The Sierpinski Carpet as a Final Coalgebra

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We advance the program of connections between final coalgebras as sources of circularity in mathematics and fractal sets of real numbers. In particular, we are interested in the Sierpinski carpet, taking it as a fractal subset of the unit square. We construct a category of *square sets* and an endofunctor on it which corresponds to the operation of gluing copies of a square set along segments. We show that the initial algebra and final coalgebra exists for our functor, and that the final coalgebra is bi-Lipschitz equivalent to the Sierpinski carpet. Along the way, we make connections to topics such as the iterative construction of initial algebras as ω-colimits, corecursive algebras, and the classic treatment of fractal sets due to Hutchinson [8].

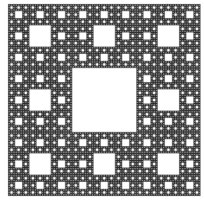
1 Introduction

This paper continues work on fractal sets modeled as final coalgebras. It builds on a line of work that began with Freyd's result [6] that the unit interval $[0,1]$ is the final coalgebra of a certain endofunctor on the category of *bi-pointed* sets. This was generalized by Leinster [9], in work which represents many of what would be intuitively called *self-similar* spaces using (a) bimodules (also called profunctors or distributors); (b) an examination of non-degeneracy conditions on functors of various sorts; (c) a construction of final coalgebras for the types of functors of interest using a notion of resolution. In addition to the characterization of fractal sets as sets, his seminal paper also characterizes them as topological spaces.

In a somewhat different direction, work related to Freyd's Theorem continues with development of *tri-pointed sets* [4, 3] and the proof that the Sierpinski gasket $\mathbb{SG}$ is related to the final coalgebra of a functor modeled on that of Freyd [6]. Although it might seem that this result is but a special case of the much better results in Leinster [9], the work on tri-pointed sets was carried out in the setting of metric spaces rather than topological spaces (and so it re-proved Freyd's result in that setting, too). Work in the metric setting is unfortunately more complicated. It originates in Hasuo, Jacobs, and Niqui [7], a paper which emphasized algebras in addition to coalgebras, and proposed endofunctors defined using quotient metrics. Following this [4, 3] show that for the unit interval, the initial algebra of Freyd's functor is also interesting, being the metric space of dyadic rationals, and thus the unit interval itself is its Cauchy completion. For the Sierpinski gasket, the initial algebra of the functor on tripointed sets is connected to the finite addresses used in building the gasket as a fractal; its completion again turns out to be the final coalgebra; and while the gasket itself is *not* the final coalgebra, the two metric spaces are bi-Lipschitz isomorphic.

In this paper, we take the next step in this area by considering the Sierpinski carpet $\mathbb{S}$. The difference between this and the gasket (or the unit interval) is that the gluing of spaces needed to define the functor involves *gluing along line segments*, not just along points. This turns out to complicate matters at every step. The main results of the paper are analogs of what we saw for the gasket: we have a category of metric spaces that we call *square metric spaces*, an endofunctor $M \otimes -$ which takes a space to 8 scaled

Submitted to:
ACT 2021



copies of itself glued along segments (the notation recalls Leinster's paper, and again we are in the metric setting), a proof that the initial algebra and final coalgebra exist, and that the latter is the completion of the former, and a verification that the actual Sierpinski carpet $\mathbb{S}$ is bi-Lipschitz isomorphic to the final coalgebra. Along the way, we need to consider a different functor $N\otimes -$ which is like $M\otimes -$ but involves 9 copies (no "hole"). The final coalgebra $N\otimes -$ turns out to be the unit square with the taxicab metric. Moreover, in much of this work we have found it convenient to work with *corecursive algebras* as a stepping stone to the final coalgebra; the unit square with the taxicab metric turns out to be a corecursive algebra for $N\otimes -$ on square metric spaces.

This extended abstract omits nearly all of the proofs and is really a very high-level view of our subject. Many of the results are much more general, since we aim to provide a foundation for future work in this area. But none of that is reflected in this abstract.

1.1 Background on the Sierpinski Carpet

In this section, we recall the definition of the Sierpinski carpet $\mathbb{S}$ (shown above) in terms of contractions of the unit square U. We also quote without proof special cases of the classical results of Hutchinson [8] on fractals.

Let $\mathscr{C}$ be the set of non-empty closed subsets of U, with the Hausdorff metric d_H. Let

$$M = \{0,1,2\}^2 \setminus \{(1,1)\}.$$

For each $m=(i,j)$ in M, let $\mathsf{shrink}(m) = \mathsf{shrink}(i,j) = (i/3, j/3)$. let $\sigma_m\colon U \to U$ be the contracting map

$$\sigma_m(x,y) = \mathsf{shrink}(m) + (x/3, y/3).$$

That is, we scale the input (x,y) by $1/3$, and then we move it by adding $\mathsf{shrink}(m)$. Then the setwise extension of σ_m is defined (as always) by taking images: for $A\subseteq X$, $\sigma_m(A) = \{\sigma_m(x) : x\in A\}$. Define $\sigma\colon \mathscr{C}\to\mathscr{C}$ by

$$\sigma(A) = \bigcup_{m\in M} \sigma_m(A)$$

This function σ is a contracting map, and we let $\mathbb{S}$ be its unique fixed point. $\mathbb{S}$ is called the *invariant set* determined by the finite set $\{\sigma_m : m\in M\}$. It is also called the Sierpinski carpet.

