causal and so must be expressed inside **FHilb**.

6.3 Inheritance of additional Categorical Structure

Not only is $\mathcal{L}[\mathcal{E}]$ a compositional constraint, but it inherits a symmetric monoidal structure whenever it exists in $\mathcal{E}$. By the algebra-homorphism property of route faketors is follows that the image $\mathcal{E}(I_{\mathbf{R}})$ of the unit object in $\mathbf{R}$ is the special semigroup defined by the following magma and its inverse co-magma.

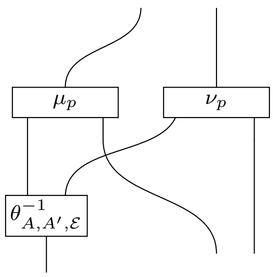


(29)

The definition of a symmetric monoidal structure for $\mathbf{Con}[\mathcal{E}]$ requires a notion of parallel composition of update structures, as well as a notion of unit object $(I_{\mathbf{R}}, I_{\mathbf{C}}, \mu)$ in which μ must be an update structure over the above algebra $\mathcal{E}(I_{\mathbf{R}})$.

Theorem 9 ($\mathbf{Con}[\mathcal{E}]$ is Symmetric Monoidal)**.** *The category* $\mathbf{Con}[\mathcal{E}]$ *is a symmetric monoidal category with*

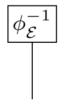
- $(A, B, \mu)\boxtimes(A', B', \nu) := (A\otimes A', B \otimes B', \mu\boxtimes\nu)$, *with* $(\mu\boxtimes\nu)_p$ *given by*



(30)

and similarly for $(\mu\boxtimes\nu)_g$.

- *Unit object (I, I, μ_I) where μ_I is given by*



(31)

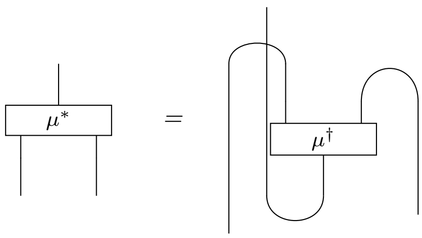
Proof. All that needs to be shown is that the above objects are well-defined in $\mathbf{Con}[\mathcal{E}]$, otherwise all symmetric mnoidal structure is inhertied directly from $\mathcal{R}$, proof is given in the Appendix, in which the result is show to hold for the generalised case of categories of the form $\mathbf{Con}[\mathcal{E}, \mathcal{F}]$. □

Theorem 10 (Monoidal Constraints from Monoidal Route-faketors)**.** *The compositional constraint $\mathcal{L}[\mathcal{E}]$ is monoidal whenever $\mathcal{E}$ is monoidal.*

Proof. Given in the appendix, again generalised to $\mathcal{L}[\mathcal{E}, \mathcal{F}]$ □

Furthermore the compositional constraint $\mathcal{L}[\mathcal{E}]$ actually inherits the entirety of the †-Compact structure of $\mathcal{E}$ whenever it exists.

Theorem 11 ($\mathcal{L}$ is Compact)**.** *For every †-Compact route faketor $\mathcal{E}$ the sub-category of $\mathbf{Con}[\mathcal{E}]$ given by restriction to partitions is a †-Compact category with the Dual $(A, B, \mu)^*$ of (A, B, μ) defined by (A^*, B^*, μ^*) where μ^* is defined by:*



(32)

Furthermore the induced compositional constraint $\mathcal{L}[\mathcal{E}]$ satisfies $\cup \subseteq \mathcal{L}[\mathcal{E}](\cup)$.

Proof. Given in the Appendix, again generalised to $\mathcal{L}[\mathcal{E}, \mathcal{F}]$ where $\mathcal{F}$ and $\mathcal{E}$ are †-Compact. □

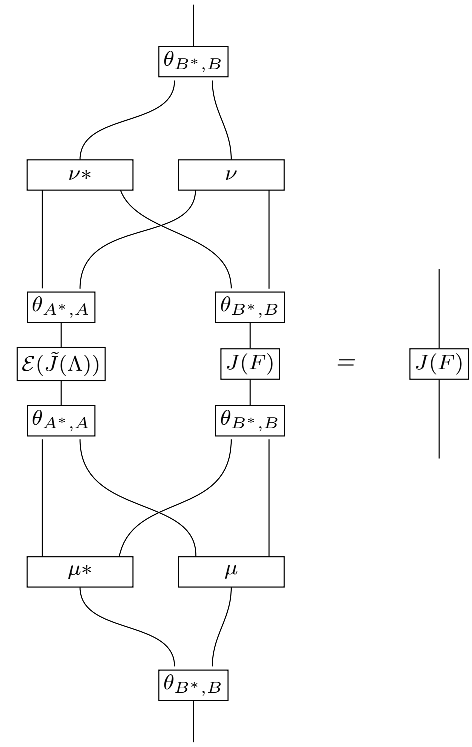
7 Calculations in Routed CPM categories

The CPM construction [39] is a well-known universal construction on dagger compact categories, which, for instance, can serve to build mixed quantum theory from pure quantum theory. From any †-compact category $\mathbf{C}$, one can build a †-compact category $\mathbf{CPM}[\mathbf{C}]$ whose objects are those of $\mathbf{C}$ and whose maps $A \to B$ are the completely positive maps $A^* \otimes A \to B^* \otimes B$ in $\mathbf{C}$. From our perspective, we already understand how to express constrains on the morphisms of $\mathbf{CPM}[\mathbf{C}]$ inside of the morphisms of $\mathbf{C}$ via the †-Compact Functor from $\mathbf{CPM}[\mathbf{C}]$ into $\mathbf{C}$. We however would like to construct an alternative in which there is a guarantee of co-ordination between the cpm construction of the route-category and the category of morphisms that will be routed.

