Exponential modalities and complementarity

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Abstract

The exponential modalities of linear logic have been used by various authors to model infinite dimensional quantum systems. This paper explains how the exponential modalities can also give rise to the complementarity principle of quantum mechanics. This uses a formulation of quantum systems based on †-linear logic, whose categorical semantics lies in mixed unitary categories, and a formulation of measurement therein. The main result exhibits a complementary system as the result of a measurements on a free exponential modality. Recalling that, in linear logic, exponential modalities have two distinct but dual components, ! and ?, this shows how these components under measurement become "compacted" into the usual notion of a complementary Frobenius algebras from categorical quantum mechanics.

1 Introduction

Linear logic introduced by Girard in his seminal paper [18] treats logical statements as resources, which cannot be duplicated or destroyed. The word "linear" refers to this resource sensitivity of the logic: a proof of a statement in linear logic may thus be regarded as a series of resource transformations. In full linear logic the classical ability to duplicate and destroy resources is recaptured by the exponential (or storage) modality written ! (pronounced the "bang"). The type $!A$ may be interpreted as an unbounded "store" from which resources of type A can be extracted an arbitrary (including 0) number of times.

The exponential modality has been proposed as a structure for modelling infinite dimensional systems: [27] used the exponential modality to model the quantum harmonic oscillator, [4] used it to model the bosonic Fock space, while [21] used it to model unbounded bit strings. However, these uses did not explain what exponential modalities have to do with the complementarity principle of quantum mechanics [11]. A pair of quantum observables (physical properties of a system) is said to be complementary if measuring one observable increases the uncertainty regarding the value of the other. The classic example is that the more one knows about position of a particle the less one knows about its momentum. The purpose of this article is to exhibit a relationship between the exponential modalities ! and its dual ? (pronounced "whimper"), and complementary observables – a relationship which suggests a possibly new perspective on measurement in quantum systems.

Linearly distributive categories (LDCs) [8] provide a categorical semantics for the multiplicative fragment of linear logic (MLL). Thus, LDCs are equipped with two distinct tensors called the "tensor", $\otimes$, and the "par", $\oplus$[1]. These are related by a linear distributor. It is not assumed that the tensor is dual to the par - which would be normal in linear logic. In an LDC having a dual is a property which an object may or may not be possess When every object possesses a dual then the category is $*$-autonomous.

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[1] In the linear logic community the par is often denoted by ⅋ but we follow the convention in [8] and use $\oplus$.

In this development, LDCs which satisfy the so called "mix law" are particularly important. The mix law provides a natural transformation from the tensor to the par called the *mixor*. When the mixor is an a natural isomorphism the LDC becomes equivalent to a monoidal category. Conversely, monoidal categories can be viewed as being degenerate or *compact* LDCs in which the mixor is the identity map. Thus, from this perspective, a compact closed category is a compact LDC with duals or a compact $*$-autonomous category.

In [6], we introduced †-linearly distributive categories (†-LDCs) with mix for modelling possibly infinite dimensional quantum processes. In a mix LDC, there is always a set of objects which cannot distinguish between the tensor and the par in the sense that $A \otimes \_ \simeq A \oplus \_$: these objects form a compact subLDC called the *core*. In a mix †-LDC, it is possible to go one step further, and identify a *unitary core* in which every object is not only in the core but isomorphic to its †-dual in a coherent way. A unitary core is equivalent to a †-monoidal category and when this category has duals it is equivalently a †-compact closed category [1, 25]. This is the main structure underlying categorical quantum mechanics (CQM) [12, 22]: finite dimensional Hilbert spaces provide the paradigmatic example.

The general notion of a *mixed unitary category* (MUC) is essentially a mix †-LDC with a specified unitary core. In particular, the unitary core may be viewed as comprising the finite dimensional processes while the larger category extends this to include infinite dimensional processes. An example of a MUC is given by the embedding of complex finite matrices into the category of finiteness matrices [17]. Another example is given by the embedding of the finite-dimensional Hilbert spaces within the category of Chu spaces [2] of vector spaces over complex numbers with the field $\mathbb{C}$ as the dualizing object. For further details see [6, 9] where completely positive maps, and environment structures for MUCs are described. In this article, we explore the notions of measurement and complementarity in MUCs.