Definition 1. For each finite sequence $\vec{m} = m_1 m_2\cdots m_k$ of elements of M, and each $A\in\mathscr{C}$, $A_{\vec{m}}$ is defined by recursion on k, starting with $k=0$ and the empty sequence ε:

$$\begin{array}{lcl} A_\varepsilon & = & A \\ A_{m_1 m_2\cdots m_k m_{k+1}} & = & \sigma_{m_1}(A_{m_2 m_3\cdots m_{k+1}}) \end{array}$$

For every infinite sequence $m_1 m_2\cdots m_k\cdots$,

$$\mathbb{S}_\varepsilon \supseteq \mathbb{S}_{m_1} \supseteq \mathbb{S}_{m_1 m_2} \supseteq \cdots \supseteq \mathbb{S}_{m_1 m_2\cdots m_k} \supseteq \cdots \tag{1.1}$$

By the Cantor Intersection Theorem, $\bigcap_{k=0}^{\infty}\mathbb{S}_{m_1 m_2\cdots m_k}$ is a singleton set, and we write its (unique) member as $a_{m_1 m_2\cdots m_k\cdots}$.

Proposition 1. *(cf. [8]) If A is a non-empty bounded set and $m_1, m_2, \ldots$ is an infinite sequence in M,*

1. *$diam(A_{m_1\ldots m_k}) \to 0$ as $k \to \infty$*
2. *$d(A_{m_1\ldots m_k}, a_{m_1 m_2\ldots m_k m_{k+1}\ldots}) \to 0$ as $k \to \infty$. In particular, $\sigma^k(A) \to \mathbb{S}$ as $k \to \infty$ in the Hausdorff metric.*

2 Square Sets and Square Metric Spaces

We start by defining the categories of interest in this paper. First, MS is the category of metric spaces of diameter 2 and short maps: $d(f(x), f(y)) \le d(x,y)$. The reason for the diameter to be 2 rather than 1 will be apparent from the next example.

Let

$$M_0 = \{(r,s) : r \in \{0,1\}, s \in [0,1]\} \cup \{(r,s) : r \in [0,1], s \in \{0,1\}\}$$

be the boundary of the unit square. A *square set* is a set X with an injective map $S_X : M_0 \to X$. The idea is that S_X designates the four sides of the square. We obtain a category $\mathsf{SquaSet}$ by taking as morphisms the functions $f\colon X \to Y$ between the sets with the property that $S_Y = f \circ S_X$.

Example 1. Here are some examples of square sets. First, M_0 itself, with $S_{M_0} = id$. Next, the unit square $U = [0,1]^2$ with S_U the inclusion. Finally, the Sierpinski carpet $\mathbb{S}$, again with $S_{\mathbb{S}}$ the inclusion.

We are most interested in square sets which are metric spaces. (X, S_X) is a *square metric space* if X is a metric space bounded by 2, a square set, and S_X satisfies the following:

(SQ_1) For $i \in \{0,1\}$ and $r,s \in [0,1]$,

$$d_X(S_X((i,r)), S_X((i,s))) = |s-r| \text{ and } d_X(S_X((r,i)), S_X((s,i))) = |s-r|.$$

That is, along each side of the square, distances coincide with distances on the unit interval. (Note that it follows from this that the image of S_X is compact.)

(SQ_2) For $(r,s),(t,u) \in M_0$, $d_X(S_X((r,s)), S_X((t,u))) \ge |r-t| + |s-u|$. This is a non-degeneracy requirement, which prevents our squares from "collapsing".

Note that we do not require that the metric on the boundary of the square coincides with the Euclidean metric (we are not requiring that opposite corners have distance $\sqrt{2}$). In fact, we will be interested in a path metric around the square.

Example 2. Here are some examples of square metric spaces:

1. (M_0, id) with the path metric: for $x, y \in M_0$, if they are on the same side, their distance coincides with the unit interval, if they are on adjacent sides which share a corner C, $d(x,y) = d(x,C) + d(C,y)$, and if they are on opposite sides, $d(x,y)$ is the minimum (between the two sides) of $d(x,C_1) + 1 + d(C_2,y)$ where C_1, C_2 are endpoints of a side not containing either x or y, with C_1 on the side containing x and C_2 on the side containing y. Note that these distances are all bounded by 2 (the distance between opposite corners is 2).
2. (M_0, id) with the taxicab metric. That is, for (x_0, y_0) and (x_1, y_1),

$$d_T((x_0,y_0),(x_1,y_1)) = |x_1 - x_0| + |y_1 - y_0|$$

Note that in the taxicab metric, the distance from $(0,1/2)$ to $(1,1/2)$ is 1, whereas with the path metric, it is 2.

3. $([0,1]^2, S)$ where S is the inclusion map, with the taxicab metric.

Let SquaMS be the category whose objects are square metric spaces (bounded by 2) whose morphisms are short maps which preserve S. That is, $f:(X,S_X)\to(Y,S_Y)$ is a map such that for $x,y\in X$, $d_X(x,y)\geq d_Y(f(x),f(y))$ and for $(r,s)\in M_0$, $f(S_X((r,s)))=S_Y((r,s))$.

Proposition 2. *(M_0, id) with the path metric is an initial object in* SquaMS. *There is no final object in* SquaMS.