Definition 24 (Completely Positive Route Construction)**.** *For every route functor between †-compact categories, the category $\mathbf{Con}_{\mathrm{CPM}}$ is defined by having objects (A, B, μ) where $\mu : \mathcal{E}(A) \rightsquigarrow B$. Morphisms and their composition are inherited from* $\mathrm{CPM}[\mathcal{R}]$ *by*

$$\mathbf{Con}_{\mathrm{CPM}}((A, B, \mu), (A', B', \mu')) := \mathrm{CPM}[\mathcal{R}](A, A')$$

which can be considered monoidal by $(A,B,M)\boxtimes(A',B',M') := (A\otimes A', B\otimes B', \mu\boxtimes\mu')$ *furthermore the compositional constraint* $\mathcal{L}_{\mathrm{CPM}} : \mathbf{Con}_{\mathrm{CPM}}[\mathcal{R}] \longrightarrow \mathrm{CPM}[\mathcal{C}]$ *is defined by* $\mathcal{L}_{\mathrm{CPM}}((A,B,M)) := B$ *and taking* $\mathcal{L}(\Lambda)$ *to be the set of all* $F \in \mathrm{CPM}(B,B')$ *such that:*



(33)

, (34)

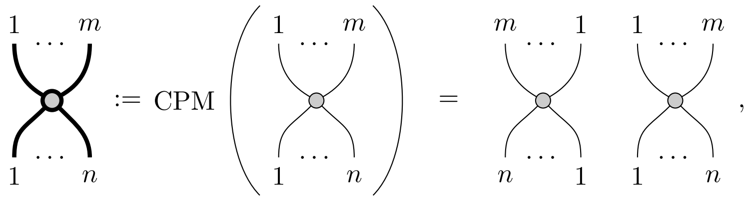
In particular, there are two important comments to be made on what $\mathrm{CPM}[\mathcal{C}]_{\mathcal{L}_{\mathrm{CPM}}[\mathcal{E}]}$ is *not* equivalent to. First, it is not equivalent to $\mathrm{CPM}[\mathcal{C}_{\mathcal{L}[\mathcal{E}]}]$; second, it is not equivalent to $\mathrm{CPM}[\mathcal{C}]_{\mathcal{L}[\mathcal{E}\circ\mathbf{J}_{\mathcal{R}},\mathbf{J}_{\mathcal{C}}]}$. These two facts can be seen for example in the construction of relational routes for **FHilb**.

7.1 Completely Positive Matching Routes in $\mathrm{CPM}[\mathbf{C}]$

The calculus in a constrained category is a double one: in the case of constraints constructed from route-faketors it is performed in parallel on the 'route' parts of the morphisms (living in **R**) on the one hand, and on the 'actual map' parts (living in **C**) on the other hand. One can often take advantage of this situation because the structure of **R** is usually much simpler than that of **C**; performing elementary calculus in **R** thus allows to directly deduce interesting properties about the result of the parallel calculus in **C**, 'bypassing' the latter. Here, we show an elementary example of this bypassing move, applied to the study of decoherence in the presence of index-matching routes. We shall see that the simple graphical calculus of **CoRel** allows one to witness the spread of decoherence in theories in a direct and intuitive way, without the need to compute anything in the category **C** itself.

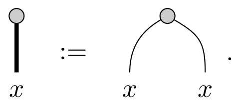
Our example will be in the case of matching route categories, as introduced in Section 5.2; we recall that a matching route category **Match**[**C**] and matching route faketor $\mathcal{E}_{\mathbf{Match}}$ is defined for any dagger compact category **C**. One can use the construction of Section 7 to define a category $\mathbf{W} := \mathrm{CPM}[\mathcal{C}]_{\mathcal{L}_{\mathrm{CPM}[\mathcal{E}_{Match}]}}$ of routed completely positive maps (Λ, F), with Λ living in CPM[**Match**[**C**]] and F living in CPM[**C**]. As **Match**[**C**] embeds into **CoRel**, the 'route part' of **W** can be understood as a corelation; the very simple calculus that the corelations are endowed with is what will unlock a simple decoherence calculus.

Let us introduce this calculus. Whereas the category of corelations **CoRel** captures the concept of perfect connections, the category of completely positive corelations CPM[**CoRel**] allows for a distinction between perfect decohered (or 'classical') and perfect coherent (or 'quantum') connections. The morphisms of CPM[**CoRel**] are generated by 1) the embedding of morphisms $\mathrm{CPM}(f) = f^* \otimes f$,



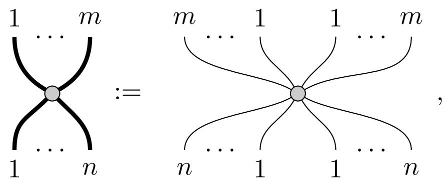
(35)

and 2) a discarding process, the cap from the †-compact structure of **CoRel**,



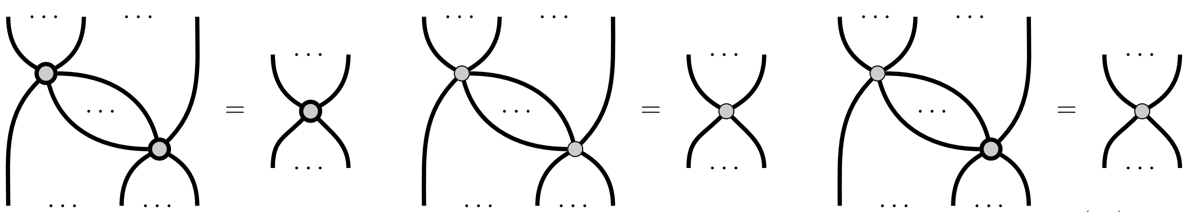
(36)

Introducing an additional generic notation for 'decoherent spiders',



(37)

the composition of any two morphisms of CPM[**CoRel**] can then be computed using 'bastard spider fusion':

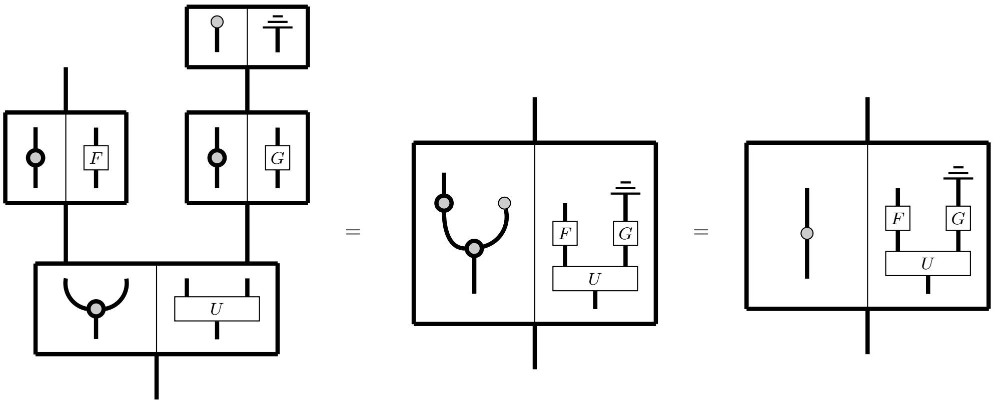


(38)

The soundness of each rule is easy to check; the crucial point is that decoherent spiders (i.e. the unbolded ones) always 'eat' bolded spiders.