In CQM, Coecke and Pavlovic [13] described a "demolition" measurement in a †-monoidal category as a map, $m : A \longrightarrow X$, with $m^\dagger m = 1_X$, to a special commutative †-Frobenius algebra, X. Interpreted in the category of finite-dimensional Hilbert spaces, the notion of the demolition measurement models the Projection Valued Measures (PVMs) in quantum mechanics. Generalizing this idea to MUCs to model measurements here is complicated by the fact that, in the †-LDC of a MUC, generally, $A \neq A^\dagger$, except in the unitary core. Thus, in a MUC, a measurement can be viewed as a two-step process in which one first "compacts" an object into the unitary core by a retraction and then one performs Coecke and Pavlovic's demolition measurement. The compaction process is discussed in Section 3 and is already quite interesting: it gives rise to a †-binary idempotent. In [26], Selinger split †-idempotents to produce classical types. Conversely, a †-binary idempotent, which is "coring" and splits, gives rise to a compaction into the "canonical" unitary core.

In CQM, quantum observables are characterized by certain †-Frobenius algebras [14] in †-monoidal categories. Two such †-FAs, $(A, \curlyvee_\circ, \eta_\circ, \curlywedge_\circ, \epsilon_\circ)$ and $(A, \curlyvee_\bullet, \eta_\bullet, \curlywedge_\bullet, \epsilon_\bullet)$ are said to be complementary [11] if $(A, \curlyvee_\circ, \eta_\circ, \curlywedge_\bullet, \epsilon_\bullet)$ and $(A, \curlyvee_\bullet, \eta_\bullet, \curlywedge_\circ, \epsilon_\circ)$ are Hopf algebras. An object which is a Frobenius algebra is always a self-dual. Conversly, any self-dual object with a monoid structure is a Frobenius algebra. In an LDC setting, a linear monoid, $A \overset{\circ}{-\!\!\!\Vdash} B$, has a $\otimes$-monoid on A and B is dual to A, hence is a $\oplus$-comonoid. A linear comonoid $A \underset{\circ}{-\!\!\!\Vdash} B$ has a $\otimes$-comonoid on A, and B is dual to A, hence is a $\oplus$-monoid. A linear monoid and a linear comonoid interacting to produce a $\otimes$-bialgebra on A and a $\oplus$-bialgebra on B is a linear bialgebra. In a MUC, the linear bialgebras in the unitary core are the base for defining complimentary systems. These structures are presented in Section 4.

Section 5 describes the connection between the free exponential modalities and complimentary systems in a †-isomix setting. An LDC is said to have exponential modalities, if it has a monoidal comonad $(!, \delta, \epsilon)$, a comonoidal monad $(?, \mu, \eta)$, for all objects A $(!A, \Delta_A, \perp_A)$ is a natural commutative $\otimes$-monoid, and $(?A, \nabla_A, \top_A)$ is a natural commutative $\oplus$-monoid. The modalities are said to be free if $(!A, \Delta_A, \perp_A)$ is cofree and $(?A, \nabla_A, \top_A)$ is free. The main result of this paper is that in a MUC, every †-complementary system in the unitary core arises as the splitting of a †-binary idempotent on the

†-linear bialgebra induced on the free exponentials. This is an interesting result since it shows that complementary observables arise from compacting dual but distinct systems of arbitrary dimensions.

Notation: Diagrammatic order of composition is used: so fg should be read as f followed by g. Circuit diagrams should be be read top to bottom: that is following the direction of gravity! Most proofs for the results of this article are in the appendices.
An extended version of this article is available in the arXiv [10].

2 Preliminaries

In this section, we recall the definitions of dagger isomix categories, unitary categories, and mixed unitary categories from [6]. To achieve this we start by recalling the definitions of linearly distributive categories and isomix categories.

A linearly distributive category (LDC), $(\mathbb{X}, \otimes, \oplus)$, is a category with two tensor products - $\otimes$ called the tensor with unit $\top$, and the $\oplus$ called the par with unit $\bot$. The tensor and the par interact by means of linear distributors which are natural transformations (which, in general, are not isomorphisms):

$$\partial^L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C \qquad \partial^R : (B \oplus C) \otimes A \to B \oplus (C \otimes A)$$

A symmetric LDC is an LDC in which both monoidal structures are symmetric, with symmetry maps $c_\otimes$ and $c_\oplus$, such that $\partial^R = c_\otimes (1 \otimes c_\oplus) \partial^L (c_\otimes \oplus 1) c_\oplus$. LDCs provide a categorical semantics for linear logic and have a graphical calculus, which is discussed in Appendix A: we use it extensively in proofs.