We will find that the concrete description of the requirements on objects in SquaMS is useful for our purposes, but alternatively we can view the relationship between X and M_0 as follows:

Corollary 3. *For every square metric space (X,S_X) and every $(r,s),(t,u)\in M_0$,*

$$d_T(S_{M_0}((r,s)),S_{M_0}((t,u)))\leq d_X(S_X((r,s)),S_X((t,u)))\leq d_{M_0}((r,s),(t,u)),$$

where d_T is the taxicab metric and d_{M_0} is the path metric.

Proof. Follows from (SQ_2) and the fact that there is a (unique) short map $f:M_0\to X$ such that $f((r,s))=S_X((r,s))$. □

Proposition 4. *Monomorphisms in* SquaMS *are the morphisms which are one-to-one.*

2.1 The functors $M\otimes -$ and $N\otimes -$

In this section we will define a functor

$$M\otimes - : \mathsf{SquaMS}\to\mathsf{SquaMS}$$

which, when applied to the initial object, will give us objects which correspond to iterations of the Sierpinski Carpet. The idea is that M will be a set of indices indicating positions to place scaled copies of X, and by showing that $M\otimes -$ is a functor, we will be able to apply it repeatedly in order to form a chain whose colimit will be an object whose completion is bi-Lipschitz equivalent to the Sierpinski Carpet.

As before, $M=\{0,1,2\}^2\setminus\{(1,1)\}$. $m=(i,j)$ will indicate a (column, row) entry in the 3×3 grid.

$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$		$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

For any set X, we naturally consider $M\times X$ as a set of eight copies of X, and (m,x) would be x in the copy labeled by m.

Let $\approx$ be the smallest equivalence relation on $M\times M_0$ such that for $r\in[0,1]$, we take

$$\begin{array}{lll}
((0,0),(r,1))\approx((0,1),(r,0)) & ((1,2),(1,r))\approx((2,2),(0,r)) & ((2,0),(0,r))\approx((1,0),(1,r))\\
((0,1),(r,1))\approx((0,2),(r,0)) & ((2,2),(r,0))\approx((2,1),(r,1)) & ((1,0),(0,r))\approx((0,0),(1,r))\\
((0,2),(1,r))\approx((1,2),(0,r)) & ((2,1),(r,0))\approx((2,0),(r,1)) &
\end{array}$$

In words, $\approx$ relates the segments which overlap in the grid pictured above. For example, $((0,0),(r,1)) \approx ((0,1),(r,0))$ tells us the top of the bottom left square is identified with the bottom of the square immediately above it. Incidentally, on $M \times M_0$, we have a characterization of $\approx$ in terms of the maps σ_m from Section 1.1, $(m,(x,y)) \approx (m',(x',y'))$ iff $\sigma_m(x,y) = \sigma_{m'}(x',y')$.

So far, we have defined $\approx$ on $M \times M_0$. But we can define a similar relation $\approx_X$ on any square set (X, S_X) by "adding S_X everywhere." For example we would want $((2,2), S_X(r,0)) \approx_X ((2,1), S_X(r,1))$. We shall drop X in our notation for $\approx$.

The square set $M \otimes X$ Let (X, S_X) be a square set. The set $M \otimes X$ is $(M \times X)/\approx$, the set of equivalence classes of $\approx$ on $M \times X$. We always write $m \otimes x$ for the equivalence class $[(m,x)]$. For the square set structure, we define each side of $S_{M\otimes X}$ by scaling the 3 copies of the corresponding side X and pasting them together at the appropriate points. For example, the left side will be given by

$$S_{M\otimes X}((0,r)) = \begin{cases} (0,0) \otimes S_X((0,3r)) & 0 \le r \le \frac{1}{3} \\ (0,1) \otimes S_X((0,3r-1)) & \frac{1}{3} \le r \le \frac{2}{3} \\ (0,2) \otimes S_X((0,3r-2)) & \frac{2}{3} \le r \le 1 \end{cases}$$

$S_{M\otimes X}$ is well-defined, and so we have a square set $(M \otimes X, S_{M\otimes X})$. If $f\colon X \to Y$ is a $\mathsf{SquaSet}$ morphism, we take $M \otimes f\colon M \otimes X \to M \otimes Y$ to be $(M \otimes f)(m,x) = (m, f(x))$. It is easy to check that this gives an endofunctor $M \otimes -$ on square sets.

$M \otimes X$ for square spaces X We next upgrade this functor to an endofunctor on SquaMS. For the metric structure on $M \otimes X$, we use the quotient metric. We start with the following metric on the set $M \times X$:

$$d((m,x),(n,y)) = \begin{cases} \frac{1}{3} d(x,y) & m = n \\ 2 & otherwise \end{cases}$$

So the distance is scaled by $\frac{1}{3}$ in the same copy of X, and otherwise, it is 2 (the maximum distance). Now we take the quotient metric on $M \otimes X$ determined by $\approx$. So given points $m \otimes x$ and $n \otimes y$, the distance between them will be defined as the infimum over all finite paths (ordered lists of elements) in $M \times X$ of the *score*, where the score is the sum of the distances (in $M \times X$) along the path, but where we count 0 for pairs of equivalent points.

First note that this is a pseudometric: clearly this is symmetric, the distance between any point and itself is 0, and it will satisfy the triangle inequality since the concatenation of two paths is a path. We will show that this is in fact a metric: distinct points will have positive distance.