7.2 Example Decoherence Calculation Using Completely Positive Index matching Routes

Let us now show in an example how this calculus can be used to reveal properties in CPM[**C**]. We will prove the following intuitive result: suppose that two parallel wires, with given partitions (in analogy with the case of **FHilb**, we interpret them as corresponding to partitions into subspaces), feature perfect and non-decohered correlations between these partitions (i.e., a state is in the k-th subspace of the left wire if and only if it is in the k-th subspace of the right wire); then tracing out one of these wires leads to the loss of any coherence between the subspaces in the other one. This property's interpretation is particularly nice in the context of quantum theory: copying an information and discarding one of the copies leads, in the other copy, to a complete loss of coherence between the alternatives that encode this information (including the case in which these alternatives correspond, not to states, but to subspaces). The proof of this fact in $\mathbf{W}$ is the following[9]:



(39)

As we can see, the proof of this intuitive yet non-trivial result has become completely straightforward; furthermore, it is now seen to be valid for any †-compact category $\mathbf{C}$. These two facts find their origin in our ability to directly use the graphical calculus of **CoRel** and bypass the need for any computation in $\mathbf{C}$ itself.

The above example is elementary, but it scales up nicely to more general situations involving perfect correlations. Implementing constraint calculus in the case of non-perfect correlations is also possible, using relational route categories. This constraint calculus would then amount to calculus in **Rel**, which is not necessarily graphical, but is still in general way easier to handle than calculus in $\mathbf{C}$ itself.

8 Conclusion

In this work, we described *constraint categories*, a general structure that allows to endow a † symmetric monoidal (or †-compact) category $\mathbf{R}$ with the interpretation of representing constraints for the morphisms of another †-symmetric monoidal (or †-compact) category $\mathbf{C}$. We constructed several general examples of a sub-class denoted *routed categories*: these are relations-like examples that exist for any †-SMC enriched in monoids; corelations-like ones that exist for any †-compact category; and examples based on delete-copy algebras, defined in cartesian monoidal categories. We showed that, given a category $\mathbf{C}$ and a route category $\mathbf{R}$ for it, one can combine them into a *routed category* that features a parallel calculus, taking place both in $\mathbf{R}$ and in $\mathbf{C}$. Extending slightly the definition of a route category, we showed that, given a route category $\mathbf{R}$ for a category $\mathbf{C}$, there exists a canonical way of taking CPM[$\mathbf{R}$] to be a route category for CPM[$\mathbf{C}$]. Finally, we showed on a simple example how the parallel calculus happening in routed categories can have practical applications, by allowing one to bypass some of the calculus of an intricate category ($\mathbf{C}$) by doing calculus in a simpler one ($\mathbf{R}$) instead.

[9] The ground symbol represents the trace [40, 41] which exists in CPM[$\mathbf{C}$] for any †-compact category $\mathbf{C}$ [39].

A first outcome of this work is to give a neat formal background to the constructions introduced by Ref. [14] to describe sectorial constraints in **FHilb**. A second outcome is to extend them in terms of their range of application: we provided an abstract framework that allows to describe such constraints for any SMC, using special semigroups rather than the much more stringent †-special commutative Frobenius algebras that were required in the case of sectorial constraints. This allowed us, for instance, to connect our constructions to apparently unrelated structures, such as lenses. A third outcome is to have shed some new light on the structural connections between well-known categories, such as **Rel** and **Hilb**.

A future direction of investigation would be to apply our constructions to the modelisation of constraints in other contexts. Another one, more focused on the study of quantum theory, would be to take advantage of their general nature in order to model more quantum scenarios. The fact that we have been describing route categories for **Hilb**, for instance, opens the way for a formalisation of sectorial constraints in infinite-dimensional Hilbert spaces. Another direction would be to study both the relation of these constructions with, and their applications to, the study of causality in quantum theory. Besides causal decompositions, whose description requires the use of matching routes, indefinite causal order, that has recently been the subject of a lot of investigation, has deep connections with routed circuits [18]; it could thus be fruitful to apply the idea of routes to recent categorical investigations into causal structure [38], following the comments made in Section 6.2 on the application of constraints to **Caus**[**C**].

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References

[1] Giulio Chiribella and Hlér Kristjánsson. Quantum Shannon theory with superpositions of trajectories. *Proceedings of the Royal Society A*, 475, 2019.

[2] Hlér Kristjánsson, Wen-Xu Mao, and Giulio Chiribella. Single-particle communication through correlated noise. *arXiv preprint arXiv:2004.06090*, 2020.

[3] Hlér Kristjánsson, Giulio Chiribella, Sina Salek, Daniel Ebler, and Matthew Wilson. Resource theories of communication. *New Journal of Physics*, 22(7):073014, jul 2020.

[4] Johan Åberg. Subspace preservation, subspace locality, and gluing of completely positive maps. *Annals of Physics*, 313(2):326–367, 2004.

[5] Johan Åberg. Operations and single-particle interferometry. *Physical Review A*, 70(1):012103, 2004.

[6] Daniel KL Oi. Interference of quantum channels. *Physical Review Letters*, 91(6):067902, 2003.

[7] Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. Quantum circuits cannot control unknown operations. *New Journal of Physics*, 16(9):093026, sep 2014.

[8] Nicolai Friis, Vedran Dunjko, Wolfgang Dür, and Hans J. Briegel. Implementing quantum control for unknown subroutines. *Phys. Rev. A*, 89:030303, Mar 2014.

[9] Nicolai Friis, Alexey A. Melnikov, Gerhard Kirchmair, and Hans J. Briegel. Coherent controlization using superconducting qubits. *Scientific Reports*, 5(1), Dec 2015.

[10] V Dunjko, N Friis, and H J Briegel. Quantum-enhanced deliberation of learning agents using trapped ions. *New Journal of Physics*, 17(2):023006, jan 2015.

[11] Jayne Thompson, Kavan Modi, Vlatko Vedral, and Mile Gu. Quantum plug n' play: modular computation in the quantum regime. *New Journal of Physics*, 20(1):013004, jan 2018.

[12] Qingxiuxiong Dong, Shojun Nakayama, Akihito Soeda, and Mio Murao. Controlled quantum operations and combs, and their applications to universal controllization of divisible unitary operations. *arXiv preprint arXiv:1911.01645*, 2019.

[13] Alastair A Abbott, Julian Wechs, Dominic Horsman, Mehdi Mhalla, and Cyril Branciard. Communication through coherent control of quantum channels. *Quantum*, 4:333, 2020.

[14] Augustin Vanrietvelde, Hlér Kristjánsson, and Jonathan Barrett. Routed quantum circuits, 2020.

[15] John-Mark A. Allen, Jonathan Barrett, Dominic C. Horsman, Ciarán M. Lee, and Robert W. Spekkens. Quantum common causes and quantum causal models. *Phys. Rev. X*, 7:031021, Jul 2017.

[16] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. Quantum causal models, 2019.

[17] Robin Lorenz and Jonathan Barrett. Causal and compositional structure of unitary transformations, 2020.