A mix category is an LDC with a mix map, $\mathsf{m} : \bot \longrightarrow \top$, when m is an isomorphism it is an isomix category. The mix map gives a natural mixor map, $\mathsf{mx} : A \otimes B \longrightarrow A \oplus B$ which, even if the mix map is an isomorphism, is usually not an isomorphism. An isomix category in which every mixor map is an isomorphism is a compact LDC. A compact LDC with $\mathsf{m} = 1$, and $\mathsf{mx} = 1$ is just a monoidal category.

The **core**, $\mathsf{Core}(\mathbb{X})$, of an isomix category $\mathbb{X}$ is the full subcategory given by the objects, U, such that for all $A \in \mathbb{X}$, the maps, $\mathsf{mx}_{U,A} : U \otimes A \longrightarrow U \oplus A$ and $\mathsf{mx}_{A,U} : A \otimes U \longrightarrow A \oplus U$, are isomorphism. The units, $\top$, and $\bot$, are always in the core. The core of an isomix category $\mathbb{X}$, $\mathsf{Core}(\mathbb{X})$, is always a compact LDC.

A **†-linearly distributive category** [6] is an LDC with a functor $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \longrightarrow \mathbb{X}$ and the following natural isomorphisms satisfying the coherence conditions which are described in [6].

tensor laxors: $A^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (A \oplus B)^\dagger \qquad A^\dagger \oplus B^\dagger \xrightarrow{\lambda_\oplus} (A \otimes B)^\dagger$

unit laxors: $\top \xrightarrow{\lambda_\top} \bot^\dagger \qquad \bot \xrightarrow{\lambda_\bot} \top^\dagger$

involutor: $A \xrightarrow{\iota} (A^\dagger)^\dagger$

In a †-LDC, $A \neq A^\dagger$ because † swaps the tensor and the par. A **†-mix category** is a †-LDC which has a mix map which satisfies, in addition:

$$[\dagger\text{-mix}] \qquad \begin{array}{ccc} \bot & \xrightarrow{\mathsf{m}} & \top \\ {\scriptstyle \lambda_\bot}\downarrow & & \downarrow{\scriptstyle \lambda_\top} \\ \top^\dagger & \xrightarrow[\mathsf{m}^\dagger]{} & \bot^\dagger \end{array}$$

If m is an isomorphism, then $\mathbb{X}$ is an **†-isomix category**. A **compact †-LDC** is a compact LDC which is also a †-isomix category.

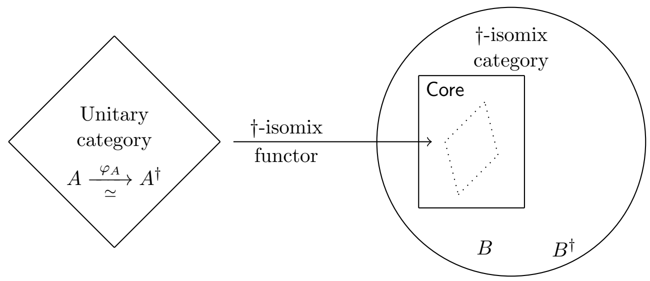
In a †-monoidal category a unitary isomorphism is an isomorphisms with $f^\dagger = f$. In a †-LDC, an object, A, does not necessarily coincide with its dagger, $A^\dagger$: this means that describing unitary isomorphism for †-LDCs is more complicated. To accomplish this the notion of **unitary structure** which is described in [6] is used. A **pre-unitary** object in a †-isomix category is an object in the core with an isomorphism $\varphi : A \longrightarrow A^\dagger$ such that $\varphi(\varphi^{-1})^\dagger = \iota$. Unitary structure for a †-isomix category is given by a family of pre-unitary objects satisfying certain closure and coherences requirements.