Proposition 5. *Let X be a square space. The following facts hold:*

- *For $(i,r),(j,r'),(i,s),(j,s') \in M \times X$, if $(i,r) \approx (j,r')$ and $(i,s) \approx (j,s')$, then $S_X(s)) = d_X(S_X(r'), S_X(s'))$.*
- *For every object X we have an injective map $S_{M\otimes X} : M_0 \to M \otimes X$, such that for all $r \in M_0$, $S_{M\otimes X}(r) = i \otimes S_X(r')$ for some $i \in M$ and $r' \in M_0$ (where i and r' only depend on r, not X).*
- *For any $x, y \in X$ and $i \in M$, for any path $(i,x) = (i_0,x_0),\ldots,(i_n,x_n) = (i,y)$,*

$$\frac{1}{3} d_X(x,y) \le \sum_{k=0}^{n-1} d_{M\times X}((i_k,x_k),(i_{k+1},x_{k+1})).$$

- *Let $i, j \in M$. Suppose that (x_p) and (y_p) are sequences with $x_p \to x$ and $y_p \to y$. If $(i,x_p) \approx (j,y_p)$ for all p, then $(i,x) \approx (j,y)$.*
- *For $i \otimes x$ and $j \otimes y$ in $M \otimes X$, there is a geodesic of the form*

$$(i,x) = (i_0,x_0), (i_0,x'_0) \approx (i_1,x_1), \ldots, (i_{n-1},x'_{n-1}) \approx (i_n,x_n), (i_n,x'_n) = (j,y)$$

(where we may omit (i_0,x_0) or (i_n,x'_n) if x or y are in the image of S_X respectively) such that $d(i \otimes x, j \otimes y) = \sum_{k=0}^{n} d((i_k,x_k),(i_k,x'_k))$.

This shows us that the distances in $M \otimes X$ are witnessed by an actual finite path, not just an infimum of paths. This gives us the following:

Corollary 6. *1. For $i \in M$, $\phi_i : X \to M \otimes X$ given by $x \mapsto i \otimes x$ is an embedding such that for $x, y \in X$, $d_{M \otimes X}(i \otimes x, i \otimes y) = \frac{1}{3} d_X(x,y)$.*

2. $M \otimes X$ is a square metric space.

We complete the definition of $M \otimes -$ as a functor as with SquaSet: for $f : X \to Y$ a morphism of SquaMS, we let $M \otimes f : M \otimes X \to M \otimes Y$ be $i \otimes x \mapsto i \otimes f(x)$. Using Proposition 5, this is a functor. It is not hard to check that this functor $M \otimes -$ preserves monomorphisms and isometric embeddings.

The functor $N \otimes -$. Let $N = M \cup \{(1,1)\}$. So $N = \{0,1,2\}^2$. We define a functor $N \otimes X$ on square spaces. The definition is just like M, except that we use the "middle point" $(1,1)$ as a possible index point. In pictures, $N \otimes X$ is *nine* copies of X (not 8, as per M). The metric again is obtained by shrinking the metric in X by $\frac{1}{3}$ and using the quotient metric obtained by "gluing on the edges of the grid". The square space structure is as for $M \otimes X$. All of the verifications for N are easier than for M.

3 Initial Algebras

We assume that the reader is familiar with the notions of *algebra* and *coalgebra* for an endofunctor on a category. We mention a few examples and then quickly mention the initial algebras of $N \otimes -$ and $M \otimes -$ on SquaMS.

The algebra $\alpha_N : N \otimes U \to U$ on SquaSet We have an algebra $\alpha_N : N \otimes U \to U$. It is defined as follows:

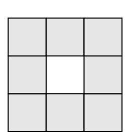
$$\alpha_N(n \otimes z) \quad = \quad \mathsf{shrink}(n) + \tfrac{1}{3}(z)$$

Notice that $n \in N$ here is a pair; earlier we had written it as (i, j). Similarly, $z \in U$; earlier we wrote it as (r,s). It takes a few routine elementary calculations to be sure that α_N is well-defined.

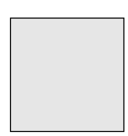
In the result below, recall that our default metric for U is the taxicab metric.

Proposition 7. *In SquaMS, $\alpha_N : N \otimes U \to U$ is an isomorphism.*

When we turn to $M \otimes -$, we have an algebra $\alpha_M : M \otimes U \to U$. It is defined the same way as N, except that the index $(1,1)$ is not used. Here is a way to picture this:



$M \otimes U$



U

This picture is misleading, because it suggests that $M \otimes U$ maps via the inclusion into U. The map is not the inclusion. This is because $M \otimes U$ is really eight copies of U, each copy with the taxicab metric, and then the overall space is given by the quotient, as discussed above.