[18] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. Cyclic quantum causal models. *Nature Communications*, 12(1), Feb 2021.

[19] Bob Coecke and Aleks Kissinger. *Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning.* Cambridge University Press, 3 2017.

[20] Chris Heunen and Jamie Vicary. Introduction to Categorical Quantum Mechanics. Technical report, 2013.

[21] Bob Coecke, Dusko Pavlovic, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 23(3):555–567, 2013.

[22] Bob Coecke and Dusko Pavlovic. Quantum measurements without sums. In *Mathematics of Quantum Computation and Quantum Technology*, pages 559–596. CRC Press, 1 2007.

[23] Samson Abramsky and Chris Heunen. H*-algebras and nonunital frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics, 2011.

[24] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1971.

[25] Brendan Fong. Decorated cospans. *Theory and Applications of Categories*, 30:1096–1120, 8 2015.

[26] Aurelio Carboni. Matrices, relations, and group representations. *Journal of Algebra*, 136(2):497–529, 2 1991.

[27] R Rosebrugh, N Sabadini, and R F C Walters. GENERIC COMMUTATIVE SEPARABLE ALGEBRAS AND COSPANS OF GRAPHS. Technical Report 6, 2005.

[28] F. Gadducci and R. Heckel. An inductive view of graph transformation. In *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, volume 1376, pages 223–237. Springer Verlag, 1998.

[29] Jason Morton. Belief propagation in monoidal categories. In *Electronic Proceedings in Theoretical Computer Science, EPTCS*, volume 172, pages 262–269. Open Publishing Association, 12 2014.

[30] Aleks Kissinger. Finite matrices are complete for (dagger-)hypergraph categories. 6 2014.

[31] Brandon Coya and Brendan Fong. Corelations are the prop for extraspecial commutative frobenius monoids, 2016.

[32] Matthew Wilson, James Hefford, Guillaume Boisseau, and Vincent Wang. The safari of update structures: Visiting the lens and quantum enclosures. In David I. Spivak and Jamie Vicary, editors, Proceedings of the 3rd Annual International *Applied Category Theory Conference 2020,* Cambridge, USA, 6-10th July 2020, volume 333 of *Electronic Proceedings in Theoretical Computer Science*, pages 1–18. Open Publishing Association, 2021.

[33] James Hefford, Vincent Wang, and Matthew Wilson. Categories of Semantic Concepts. *arXiv*, 4 2020.

[34] J. Nathan Foster, Michael B. Greenwald, Jonathan T. Moore, Benjamin C. Pierce, and Alan Schmitt. Combinators for bidirectional tree transformations: A linguistic approach to the view-update problem. In *ACM Transactions on Programming Languages and Systems*, volume 29, page 17, 5 2007.

[35] Mitchell Riley. Categories of Optics. 9 2018.

[36] F. Bancilhon and N. Spyratos. Update Semantics of Relational Views. *ACM Transactions on Database Systems (TODS)*, 6(4):557–575, 12 1981.

[37] Georg Gottlob, Paolo Paolini, and Roberto Zicari. Properties and update semantics of consistent views. *ACM Transactions on Database Systems (TODS)*, 13(4):486–524, 1 1988.

[38] Aleks Kissinger and Sander Uijlen. A categorical semantics for causal structure. *Logical Methods in Computer Science*, 15(3), 2019.

[39] Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 03 2007.

[40] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Quantum from Principles. In *Fundamental Theories of Physics*, volume 181, pages 171–221. Springer, 2016.

[41] Bob Coecke and Raymond Lal. Causal Categories: Relativistically Interacting Processes. *Foundations of Physics*, 43(4):458–501, 4 2013.

Appendix

A Relation with CP* and the Karoubi envelope of CPM

In this Appendix, we discuss how our constructions relate to two previous categorical constructions: the CP* construction, and the Karoubi envelope of the CPM construction. More precisely, our aim is, in the light of discussions with colleagues, to provide a reply to worries that the structures we capture might have been formalised already by these earlier frameworks, or might be relatively straightforward to formalise when taking them as a starting point. For concreteness, we will here take $\mathbf{C} = \mathbf{FHilb}$.

It might be argued, for instance, that the CP* construction, when applied to **FHilb**, already provides a way of modelling orthogonal partitions of Hilbert spaces (or equivalently, finite-dimensional C*-algebras)[10], and that it could thus provide a sufficient basis for the formalisation of our two main examples of practical applications (superpositions of paths and causal decompositions). However, CP*[**FHilb**] does not include a large portion of the maps about which we wish to express sectorial constraints.

Indeed, suppose we take a first object A in CP*[**FHilb**] corresponding to a non-partitioned Hilbert space $\mathcal{H}_A$, and another object B that corresponds to a Hilbert space with a non-trivial decomposition into sectors, $\mathcal{H}_B = \oplus_k \mathcal{H}_B^k$. Then, the maps $A \to B$ in CP*[**FHilb**] are those CP maps from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ whose outputs are completely decoherent with respect to the partition of $\mathcal{H}_B$. On the contrary, it is crucial for our purposes to be able to write CP maps $A \to B$ whose output feature coherence between B's sectors; such maps are for example those that 'create an index' at the bottom of the diagrams for superpositions of paths and causal decompositions in Ref. [14][11].

[10] Namely, any object in CP*[**FHilb**] can be characterised as a finite-dimensional C* algebra, or, equivalently, as a Hilbert space $\mathcal{H}_A$ with a preferred partition into sectors, $\mathcal{H}_A := \oplus_{k\in K} \mathcal{H}_A^k$.

[11] In unitary scenarios, it could be argued that one could just write down a different formalisation, in which the domain of such a map is also partitioned with 'the same index' as its codomain, so that the maps don't need to create the index anymore. However, this is not suitable for us as 1) these partitions of the domain will in general

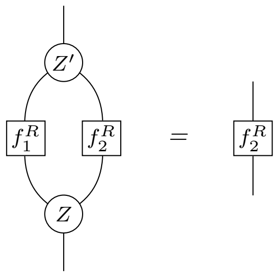
The same argument can be made about the use of the Karoubi envelope of CPM[**FHilb**]. Indeed, the objects of the latter are able to encode orthogonal partitions of Hilbert spaces only inasmuch as they forbid the presence of coherence between the sectors of said partitions.

B 2-Categorical Structure of Route-Faketors

To show that under certain conditions a route functor defines an oplax functor, it must first be shown that a 2-Categorical structure emerges from those conditions.