A **unitary category** is a compact †-LDC equipped with unitary structure which makes every object a (pre)unitary object. A †-monoidal category [22] is a unitary category with $\otimes = \oplus$ and the unitary structure given by the identity map. Conversely, every unitary category is †-linearly equivalent to a †-monoidal category via the †-linear functor $\mathsf{Mx}_\downarrow : (\mathbb{X}, \otimes, \oplus) \longrightarrow (\mathbb{X}, \otimes, \otimes)$, see [6, Prop. 5.11].

A **mixed unitary category** (MUC) [6] is a †-isomix category, $\mathbb{C}$, equipped with a strong †-isomix functor $M : \mathbb{U} \longrightarrow \mathbb{C}$ from a unitary category $\mathbb{U}$ such that there are a natural transformations:

$$\mathsf{mx}' : M(U) \otimes X \longrightarrow M(U) \oplus X \text{ with } \mathsf{mx}\,\mathsf{mx}' = 1,\ \mathsf{mx}'\mathsf{mx} = 1$$

Thus, a mixed unitary category can be visualized schematically as:



Within the unitary category, $A \simeq A^\dagger$ by the means of the unitary structure map. However, outside the unitary core, an object is not in general isomorphic to its dagger.

Given any †-isomix category, the preunitary objects always form a unitary category, $\mathsf{Unitary}(\mathbb{X})$ with a forgetful †-isomix functor $U : \mathsf{Unitary}(\mathbb{X}) \longrightarrow \mathbb{X}$ which produces a MUC. $\mathsf{Unitary}(\mathbb{X}) \longrightarrow \mathbb{X}$ satisfies a couniversal property, see [6, Section 5.2], and is the "largest" possible unitary core for the †-isomix category $\mathbb{X}$. We shall call $\mathsf{Unitary}(\mathbb{X})$ the **canonical** unitary core of $\mathbb{X}$.

3 Measurement

A measurement in a MUC can be broken into two steps: a compaction step into an object in the unitary core followed by a demolition measurement within the unitary core.

Definition 3.1. *Let $M : \mathbb{U} \longrightarrow \mathbb{C}$ be a MUC. A* **compaction** *to U of an object $A \in \mathbb{C}$ is a retraction, $r : A \longrightarrow M(U)$. This means that there is a section $s : M(U) \longrightarrow A$ such that $sr = 1_{M(U)}$. A compaction is said to be* **canonical** *when $\mathbb{U} = \mathsf{Unitary}(\mathbb{X})$ (so U is a preunitary object).*

The compact object, $M(U)$, has a unitary structure map which is an isomorphism between $M(U)$ and $M(U)^\dagger$ given by composing the unitary structure map of U with the preservator:

$$\psi := M(U) \xrightarrow{M(\varphi)} M(U^\dagger) \xrightarrow{\rho} M(U)^\dagger$$

Once one has reached $M(U)$, one can follow with a classical demolition measurement $U \xrightarrow{w} X$ to obtain an overall compaction $A \xrightarrow{rM(w)} M(X)$, which gives a (demolition) measurement in a MUC.

We start by showing how a compaction gives rise to a binary idempotent:

Definition 3.2. *A* **binary idempotent** *in any category is a pair of maps (u, v) with $\mathsf{u} : A \longrightarrow B$, and $\mathsf{v} : B \longrightarrow A$ such that $\mathsf{uvu} = \mathsf{u}$, and $\mathsf{vuv} = \mathsf{v}$.*

A binary idempotent, $(\mathsf{u}, \mathsf{v}) : A \longrightarrow B$ gives a pair of idempotents : $e_A := \mathsf{uv} : A \longrightarrow A$, and $e_B := \mathsf{vu} : B \longrightarrow B$. We say a binary idempotent **splits** in case the idempotents e_A and e_B split.

Lemma 3.3. *In any category the following are equivalent:*

(i) $(\mathsf{u}, \mathsf{v}) : A \longrightarrow B$ *is a binary idempotent which splits.*

(ii) A pair of idempotents $e : A \longrightarrow A$, and $d : B \longrightarrow B$ which split through isomorphic objects.

