Network Sheaves Valued in Categories of Adjunctions and their Laplacians

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Abstract

We report past and ongoing efforts to compute (global) sections of cellular sheaves valued in categories of adjunctions. First, we discuss previous work on sheaves valued in $\mathcal{L}tc$, the category of complete lattices and Galois connections. Then, we survey ongoing efforts to generalize the fixed point theorem (Theorem 1) to (i) $\mathcal{C}at\mathcal{A}dj$, the category of (sufficiently small) categories and adjunctions, and (ii) $\mathbb{M}$-$\mathcal{A}dj$, the category of ordered monoid -enriched categories and adjunctions.

Cellular sheaves are sheaves with coeficients in a category $\mathcal{D}$ whose base space is an Alexandrov space of a certain poset—a face relation poset—encoding the gluing of cells in a cell complex X (vertices and edges in a graph, perhaps) onto one another [1, 2]. If $\mathcal{D}$ is complete, a folk theorem (full proof is supplied in [3]) is that the category of sheaves over the Alexandrov topology of a face relation poset $\mathbf{P}_X$ is equivalent to the category of functors and natural transformations,

$$[\mathbf{P}_X, \mathcal{D}].$$

One motivation for extending the theory of cellular sheaves beyond categories of vector spaces [1] and categories of inner product spaces [4] lies in graph signal proccessing. Standard graph filtering techniques not only require input signals to be collected in vector spaces with a homogeneous number of features, but are unnameable to data types that are not vector-valued, such as set-valued data types e.g. arising in semantics of multi-agent systems [5] or recommendation systems [6].

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In this extended abstract, we consider cellular sheaves over a graph $G = (V_G, E_G)$ valued in a category of adjunctions, i.e. functors

$$\mathcal{F} : \mathbf{P}_G \to \mathcal{A}dj$$

—for brevity, we narrow our focus to sheaves over graphs, but many of the results readily generalize to sheaves over cell complexes. Previous work [7] has addressed the particular case where $\mathcal{A}dj$ is $\mathcal{L}tc$, the category of lattices and Galois connections. Objects of $\mathcal{L}tc$ are complete lattices, i.e. posets that are complete (limits denoted $\bigvee$) in the categorical sense—and consequently cocomplete (colimits denoted $\bigwedge$) [9, Theorem 2.2]. Morphisms in $\mathcal{L}tc$ consist of adjunctions (i.e. monotone Galois connections [8]) $F_\bullet \dashv F^\bullet$ between complete lattices,

$$\mathbf{L} \underset{F_\bullet}{\overset{F^\bullet}{\leftrightarrows}} \mathbf{L}'.$$

Adjunctions are composed in the usual way; the identitiy morphism is the identity adjunction, $1_{\mathbf{L}} \dashv 1_{\mathbf{L}}$.

We summarize some key results of our work [7] as follows:

- We construct a Laplacian—the **Tarksi Laplacian** denoted L—associated to a given sheaf

  $$\mathcal{F} : \mathbf{P}_G \to \mathcal{L}tc$$

  which acts as a diffusion operator on the (product) lattice of 0-cochians,

  $$C^0(G; \mathcal{F}) := \prod_{v \in V_G} \mathcal{F}(v).$$

  Explicitly, the Tarski Laplacian is an order-preserving map,

  $$L : \prod_{v \in V_G} \mathcal{F}(v) \to \prod_{v \in V_G} \mathcal{F}(v),$$

  $$(L\mathbf{x})_v = \bigwedge_{e \in \delta(v)} (\mathcal{F}_{v \triangleleft e})^\bullet \left(\bigwedge_{w \in \partial(e)} (\mathcal{F}_{w \triangleleft e})_\bullet (x_w) \right).$$