Lemma 5. *Any category* $\mathbf{EW}[\mathbf{C}]$ *with loosened-re-normalisation can be viewed as a* 2*-category by poset enrichment of each* $\mathbf{EW}[\mathbf{C}](Z, Z')$ *via the relation*

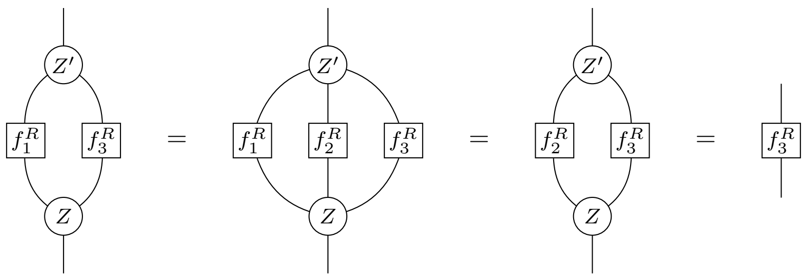
$f \leq f' \iff$



Proof. For vertical composition of 2-Morphisms to be defined translates to the requiremetn that

$$f_1 \leq f_2 \text{ and } f_2 \leq f_3 \implies f_1 \leq f_3$$

Which is easy to check.

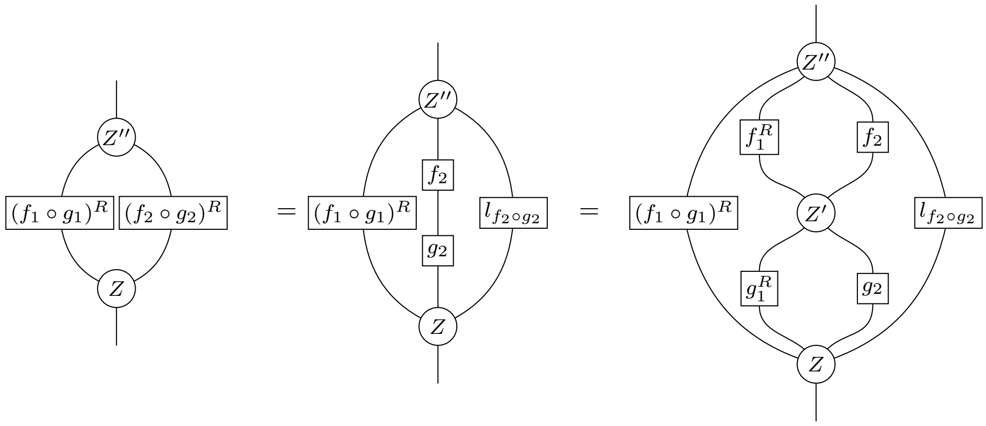


For horizontal composition of 2-Morphisms to be defined translates to requiring that

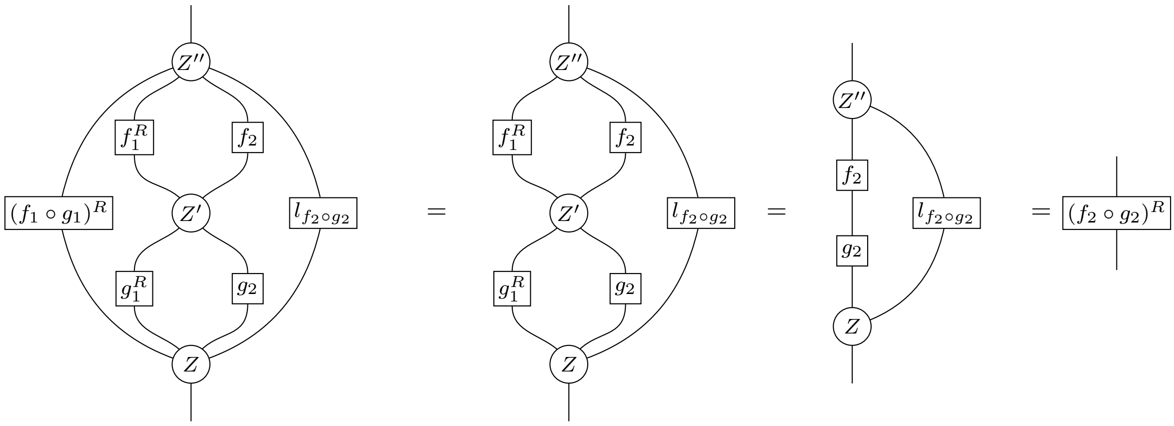
$$f_1 \leq f_2 \text{ and } g_1 \leq g_2 \implies (f_1 \circ g_1) \leq (f_2 \circ g_2)$$

have no physical significance, 2) this will not generalise to the non-unitary case, and 3) even in the unitary case, this will not be possible in general if the domain is a tensor product of several objects, as the wanted partition will not necessarily intersect this factorisation nicely.

This can be confirmed first by using the re-normalisation condition along with [$f_1 \leq f_2$ and $g_1 \leq g_2$



and using the loosening condition and undoing the previous steps



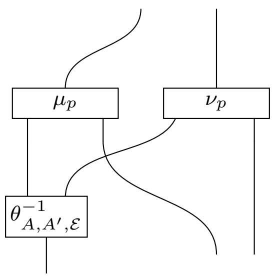
The coherence conditions required for the definition of a 2-category including the interchange law are all immediately satisfied by the uniqueness of the defined 2-Morphisms. $\square$

C Symmetric Monoidal Structure of Routed Categories

Where appropriate we will often include dotted lines to represent identity systems for readability.

Theorem 12 (**Con**[$\mathcal{E}$] is Symmetric Monoidal)**.** *The category* **Con**[$\mathcal{E}$] *is a symmetric monoidal category with*

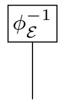
- $(A, B, \mu) \boxtimes (A', B', \nu) := (A \otimes A', B \otimes B', \mu \boxtimes \nu)$, *with* $(\mu \boxtimes \nu)_p$ *given by*



(40)

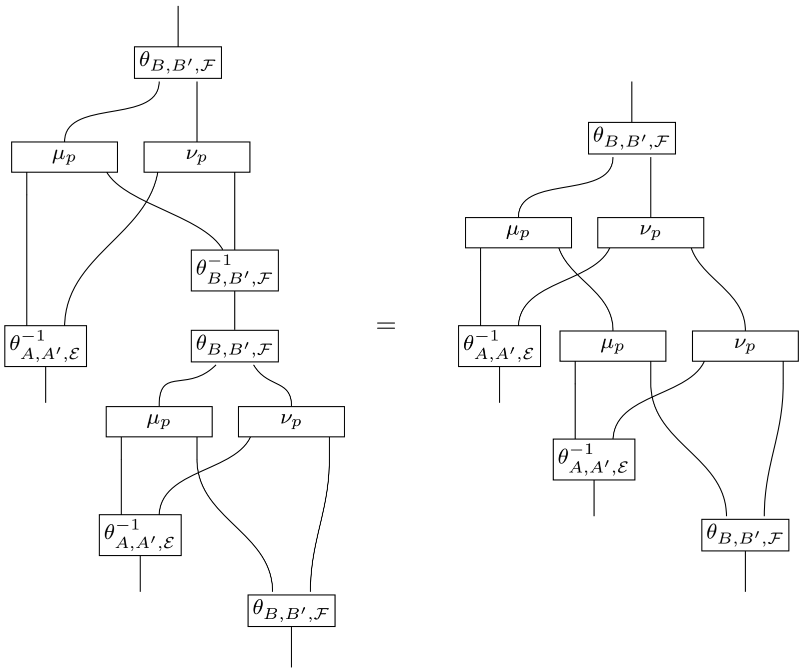
and similarly for $(\mu \boxtimes \nu)_g$.