Observe that a compaction of an object, say A, in any MUC, gives the following system of maps:

$$A \underset{s}{\overset{r}{\rightleftarrows}} M(U) \underset{\psi^{-1}}{\overset{\psi:=M(\varphi)\rho}{\rightleftarrows}} M(U)^\dagger \underset{s^\dagger}{\overset{r^\dagger}{\rightleftarrows}} A^\dagger$$

Thus the compaction gives rise to a binary idempotent $(\mathsf{u}, \mathsf{v}) : A \longrightarrow A^\dagger$ where $\mathsf{u} := r\psi r^\dagger$ and $\mathsf{v} := s^\dagger \psi^{-1} s$

Because U is a unitary object, we have that $\varphi(\varphi^{-1\dagger}) = \iota$. The preservator, on the other hand, satisfies $\iota\rho^\dagger = M(\iota)\rho$ (see after Definition 3.17 in [6]). Thus $\iota\rho^\dagger = M(\iota)\rho = M(\varphi\varphi^{-1\dagger})\rho = M(\varphi)\rho M(\varphi^{-1})^\dagger$ and whence $\psi = M(\varphi)\rho = \iota\rho^\dagger M(\varphi)^\dagger = \iota(M(\varphi)\rho)^\dagger = \iota\psi^\dagger$. This allows us to observe:

$$\begin{aligned} \iota\mathsf{u}^\dagger &= \iota(r\psi r^\dagger)^\dagger = \iota r^{\dagger\dagger}\psi^\dagger r^\dagger = r\iota\psi^\dagger r^\dagger = r\psi r^\dagger = \mathsf{u} \\ \mathsf{v}^\dagger &= (s^\dagger\psi^{-1}s)^\dagger = s^\dagger(\psi^\dagger)^{-1}s^{\dagger\dagger} = s^\dagger(\iota^{-1}\psi)^{-1}s^{\dagger\dagger} = s^\dagger\psi^{-1}\iota s^{\dagger\dagger} = s^\dagger\psi^{-1}s\iota = \mathsf{v}\iota \end{aligned}$$

This leads to the following definition:

Definition 3.4. *A binary idempotent,* $(\mathsf{u}, \mathsf{v}) : A \longrightarrow A^\dagger$ *in a* †*-LDC, is a* **†-binary idempotent***, written* $\dagger(\mathsf{u}, \mathsf{v})$*, if* $\mathsf{u} = \iota\mathsf{u}^\dagger$*, and* $\mathsf{v}^\dagger = \mathsf{v}\iota$*.*

In a †-monoidal category, where $A = A^\dagger$ and $\iota = 1_A$ this make $\mathsf{u} = \mathsf{u}^\dagger$ and $\mathsf{v} = \mathsf{v}^\dagger$, thus $\mathsf{uv} = (\mathsf{vu})^\dagger$. This means that if we require $\mathsf{uv} = \mathsf{vu}$ we obtain a dagger idempotent in the sense of [26].

Splitting a dagger binary idempotent almost produces a preunitary object. In a †-LDC, we shall call an object A with an isomorphism $\varphi : A \longrightarrow A^\dagger$ such that $\varphi\varphi^{\dagger-1} = \iota$ a **weak preunitary object**. Clearly, in a †-isomix category, a weak preunitary object (A, φ) is a preunitary object when, in addition, A is in the core. We next observe that dagger binary idempotent split through weak preunitary objects:

Lemma 3.5. *In a* †*-LDC with a* †*-binary idempotent* $\dagger(\mathsf{u}, v) : A \longrightarrow A^\dagger$*:*

(i) $e_{A^\dagger} := \mathsf{vu} = (\mathsf{uv})^\dagger =: (e_A)^\dagger$*;*

(ii) if $\dagger(\mathsf{u}, \mathsf{v})$ *splits with* $e_A = A \xrightarrow{r} E \xrightarrow{s} A$ *then* E *is a weak preunitary object.*

Thus, in a †-isomix category, an object which splits a †-binary idempotent is always weakly preunitary. In order to ensure that the splitting of a dagger binary idempotent is a preunitary object – and so a canonical compaction – it remains to ensure that the splitting is in the core. This leads to the following definition:

Definition 3.6. *An idempotent* $A \xrightarrow{e} A$ *in an isomix category,* $\mathbb{X}$*, is a* **coring idempotent** *if it is equipped with natural* $\kappa^L_X : X \oplus A \longrightarrow X \otimes A$ *and* $\kappa^R_X : A \oplus X \longrightarrow A \otimes X$ *such that the following diagrams commute:*

$$\begin{array}{ccc} X \otimes A & \xrightarrow{1\otimes e} & X \otimes A \\ {\scriptstyle 1\otimes e}\downarrow & [\mathbf{KL.1}] & \downarrow{\scriptstyle \mathsf{mx}} \\ X \otimes A & \xleftarrow[\kappa^L_X]{} & X \oplus A \end{array} \qquad \begin{array}{ccc} X \oplus A & \xrightarrow{1\oplus e} & X \oplus A \\ {\scriptstyle 1\oplus e}\downarrow & [\mathbf{KL.2}] & \downarrow{\scriptstyle \kappa^L_X} \\ X \oplus A & \xleftarrow[\mathsf{mx}]{} & X \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes X & \xrightarrow{e\otimes 1} & A \otimes X \\ {\scriptstyle e\otimes 1}\downarrow & [\mathbf{KR.1}] & \downarrow{\scriptstyle \mathsf{mx}} \\ A \otimes X & \xleftarrow[\kappa^R_X]{} & A \oplus X \end{array} \qquad \begin{array}{ccc} A \oplus X & \xrightarrow{e\oplus 1} & A \oplus X \\ {\scriptstyle e\oplus 1}\downarrow & [\mathbf{KR.2}] & \downarrow{\scriptstyle \kappa^R_X} \\ A \oplus X & \xleftarrow[\mathsf{mx}]{} & A \otimes X \end{array}$$

For a coring idempotent $A \xrightarrow{e} A$, the transformations κ^{-}_X act on a splitting as the inverse of the mixor, mx. Thus, a coring idempotent always splits through the core:

Lemma 3.7. *In a mix category:*

(i) An idempotent splits through the core if and only if it is coring;

(ii) If (u, v) *is a binary idempotent then* uv *is coring if and only if* vu *is coring.*

This allows:

Definition 3.8. *A* **coring binary idempotent** *in a mix category is a binary idempotent,* (u,v)*, for which either* uv *or* vu *is a coring idempotent.*

These observations can be summarized by the following:

Theorem 3.9. *In the MUC* $M{:}\mathsf{Unitary}(\mathbb{X}) \longrightarrow \mathbb{X}$*, with* †*-isomix category* $\mathbb{X}$*, an object* U *is a compaction of* A *if and only if* U *is the splitting of a coring* †*-binary idempotent* $\dagger(\mathsf{u},\mathsf{v}) : A \longrightarrow A^\dagger$*.*

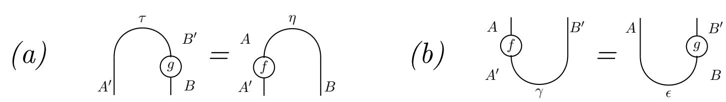
Using this characterization of canonical compaction, we will show that, in the presence of free †-exponential modalities, complementarity always arises as a canonical compaction of a †-linear bialgebra on the exponential modality.

4 Complementarity

The objective of this section, is to describe strong complementarity within a †-isomix category. Strong complementarity classically is, in a †-monoidal setting, between two special commutative †-Frobenius Algebras. In a linear setting with two distinct tensor products, Frobenius Algebras are generalized by linear Frobenius algebras, a.k.a. linear monoids [16, 7] which consist of a ⊗-monoid and a dual ⊕-comonoid. The directionality of the linear distributor makes a bialgebraic interaction between two †-linear monoids impossible. However, such an interaction is possible between a †-FA and a †-linear coFrobenius algebra and this gives a †-linear bialgebra. These †-linear bialgebras provide the basis for complementarity in a †-isomix category. In a MUC, one can, furthermore, consider the effect of a compaction which preserves these structures to arrive back at the classical CQM notion of a complementary system.

4.1 Duals

Definition 4.1. *A* **dual** *in an LDC,* $(\eta, \epsilon) : A \dashv\!\!\dashv B$*, consists of maps,* $\eta : \top \longrightarrow A \oplus B$*, and* $\epsilon : B \otimes A \longrightarrow \bot$ *such that the snake diagrams hold. A* **morphism of duals***,* $(f, g) : (\eta, \epsilon) : A \dashv\!\!\dashv B \longrightarrow (\tau, \gamma) : A' \dashv\!\!\dashv B'$*, is given by a pair of maps* $f : A \longrightarrow A'$*, and* $g : B' \longrightarrow B$ *such that:*