Proposition 8. *$\alpha_M \colon M \otimes U \to U$ is a (short) injective map.*

Here is what is happening. In $M \otimes U$, we have to navigate around the hole, potentially making the distance longer. For example, if we consider the midpoints of the bottom and top of the middle square,

$$d_{M\otimes U}((1,0)\otimes S_U(\frac{1}{2},1),(1,2)\otimes S_U((\frac{1}{2},0))) = \frac{2}{3}$$

whereas

$$d_U(\alpha_M((1,0)\otimes S_U(\frac{1}{2},1)),\alpha_M((1,2)\otimes S_U((\frac{1}{2},0)))) = d_U((\frac{1}{2},\frac{1}{3}),(\frac{1}{2},\frac{2}{3})) = \frac{1}{3}.$$

We construct the initial algebra of these functors by iteration in ω steps. For example, consider N. We have the initial ω-sequence of the functor $N \otimes -$:

$$M_0 \xrightarrow{!=S_{N\otimes M_0}} N\otimes M_0 \xrightarrow{N\otimes !} N^2 \otimes M_0 \xrightarrow{N^2\otimes !} N^3\otimes M_0 \xrightarrow{N^3\otimes !} \cdots N^k \otimes M_0 \xrightarrow{N^k\otimes !} N^{k+1}\otimes M_0 \cdots \qquad (3.1)$$

Unlike the situation with bipointed and tripointed sets, the maps in the chain are not isometric embeddings. Nevertheless, the colimit W exists, and (by an argument) the colimit maps $i_k \colon N^k \otimes M_0 \to W$ are injective. (The key point here is that each space $N^k \otimes M_0$ has an injective short map into the carrier of an injective algebra, namely the unit square U.)

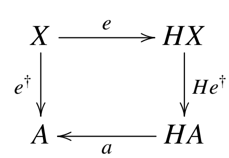
Theorem 9. *The colimits of the initial ω-sequences exist, and the functors preserve these colimits. Thus by Adámek's Theorem [1] there are initial algebras $M \otimes G \to G$ and $N \otimes W \to W$.*

4 Final Coalgebras

4.1 Corecursive algebras

As a technical tool to obtain the final coalgebras, it will be useful to use a different kind of structure.

Definition 2 (Capretta, Uustalu and Vene [5])**.** Let $H \colon \mathscr{A} \to \mathscr{A}$ be an endofunctor on any category. An algebra $a \colon HA \to A$ is *corecursive* if for every coalgebra $e \colon X \to HX$ there is a unique *coalgebra-to-algebra morphism* $e^\dagger \colon X \to A$. This means that $e^\dagger = a \circ He^\dagger \circ e$:



The map $e^\dagger$ is also called *the solution to e in the algebra* (A,a).

This section provides a few examples, beginning with the following one on Set. First fix a real number $0 \le \delta < 1$. The rest of this example depends on this parameter, and in later sections we are going to take $\delta = \frac{1}{3}$. Let $K = [0, 1-\delta]$. (We mean this as a subset of the reals.) For the functor, we take $H \colon \mathrm{Set} \to \mathrm{Set}$ to be given by $HX = K \times X$. For a function $f \colon X \to Y$, $Hf \colon HX \to HY$ is given by $Hf(\xi,x) = (\xi, f(x))$. We have an algebra (I,ι), where I is the unit interval $[0,1]$, and

$$\iota \colon H[0,1] = K \times [0,1] \to [0,1]$$

is given by $\iota(\xi,x) = \xi + \delta x$, for $\xi \in K$ and $x \in I$.

Proposition 10 (cf. [2])**.** *$\iota\colon HI \to I$ is a corecursive algebra for H.*

Proposition 11. *[5] If a corecursive H-algebra (A,a) has an invertible structure map a, then (A,a^{-1}) is a final coalgebra for the same functor. And if (A,a) is a final coalgebra, then (A,a^{-1}) is a corecursive algebra.*

Lemma 12. *Let $e\colon X \to HX$ and $f\colon Y \to HY$ be coalgebras, and let $h\colon X \to Y$ be a coalgebra morphism. Let $a\colon HA \to A$ be corecursive algebra. Then $e^\dagger = f^\dagger \circ h$.*

4.2 Square Sets

Lemma 13. *$(U, \alpha_N\colon N\otimes U \to U)$ is a corecursive algebra for $N\otimes X$ on* SquaSet.

4.3 Square spaces

The main result next is that (U,α_N^{-1}) is a final $N\otimes -$coalgebra on square *spaces*. As above, the metric on U is the taxicab metric. We need a few preliminary lemmas. In these, we fix an $N\otimes -$coalgebra on SquaMS, $(B,\beta : B \to N\otimes B)$. We already know that there is a unique SquaSet morphism $\beta^\dagger\colon B \to U$. Also, α_N is an isometry, hence α_N^{-1} is short. Our main work in this section shows that $\beta^\dagger$ is short (on all of B), of course using that β is a short map. The surprising feature of our proof is that we must consider other coalgebras in order to prove the shortness of $\beta^\dagger$. Notice that $(N\otimes B, N\otimes\beta)$ is also an $N\otimes -$coalgebra. Furthermore, $\beta\colon B\to N\otimes B$ is a coalgebra morphism.

Lemma 14. $(N\otimes\beta)^\dagger = \alpha_N\circ(N\otimes\beta^\dagger)$.

Definition 3. Let $Z\subseteq B$. We say that $\beta^\dagger$ is *short on Z* if for all $b,c\in Z$, $d_U(\beta^\dagger b, \beta^\dagger c) \le d_B(b,c)$.

Also, we write $N\otimes Z$ for $\{n\otimes b : n\in N \text{ and } b\in Z\}$.