- *Unit object (I, I, μ_I) where μ_I is given by*



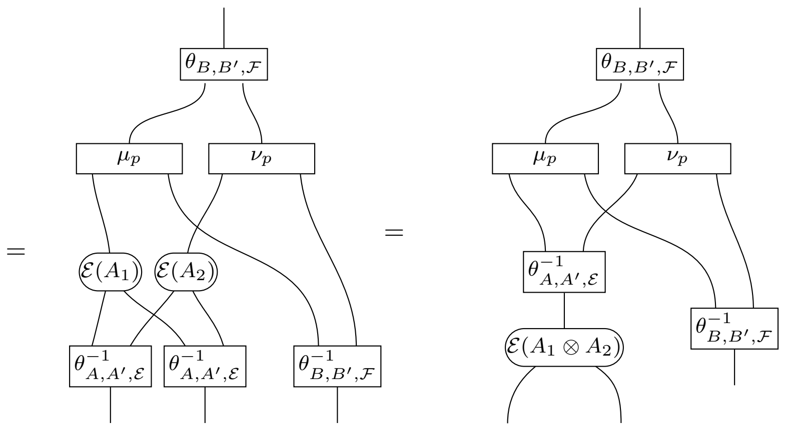
(41)

Proof. A fully algebraic proof is simple but tedious, we give the reader the outline in terms of string diagrams. We first show that the tensor product of two update structures is an update structure of the required type. beginning with the module law.



(42)

By definition of a frobenius erstaz functor θ must be a special semigroup homomorphism

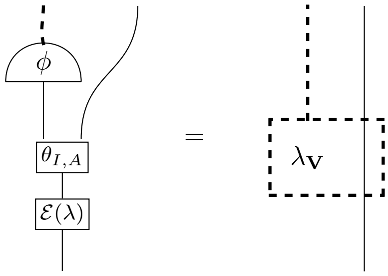


(43)

The proof that the co-module equation is satisfied is identical, as are the proofs of the `GetPut` and `PutGet` laws. □

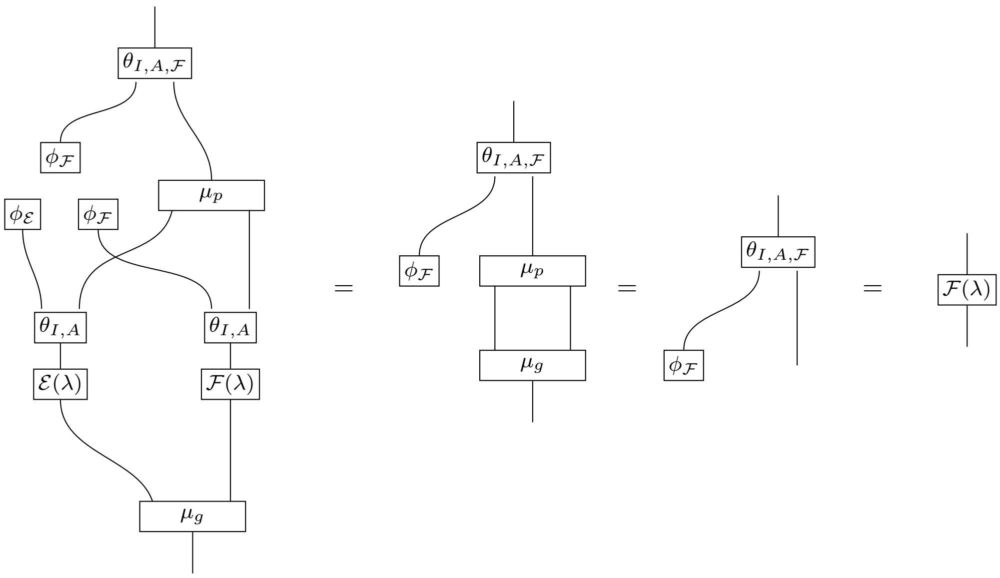
Theorem 13 (Monoidal Constraints from Monoidal Route-faketors)**.** *The compositional constraint $\mathcal{L}[\mathcal{E}, \mathcal{F}]$ is monoidal whenever $\mathcal{E}$ and $\mathcal{F}$ are monoidal.*

Proof. One can use the fact that $\mathcal{E}$ is a route faketor and that $\mathcal{F}$ is a functor to show that $\lambda_{\mathcal{V}} \in \mathcal{L}(\lambda_{\mathbf{Con}})$, $\alpha_{\mathcal{V}} \in \mathcal{L}(\alpha_{\mathbf{Con}})$ and $\beta_{\mathcal{V}} \in \mathcal{L}(\beta_{\mathbf{Con}})$. Since $\mathcal{E}$ is strong monoidal as a faketor:



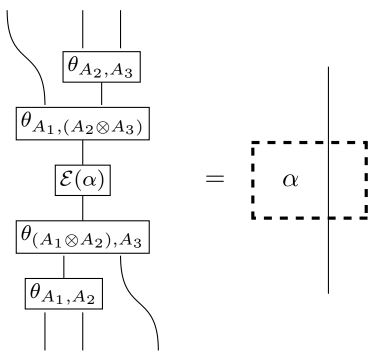
(44)

and similarly for $\mathcal{F}$ as a functor. In turn this coherence condition can be used to confirm that $\lambda_{\mathcal{V}} \in \mathcal{L}(\lambda_{\mathbf{Con}})$ I.E