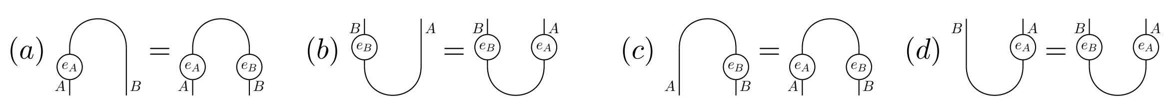


A **self-duality** *is a dual,* $(\eta, \epsilon) : A \dashv\!\!\dashv B$ *in which* A *is isomorphic to* B *(or* $A = B$*)).*

A morphism of duals is determined by either of the maps, as f is dual to g. They are, thus, Australian mates, see [7]. In a †-LDC, if A is dual to B, then $B^\dagger$ is dual to $A^\dagger$:

A binary idempotent can implicitly express a morphism of duals, which becomes explicit when the idempotent splits.

Definition 4.2. *A binary idempotent,* (u,v) *is* **retractional** *on a dual* $(\eta, \epsilon) : A \dashv\!\!\dashv B$ *if equations* (a) *and* (b)*, below, hold. On the other hand,* (u, v) *is* **sectional***, if equations* (c) *and* (d) *hold:*



where $e_a := \mathsf{uv}$ *and* $e_B := \mathsf{vu}$*.*

The idempotent pair (e_A, e_B) is a morphism of duals only when the binary idempotent is both sectional and retractional.

Lemma 4.3. *In an LDC, a binary idempotent,* (u, v) *on a dual* $(\eta, \epsilon) : A \dashv\!\vdash B$*, with splitting* $A \xrightarrow{r} E \xrightarrow{s} A$*, and* $B \xrightarrow{r'} E' \xrightarrow{s'} B$ *is sectional (respectively retractional) if and only if the section* (s, r') *(respectively the retraction* (r, s')*) is a morphism for* $(\eta(r \oplus r'), (s' \otimes s)\epsilon) : E \dashv\!\vdash E'$*.*

Splitting binary idempotents which are either sectional or retractional on a dual produces a self-duality.

We next observe that the dagger of a dual is itself a dual:

Lemma 4.4. *Suppose* $\mathbb{X}$ *is a* †*-LDC, and* $(\eta, \epsilon) : A \dashv\!\vdash B$ *is a dual in* $\mathbb{X}$*. Then,* $(\epsilon\dagger, \eta\dagger) : B^\dagger \dashv\!\vdash A^\dagger$ *is a dual where:*

$$\epsilon\dagger := \top \xrightarrow{\lambda_\top} \bot^\dagger \xrightarrow{\epsilon^\dagger} (B \otimes A)^\dagger \xrightarrow{\lambda_\oplus^{-1}} B^\dagger \oplus A^\dagger$$

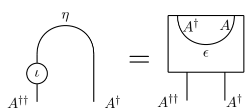
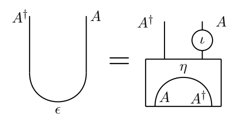
$$\eta\dagger := A^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (A \oplus B)^\dagger \xrightarrow{\eta^\dagger} \top^\dagger \xrightarrow{\lambda_\bot^{-1}} \bot$$

Definition 4.5. *In a* †*-LDC, a* **†-dual***,* $A \overset{\dagger}{\dashv\!\vdash} A^\dagger$ *is a dual* $(\eta, \epsilon) : A \dashv\!\vdash A^\dagger$ *such that*

$$(\iota_A, 1_{A^\dagger}) : (\eta, \epsilon) : A \dashv\!\vdash A^\dagger \longrightarrow (\eta^\dagger, \epsilon^\dagger) : A^{\dagger\dagger} \dashv\!\vdash A^\dagger$$

is an isomorphism of duals (see 4.5.1 (a), (b)*). A* **self †-dual** *is a right* †*-dual with an isomorphism* $\alpha : A \longrightarrow A^\dagger$ *such that* $\alpha\alpha^{-1\dagger} = \iota$*. A* **morphism of †-duals** *consists of a pair of maps* $(f, f^\dagger) : ((\eta, \epsilon) : A \overset{\dagger}{\dashv\!\vdash} A^\dagger) \longrightarrow ((\eta', \epsilon') : B \overset{\dagger}{\dashv\!\vdash} B^\dagger)$ *which are morphism of duals.*