Lemma 15. *For all $n_1,n_2\in N$ and $(r_1,s_1),(r_2,s_2)\in M_0$,*

$$d_{N\otimes B}(n_1\otimes S_B((r_1,s_1)), n_2\otimes S_B((r_2,s_2))) \ge d_{N\otimes U}(n_1\otimes(r_1,s_1), n_2\otimes(r_2,s_2))$$

Lemma 16. *Let $Z\subseteq B$ be any set that includes the image $S_B[M_0]$. If $\beta^\dagger$ is short on Z, then $(N\otimes\beta)^\dagger$ is short on $N\otimes Z$.*

Lemma 17. *Let $(B,\beta\colon B\to N\otimes B)$, and let $k\in\omega$. There is a coalgebra $(C,\gamma\colon C\to N\otimes C)$, a coalgebra morphism $h\colon B\to C$, and a set $Z\subseteq C$ so that*

1. *$S_C[M_0]\subseteq Z$.*
2. *$\gamma^\dagger$ is short on Z.*
3. *For every $c_1\in C$ there is some $c_2\in Z$ such that $d_C(c_1,c_2)\le \frac{2}{3^k}$, and also $d_U(\gamma^\dagger(c_1),\gamma^\dagger(c_2))\le\frac{2}{3^k}$.*

Lemma 18. *$\beta^\dagger\colon B\to U$ is short.*

Proof. Fix $\varepsilon>0$. Let $b_1,b_2\in B$. Let k be large enough so that $2/3^k<\varepsilon/4$. Let C, h, Z, c_1 and c_2 be as in Lemma 17 so that $c_1,c_2\in Z$, $d_C(hb_1,c_1)$ and $d_C(hb_2,c_2)$ are each $\le\varepsilon/4$, and also $d_U(\gamma^\dagger(hb_i),\gamma^\dagger(c_i))\le\varepsilon/4$ for $i=1,2$. Then $d_C(c_1,c_2)\le d_C(hb_1,hb_2)+\varepsilon/2$. And

$$\begin{array}{lll}
 & d_U(\beta^\dagger b_1,\beta^\dagger b_2) & \\
= & d_U(\gamma^\dagger(hb_1),\gamma^\dagger(hb_2)) & (1)\\
\le & d_U(\gamma^\dagger(hb_1),\gamma^\dagger c_1)+d_U(\gamma^\dagger c_1,\gamma^\dagger c_2)+d_U(\gamma^\dagger c_2,\gamma^\dagger(hb_2)) & \\
\le & \varepsilon/4+d_C(c_1,c_2)+\varepsilon/4 & (2)\\
\le & \varepsilon/2+(d_C(hb_1,hb_2)+\varepsilon/2) & \\
\le & \varepsilon+d_B(b_1,b_2) & (3)
\end{array}$$

Point (1) uses Lemma 12. Point (2) uses the shortness of $\gamma^\dagger$ on Z. Point (3) uses the shortness of h. This for all $\varepsilon > 0$ proves our result. □

Theorem 19. *(U,α_N) is a corecursive algebra for $N\otimes -$ on* SquaMS*, and (U,α_N^{-1}) is a final coalgebra for this same functor.*

Proof. We already know that if we forget the metric, (U,α) is a corecursive algebra for $N\otimes -$ on SquaSet. In the case that we have a short coalgebra structure, (B,β), the unique SquaSet map $\beta^\dagger$ is short, by Lemma 18. The forgetful functor SquaMS $\to$ SquaSet is faithful, and so $\beta^\dagger$ is the unique coalgebra-to-algebra map in SquaMS. This shows the first assertion in our result. The second follows since α_N is invertible (Proposition 11). □

4.4 U is isomorphic to the completion of the initial algebra $N\otimes W \to W$

Recall the initial sequence of the functor $N\otimes -$ in (3.1). We write W for the colimit. And we write $i_k\colon N^k\otimes M_0 \to W$ for the canonical injection. There are canonical maps

$$\ell_k\colon N^k\otimes M_0 \to U$$

given by: $\ell_0 = S_U$, and $\ell_{k+1} = \alpha_N\circ(N\otimes \ell_k)$. The maps $(\ell_k)_{k\in\omega}$ are a cocone. It is easy to see that $\ell_k = \beta_k^\dagger$, where

$$\beta_k\colon N^k\otimes M_0 \to N^{k+1}\otimes M_0 \tag{4.1}$$

is given by $\beta_k = N^k\otimes ! = N\otimes N\otimes\cdots\otimes N\otimes S_{N\otimes M_0}$. Thus ℓ_k is short. By the colimit property of W, we have a unique short map

$$\psi\colon W\to U$$

so that for all k, $\psi\circ i_k = \ell_k$.

Lemma 20. *The Cauchy completion operation on* MS *has a lift to C:* SquaMS $\to$ SquaMS*. For all X, $C(N\otimes X)\cong N\otimes C(X)$.*

Observe that since $N\otimes W\cong W$ (by Lambek's Lemma), we have an isomorphism

$$\eta : N\otimes C(W)\cong C(N\otimes W)\cong C(W).$$

Lemma 21. *$\psi\colon W\to U$ is an isometry, and ψ extends to an isomorphism $\overline{\psi}\colon C(W)\to C(U) = U$.*

Although we lack the space to show it, this last result is quite involved, requiring work with maps that are not short and also requiring special work on the relation between each space $N^k\otimes U$ and it's "cousin" $N^k\otimes M_0$.