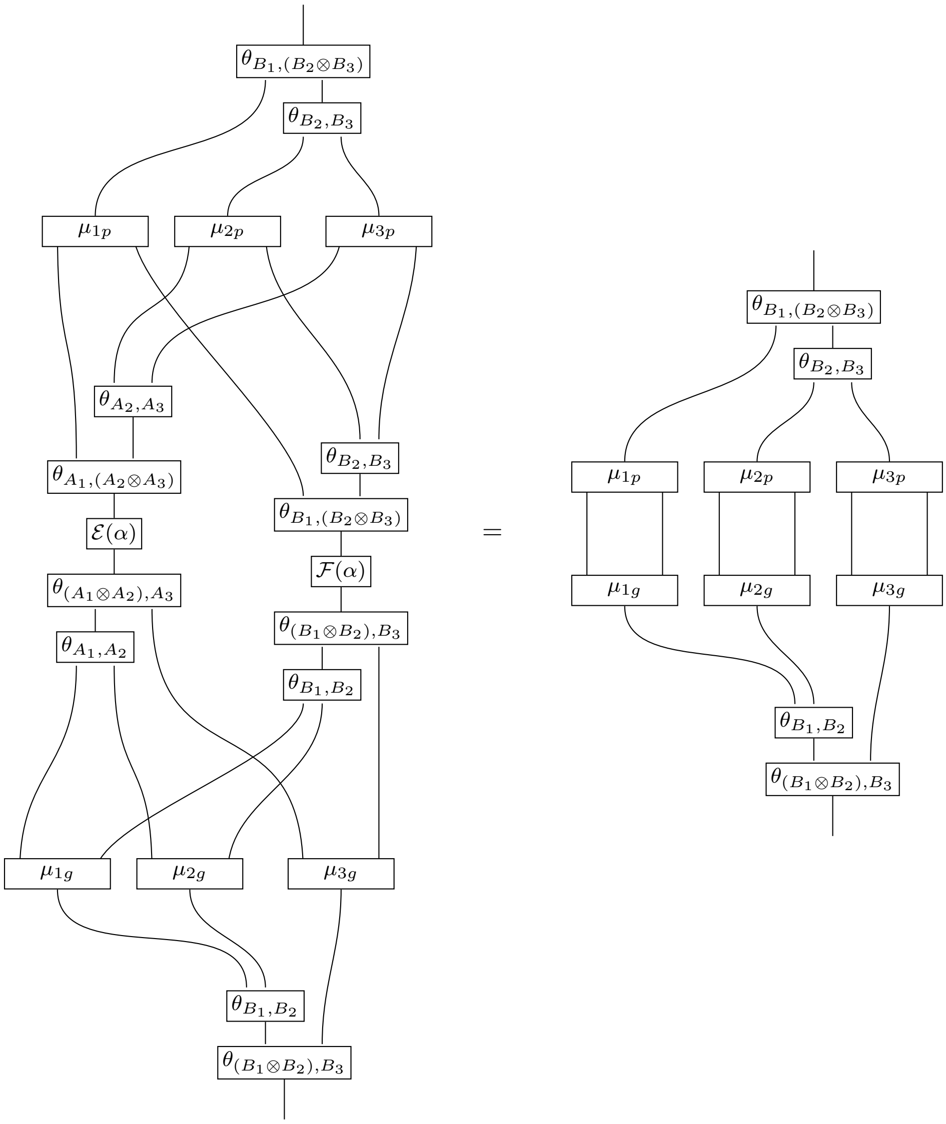
(45)

Next we consider the associator, similarly again since $\mathcal{E}$ is strong monoidal



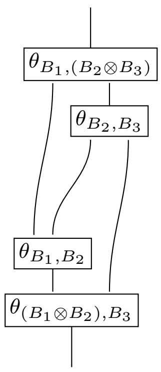
(46)

Which in turn implies that $\alpha_{\mathcal{V}} \in \mathcal{L}(\alpha_{\mathbf{Con}})$ since using the fact that $\mathcal{E}$ and $\mathcal{F}$ are monoidal

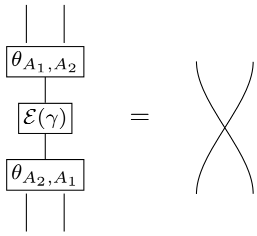


(47)

and using the `GetPut` condition

$=$  $= \boxed{\mathcal{F}(\alpha)}$ (48)

Finally since $\mathcal{E}$ is symmetric

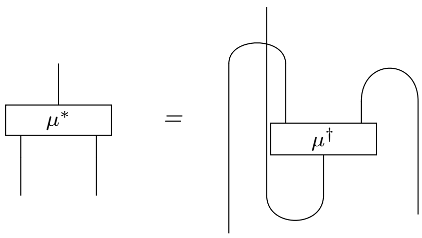


(49)

Which entails that $\beta_{\mathcal{V}} \in \mathcal{L}(\beta_{\mathbf{Con}})$ by the same steps as for unitors and associators. The last condition to be checked is that when f follows λ and g follows τ then $f \otimes g$ indeed follows $\lambda \otimes \tau$ in other words the monoidal laxity condition $\mathcal{L}(\tau) \otimes \mathcal{L}(\lambda) \subseteq \mathcal{L}(\tau \otimes \lambda)$. This in fact follows from the naturality conditions for $\theta_{\mathcal{E}}$ and $\theta_{\mathcal{F}}$ and the same series of steps as for the above proofs. $\square$

C.1 Compact Structure of Routed Categories

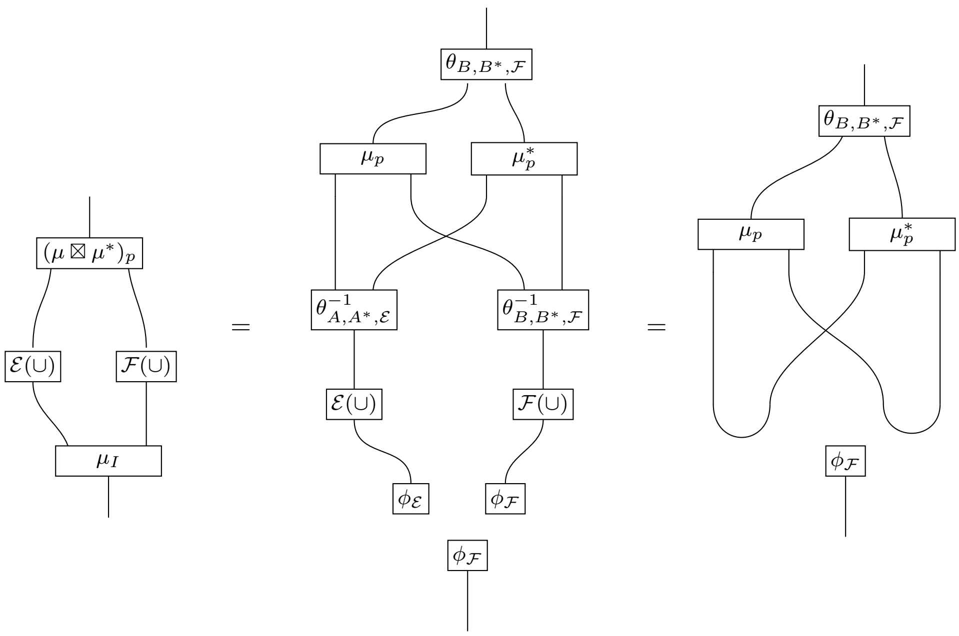
Theorem 14 ($\mathcal{L}$ is Compact)**.** *For every †-Compact route faketor $\mathcal{E}$ the sub-category of* $\mathbf{Con}[\mathcal{E}]$ *given by restriction to partitions is a †-Compact category with the Dual $(A, B, \mu)^*$ of (A, B, μ) defined by (A^*, B^*, μ^*) where μ^* is defined by:*