- Via the Tarski Laplacian, we show how to compute (guaranteeing existence *gratis* by invoking the Tarski Fixed Point Theorem [10]) a limit,

  $$\lim \left(\underline{\mathcal{F}} : \mathbf{P}_G \to \mathcal{S}up \right),$$

  (a complete lattice) whose elements are called **sections**. In order that sections exist, we pass to the category $\mathcal{S}up$ of complete lattices and completely $\bigvee$-preserving maps (continuous functors). The category $\mathcal{S}up$ being complete [11], functors factoring through $\mathcal{F}$, i.e. functors $\underline{\mathcal{F}}$

  $$\begin{array}{ccc} \mathbf{P}_G & \xrightarrow{\mathcal{F}} & \mathcal{L}tc \\ & \underset{\underline{\mathcal{F}}}{\searrow} & \downarrow U_{\text{left}} \\ & & \mathcal{S}up \end{array},$$

into $\mathcal{S}up$, have all limits.[1]

- We prove the following fixed point theorem:

  Theorem 1 ([7, Theorem 3.1])**.** *Let $\mathcal{F} : \mathbf{P}_G \to \mathcal{L}tc$ be a cellular sheaf over G. Suppose $\underline{\mathcal{F}} : \mathbf{P}_G \to \mathcal{S}up$ is the functor $U_{\text{left}} \circ \mathcal{F}$. Then,*

  $$\lim \underline{\mathcal{F}} = \mathrm{Post}(L)$$

  where $\mathrm{Post}(L) := \mathrm{Fix}(L \wedge id) = \{\mathbf{x} \in C^0(G;\mathcal{F}) : L(\mathbf{x}) \geq \mathbf{x}\}$.

In work in progress, we seek to generalize the above results (especially the fixed point theorem) for various choices of $\mathcal{A}dj$.

1. Suppose $\mathcal{A}dj$ is the 2-category $\mathcal{C}at\mathcal{A}dj$, the category of ($\mathbb{U}$-small) categories (for some Grothendieck universe $\mathbb{U}$) and adjunctions (cf. [12]).

   (a) A **cellular stack** over G is a functor $\mathcal{C} : \mathbf{P}_G \to \mathcal{C}at\mathcal{A}dj$.

   (b) The Laplacian L is the endofunctor $L : \prod_{v \in V_G} \mathcal{C}(v) \to \prod_{v \in V_G} \mathcal{C}(v)$ given by

   $$(L\mathbf{X})_v := \prod_{e \in \delta(v)} \prod_{w \in \delta(e)} (\mathcal{F}_{v \lhd e})^{\bullet} (\mathcal{F}_{w \lhd e})_{\bullet} (X_w).$$

   For convenience, we supply an additional endofunctor called the **parison functor**[2],

   $$(\Delta\mathbf{X})_v := \prod_{e \in \delta(v)} (\mathcal{F}_{v \lhd e})^{\bullet} (\mathcal{F}_{v \lhd e})_{\bullet} (X_v),$$

   which—in $\mathcal{L}tc$ coefficients—is the (point-wise) meet ($\wedge$) of expanding maps.

   (c) The following imitates Theorem 1.

   Theorem 2. *Suppose $\mathcal{C}$ is a cellular stack over G. Then, the category* $\mathrm{Post}(L) := \{f : \mathbf{X} \to L(\mathbf{X}) \mid \mu_{\mathbf{X}} \circ Lf \circ f = \eta_{\mathbf{X}}\}$*—as in the following commutative diagram*[3]

   $$\begin{array}{ccccc} \mathbf{X} & \xrightarrow{f} & L(\mathbf{X}) & \xrightarrow{Lf} & L^2(\mathbf{X}) \\ & \searrow^{\eta_{\mathbf{X}}} & & \swarrow_{\mu_{\mathbf{X}}} & \\ & & \Delta(\mathbf{X}) & & \end{array}$$

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[1] U_{left} is the functor that forgets right adjoints (remembers left adjoints) in $\mathcal{L}tc$. In slight abuse of notation, we denote U_{left} for the functor that forgets right adjoints in any category of categories and adjunctions and $\underline{\mathcal{F}}$ for postcomposition of some functor $\mathcal{F}$ in such a category with U_{left}.

[2] A *parison* is an expanding bubble of glass formed in the process of glassblowing.

[3] $\eta_{\mathbf{X}}$ is the product of units

$$X_v \xrightarrow{\eta} (\mathcal{F}_{v \lhd e})^{\bullet} (\mathcal{F}_{v \lhd e})_{\bullet} (X_v).$$

$\mu_{\mathbf{X}}$ comes from the monads $((\mathcal{F}_{v \lhd e})^{\bullet} (\mathcal{F}_{v \lhd e})_{\bullet}, \mu_{v \lhd e}, \eta_{v \lhd e})$.