Theorem 22. *$(H,\eta^{-1}\colon H\to N\otimes H)$ is the final coalgebra, where $H = C(W)$ and η is the map above.*

Proof. Let $(B,\beta\colon B\to N\otimes B)$ be a coalgebra. Consider the metric space H^B. As usual, this is complete because H is. The subspace $V\subseteq H^B$ of short maps which preserve the square space structure is a closed subset, and (crucially) it is non-empty. This is because we have a short map $B\to U$ by Lemma 18, and a short map $U\to H$ by Lemma 21. We also have a $\frac{1}{3}$-contracting map $\Phi\colon V\to V$ given by $\Phi(f) = e\circ(N\otimes f)\otimes\eta$. Thus, Φ has a unique fixed point. The fixed points of Φ are exactly the coalgebra morphisms from $B\to H$. Thus, there is a unique coalgebra morphism from $B\to H$. □

4.5 Final coalgebra for $M \otimes -$

Recall that $M \otimes G \cong G$ is the initial $M \otimes -$ algebra. Let $Q = C(G)$ be its completion and let $\gamma : Q \to M \otimes Q$ be the completion functor applied to the inverse of the isomorphism from $M \otimes G$ to G. We aim to show that (Q, γ) is the final $M \otimes -$ coalgebra.

The main goal in this section is to exhibit a short map $h\colon B \to Q$, where (B, β) is an arbitrary $M \otimes -$ coalgebra. With this in mind, let $b \in B$ be given. Then we can choose $m_0, m_1, \ldots \in M$ and $b = b_0, b_1, \ldots \in B$ be such that

$$(M^{k-1} \otimes \beta) \circ \ldots \circ (M \otimes \beta) \circ \beta(b) = m_0 \otimes \ldots \otimes m_{k-1} \otimes b_k \in M^k \otimes B.$$

This is where we will use our work on the functor $N \otimes -$. Note that the inclusion $M \otimes B \hookrightarrow N \otimes B$ is a short map, since every path in $M \otimes B$ is a path in $N \otimes B$, and thus, the inclusion is a morphism. So we can view any $M \otimes -$ coalgebra (B, β) as an $N \otimes -$ coalgebra by taking the composition of the inclusion morphism with β.

Let $i_k : M^k \otimes U \to U$ be $M^{k-1} \otimes \alpha_M \circ \ldots \circ \alpha_M$.

By Lemmas 13 and 18, there is a morphism $\beta^\dagger : B \to U$, and for all k,

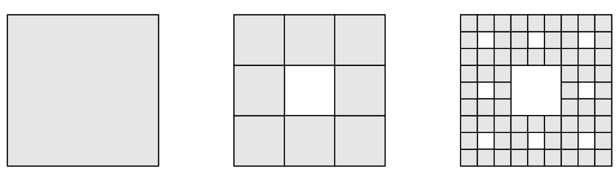
$$\beta^\dagger(b) = i_k(m_0 \otimes \ldots \otimes m_{k-1} \otimes \beta^\dagger(b_k))$$

Now we have a short map from B to U. Our aim is to get to Q, which is the completion of G, which is a colimit of the chain $M^k \otimes M_0$; see (3.1), but with M instead of N. So to connect these, we will restrict our attention to corner points in $M^k \otimes M_0$, and show that the inclusion into $M^k \otimes U$ restricted to these corner points is an isometry. This way, we will be able to define approximate maps from B to $M^k \otimes U$ and in turn, to $M^k \otimes M_0$ whose limit will be our required map from B to Q. Note that we cannot expect to define a morphism directly from the image of $\beta^\dagger$ in U to Q, since (as we will see in our discussion of bi-Lipschitz equivalence), such a map with the required properties will not be a short map.

Definition 4. The set CP_k^M of *corner points* of $M^k \otimes M_0$ is defined as follows:

$$\begin{aligned} CP_0^M &= \{(0,0), (0,1), (1,0), (1,1)\} \\ CP_{k+1}^M &= M \otimes CP_k^M \quad (= \{m \otimes x : m \in M, x \in CP_k^M\}) \end{aligned}$$

For example, CP_0^M, CP_1^M, and CP_2^M are the intersections of segments in each of these squares, respectively.



Lemma 23. *Let x and y be corner points in $M^k \otimes U$ (via the inclusion $M^k \otimes M_0 \hookrightarrow M^k \otimes U$). Then there exists a geodesic as in Proposition 5 such that every entry on the path is also a corner point in $M^k \otimes U$.*

Corollary 24. *Let $x, y \in CP_k^M$. Then for $\iota = S_U : M_0 \to U$,*

$$d_{M^k \otimes U}(M^k \otimes \iota(x), M^k \otimes \iota(y)) = d_{M^k \otimes M_0}(x, y)$$

That is, the distance between corners in $M^k \otimes U$ coincides with the distance in $M^k \otimes M_0$.

For each natural number k, define $h_k : B \to M^k \otimes M_0$ by

$$h_k(b) = m_0 \otimes \ldots \otimes m_{k-1} \otimes (0,0).$$

And now the map

$$h'_k := i_k \circ (M^k \otimes \iota) \circ h_k : B \to U$$

is not a short map itself, but it approximates $\beta^\dagger$ in the following sense: for $b \in B$,

$$\begin{array}{ll} & d_U(h'_k(b), \beta^\dagger(b)) \\ = & d_U(i_k(m_0 \otimes \ldots \otimes m_{k-1} \otimes S_U(0,0)), i_k(m_0 \otimes \ldots \otimes m_{k-1} \otimes \beta^\dagger(b_k))) \\ \leq & d_{M^k \otimes U}(m_0 \otimes \ldots \otimes m_{k-1} \otimes S_U(0,0), m_0 \otimes \ldots \otimes m_{k-1} \otimes \beta^\dagger(b_k)) \qquad \text{since } i_k \text{ is short} \\ \leq & \frac{2}{3^k} \end{array}$$

Finally, we define $h : B \to Q$ by $b \mapsto ([h_k(b)])_k \in Q$. This is a Cauchy sequence of elements of the initial algebra G, since for $m,n < \omega$, $d_G(h_n(b), h_m(b)) \leq (\frac{2}{3^{\min(m,n)}})$.