(50)

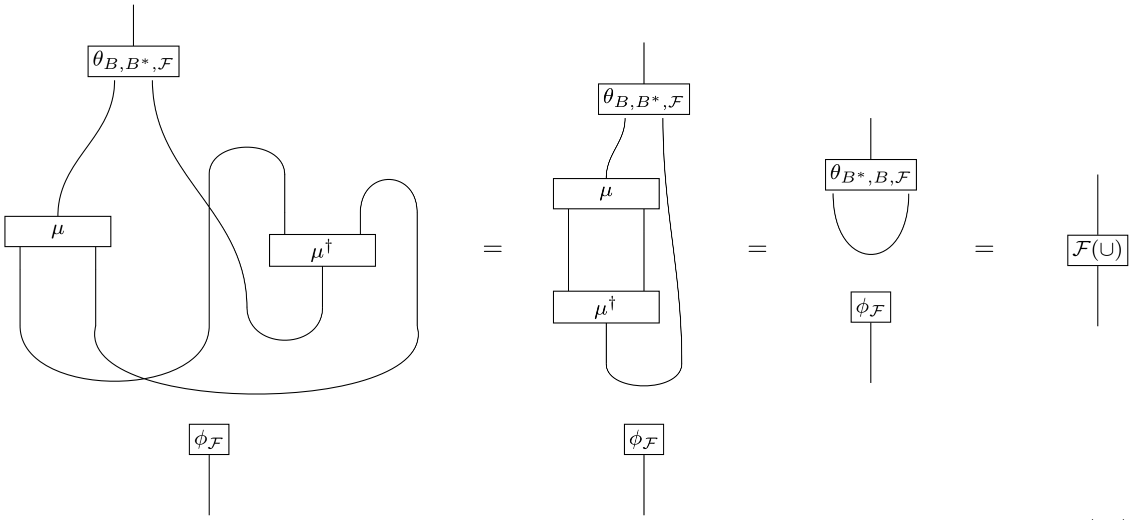
Furthermore the induced compositional constraint $\mathcal{L}[\mathcal{E}]$ satisfies $\cup \subseteq \mathcal{L}[\mathcal{E}](\cup)$.

Proof. We begin by using the fact that $\mathcal{E}$ is †-Compact



(51)

and then use the definition of μ^* and $\dagger$-module equations to show that $\cup \in \mathcal{L}(\cup)$



(52)

The proof is identical for $\cap \in \mathcal{L}(\cap)$. ☐

D Existence of a route-faketor from fRel

Theorem 15. *Let* $\mathbf{C}$ *be a dagger-SMC with a* $\mathbf{0}$ *object. For every component-full subset* $S \subseteq C_{\perp \mathbf{C}}$ *such that* $\mathcal{E}(Z_\lambda) \in S$ *there exists a route faketor*

$$\mathcal{E} : \mathbf{fRel}_{\times S} \longrightarrow \mathbf{Sg}[\mathbf{C}]$$

Proof. A faketor $\mathcal{E} : \mathbf{fRel}_{\times S} \longrightarrow \mathbf{Sg}[\mathbf{C}]$ on objects can be defined inductively using the function $\mathcal{E}$ by

$$\mathcal{E} : S_1 \times S_2 \mapsto \mathcal{E}(S_1) \otimes \mathcal{E}(S_2)$$

Which is well defined since the cartesian product decomposition of any object in **Rel** is unique. Each object S_A is a cartesian product $\times_i S_i$ and so each element $a \in S_A$ is a tuple of copyable states $a = \times_i |i\rangle$. From each element a one can uniquely define $|a\rangle$ by $|a\rangle = \otimes_i |i\rangle$ with the bracketing of $|a\rangle$ inherited from the bracketing of a. For a relation $\lambda : S_A \to S_B$, I.E a relation $\lambda : \{a\} \to \{b\}$ then define $\mathcal{E}(\lambda)$ by

$$\langle b| \mathcal{E}(\lambda) |a\rangle := 1 \text{ if } a \overset{\lambda}{\sim} b$$
$$\langle b| \mathcal{E}(\lambda) |a\rangle = 0 \text{ if } \textbf{Else}$$

Then $\mathcal{E}$ is a faketor since for all a, b:

$$\langle b| \mathcal{E}(1) |a\rangle = \delta_a^b = \langle b| 1 |a\rangle$$

and so since a, b are orthonormal bases $\mathcal{E}(1) = 1$. $\mathcal{E}$ is Symmetric Monoidal since $\mathcal{E}(\{1\}) = Z_\lambda$ and by definiion $\mathcal{E}(S_1 \times S_2) = \mathcal{E}(S_1) \otimes \mathcal{E}(S_2)$. Furthermore if $\mathbf{C}$ is $\dagger$-Compact then its easy to show that $\mathcal{E}(\cup) = \cup$. Finally using any orthonormal bases (or tuples there-of) $\{a\}, \{b\}, \{c\}$ the loosening condition reads

$$\langle c| \mathcal{E}(\sigma \circ \lambda) |a\rangle \langle c| \mathcal{E}(\sigma) |b\rangle \langle b| \mathcal{E}(\lambda) |a\rangle = \langle c| \mathcal{E}(\sigma) |b\rangle \langle b| \mathcal{E}(\lambda) |a\rangle$$

By translating to definition of $\mathcal{E}$ we see that $\langle c| \mathcal{E}(\sigma \circ \lambda) |a\rangle = 0 \implies \nexists b$ such that $\langle c| \mathcal{E}(\sigma) |b\rangle = \langle b| \mathcal{E}(\lambda) |a\rangle = 1$ and so $\langle c| \mathcal{E}(\sigma \circ \lambda) |a\rangle \langle c| \mathcal{E}(\sigma) |b\rangle \langle b| \mathcal{E}(\lambda) |a\rangle = 0 = \langle c| \mathcal{E}(\sigma) |b\rangle \langle b| \mathcal{E}(\lambda) |a\rangle$. Clearly when $\langle c| \mathcal{E}(\sigma \circ \lambda) |a\rangle = 1$ the condition is trivially satisfied, so indeed $\mathcal{E}$ is a route faketor. $\square$