—*coincides with the* 2*-categorical limit*

$$\lim\left(\underline{\mathcal{C}} : \mathbf{P}_G \to \mathcal{C}at\right).$$

2. In an effort to (i) generalize Theorem 1 to weighted graphs and (ii) facilitate the notion of an approximate section (more on this [13, 4]), we look to enriched category theory [14]. For the remainder of this extended abstract, we explore network sheaves of categories and adjunctions for $\mathcal{A}dj$ the category $\mathbb{M}$-$\mathcal{A}dj$ of certain monoidal categories and monoidal adjunctions.

   (a) An ordered monid is a tuple $\mathbb{M} = (M, \cdot, 1, \leq)$ for which $(M, \cdot, 1)$ is a monoid (small monoidal category) and $(M, \leq)$ is a partial order. An $\mathbb{M}$-category is a category enriched in $\mathbb{M}$. An $\mathbb{M}$-functor F between $\mathbb{M}$-categories is a 1-functor $F : \mathcal{A} \to \mathcal{B}$ such that

   $$\hom_{\mathcal{A}}(x, x') \leq \hom_{\mathcal{B}}(F(x), F(x'))$$

   for all $x, x' \in \mathcal{A}$. An $\mathbb{M}$-adjunction is a pair of opposing $\mathbb{M}$-functors $\mathcal{A} \underset{F_\bullet}{\overset{F^\bullet}{\leftrightarrows}} \mathcal{B}$ such that

   $$\hom_{\mathcal{B}}(F_\bullet(x), y) = \hom_{\mathcal{A}}(x, F^\bullet(y))$$

   for all $x \in \mathcal{A}$, $y \in \mathcal{B}$. Now, let $\mathbb{M}$-$\mathcal{A}dj$ denote the category of $\mathbb{M}$-categories and $\mathbb{M}$-adjunctions.

   (b) A **cellular $\mathbb{M}$-stack** over G is a functor $\mathcal{F} : \mathbf{P}_G \to \mathbb{M}$-$\mathcal{A}dj$.

   (c) Suppose $W : \mathbf{P}_G \to \mathbb{M}$ is a weighting of G. Then, the **weighted Tarski Laplacian** is the endomorphism on the product of $\mathbb{M}$-categories (again, an $\mathbb{M}$-category) $\prod_{v \in G} \mathcal{F}_v$,

   $$(L\mathbf{X})_v := \prod^{W}_{\substack{e \in \partial(v) \\ w \in \delta(e)}} (\mathcal{F}_{v \triangleleft e})^\bullet (\mathcal{F}_{w \triangleleft e})_\bullet (X_w)$$

   where the product is a limit (indexed or) weighted by W [14, p. 37].

   (d) We end with another generalization of Theorem 1—again relating sections to fixed points. In the special case, $\mathbb{M}$ is $\mathbb{B} = (\{\mathtt{T}, \mathtt{F}\}, \wedge, 1, \leq)$, $W(e) = 1$ for all $e \in E_G$, and $m = 1$, we recover Theorem 1 on the nose. We specialize to closed $\mathbb{M}$-categories[4] with internal hom in $\mathbb{M}$,

   $$[a, b] := \bigvee \{c \mid a \cdot c \leq b\}.$$

   Theorem 3. *Let $\mathcal{F} : \mathbf{P}_G \to \mathbb{M}$-$\mathcal{A}dj$ be cellular $\mathbb{M}$-stack over G. Suppose $m \in \mathbb{M}$. Then,* $\hom(\mathbf{X}, L(\mathbf{X})) \geq m$ *if and only if*

   $$[W(e), \hom((\mathcal{F}_{w \triangleleft e})_\bullet (X_w), (\mathcal{F}_{v \triangleleft e})_\bullet (X_v))] \geq m$$

   for all $e \in E_G$, $v, w \in \partial(e)$.

[4] Examples of closed $\mathbb{M}$-categories include categories enriched in $\mathbb{B}$, the interval $\mathbb{I} = ([0,1], \cdot, 1, \leq)$, and Heyting algebras [15].

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