Please note that h_k is not a short map.

Proposition 25. *$h\colon B \to Q$ is a short map.*

Theorem 26. *$(Q, \gamma : Q \to M \otimes Q)$ is a final $M \otimes -$ coalgebra.*

5 Bi-Lipschitz Equivalence

Two metric spaces M and N are *bi-Lipschitz equivalent* if there is a bijection $b : M \to N$ and a number K such that

$$\frac{1}{K} d_M(x,y) \leq d_N(b(x), b(y)) \leq K d_M(x,y).$$

Here we will show that the final $M \otimes -$ coalgebra Q is bi-Lipschitz equivalent to $\mathbb{S}$, the Sierpinski Carpet as a subset of U with the taxicab metric. Note that $\mathbb{S}$ with the taxicab metric is bi-Lipschitz equivalent to $\mathbb{S}$ with the Euclidean metric.

As in our proof that Q is the final $M \otimes -$ coalgebra, we can view it as an $N \otimes -$ coalgebra, so by Lemmas 13 and 18, there is a morphism $\gamma^\dagger : Q \to U$ such that $\gamma^\dagger = \alpha_N \circ N \otimes \gamma^\dagger \circ \gamma$.

Proposition 27. *$\gamma^\dagger$ is injective.*

Clearly the image of Q under $\gamma^\dagger$ is non-empty, so to see that $\gamma^\dagger$ is a bijection between Q and $\mathbb{S}$, we show that the image is compact with respect to the taxicab metric on U, and that it is fixed under σ.

Proposition 28. $\gamma^\dagger(Q) = \mathbb{S}$.

Theorem 29. *The metric space Q is bi-Lipschitz equivalent to the Sierpinski Carpet as a subset of the plane with the taxicab metric, and thus, the Euclidean metric. Specifically, for $x,y \in Q$,*

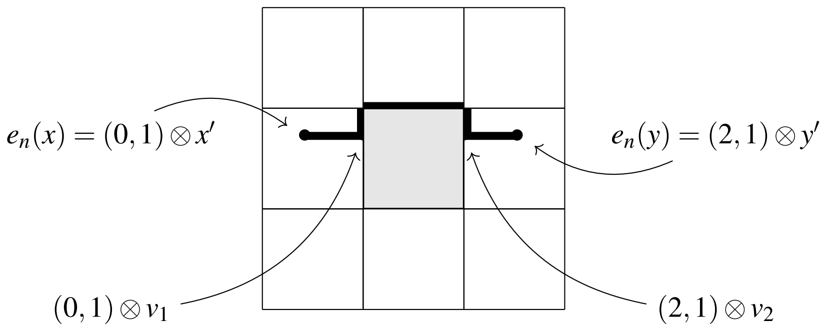
$$\frac{1}{4} d_Q(x,y) \leq d_U(\gamma^\dagger(x), \gamma^\dagger(y)) \leq 4 d_Q(x,y).$$

Since we can view the taxicab metric as the sum of the horizontal and vertical components of the distance, to prove this theorem, we will focus our attention on these. We use the following lemma, comparing distances in $M^n \otimes M_0$ to those in U.

Recall we have defined the morphisms $i_n : M^n \otimes U \to U$ using α_M. Then if $\iota : M_0 \to U$ is the inclusion map (that is, $\iota = S_U$), then we can define a morphism $e_n : M^n \otimes M_0 \to U$ by $e_n = i_n \circ M^n \otimes \iota$.

Lemma 30. *Let $n \geq 0$ and $x, y \in M^n \otimes M_0$ be such that $e_n(x)$ and $e_n(y)$ are on a horizontal or vertical line segment in the unit square, on which they are distance d apart (via the Euclidean metric). Then* $\frac{1}{2} d_{M^n \otimes M_0}(x, y) \leq d$.

The idea is that a path between points on a line segment may require navigating around a hole, but the score of this path is not more than 2 times the length of the segment. Here is a typical case:



Then, in calculating the taxicab metric, we may run into the situation where we must go around a corner to avoid a hole, but we find that this does not affect the score. The full proof is an induction on k.

To prove Theorem 26 we approximate the distances in Q to get the required inequality by looking at corresponding points in $M^k \otimes M_0$ as $k \to \infty$.

Conclusion Stepping back, the main point of this paper has been to further the interaction between the subject of coalgebra broadly considered (including corecursive algebras) and continuous mathematics. The questions that we asked in this paper concerned the relationship between very natural and very concrete fractal sets on the one hand, and more abstract ideas like initial algebras and final coalgbebras on the other. We came to this work in order to explore these general issues. What we found in the expolaration was a set of ideas connecting category-theoretic and analytic concepts: colimits in metric spaces, short maps approximated by non-short maps, corecursive algebras as an alternative to infinite sums, and the like. We hope that the results in this paper further these connections.

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