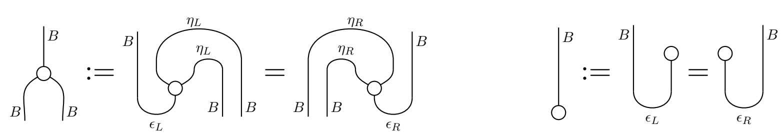
$(\iota_A, 1_{A^\dagger})$ being an isomorphism of the duals means that the following equations hold:

(a)  (or equivalently) (b)  (4.5.1)

In a †-LDC, splitting a †-binary idempotent on a †-dual gives a †-self-duality if the binary idempotent is either sectional or retractional (See Lemma C.4).

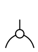
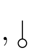
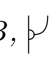
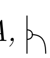
4.2 Linear monoid

Definition 4.6. *A* **linear monoid** *[7, 16],* $A \overset{\circ}{\dashv\!\vdash} B$*, in an LDC consists of a monoid* $(A, e : \top \longrightarrow A, m : A \otimes A \longrightarrow A)$*, a left dual* $(\eta_L, \epsilon_L) : A \dashv\!\vdash B$*, and a right dual* $(\eta_R, \epsilon_R) : B \dashv\!\vdash A$ *such that:*

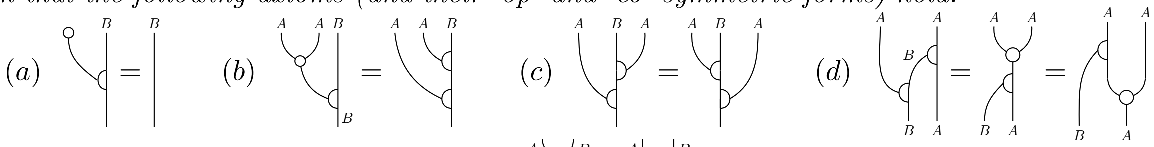
 (4.6.1)

In a symmetric LDC, a linear monoid is **symmetric** when its duals are symmetric, i.e, $\eta_R = \eta_L c_\oplus$, and $\epsilon_R = c_\otimes \eta_L$. A symmetric linear monoid is determined by a monoid, (A, m, u) and a dual $(\eta, \epsilon) : A \dashv\!\vdash B$. There is a more useful form for linear monoids in which their similarity to the usual description of Frobenius algebras in CQM is evident:

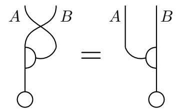
Proposition 4.7. *A linear monoid,* $A \overset{\circ}{\dashv\!\vdash} B$*, in an LDC is equivalent to the following data:*

- *a monoid* $(A,$  $: A \otimes A \longrightarrow A,$  $: \top \longrightarrow A)$
- *a comonoid* $(B,$  $: B \longrightarrow B \oplus B,$  $: B \longrightarrow \bot)$
- *actions,*  $: A \otimes B \longrightarrow B,$  $: B \otimes A \longrightarrow B$*, and coactions*  $: A \longrightarrow B \oplus A,$  $: A \longrightarrow A \oplus B$*,*

such that the following axioms (and their 'op' and 'co' symmetric forms) hold:



If the linear monoid is symmetric, then:



To keep the terminologies distinct, we will refer to a linear monoid as a linear monoid. A linear monoid $A \overset{\circ}{-\!\!\!\dashv} B$, in a monoidal category gives a Frobenius Algebra when it is a self-linear monoid, that is $A = B$, and the dualities coincide with the self-dual cup, and the cap. Note that while a Frobenius Algebra is always on a self-dual object, a linear monoid allows Frobenius interaction between distinct objects which are duals of one another.

Definition 4.8. *A* **morphism of linear monoids** *is a pair of maps, $(f, g) : (A \overset{\circ}{-\!\!\!\dashv} B) \longrightarrow (A' \overset{\circ}{-\!\!\!\dashv} B')$, such that $f : A \longrightarrow A'$ is a monoid morphism (or equivalently $g : B' \longrightarrow B$ is a comonoid morphism), and (f, g) and (g, f) preserves the left and the right duals respectively.*

Note that a morphism of Frobenius algebras is usually given by a single monoid morphism which is an isomorphism. However, in the case of a morphism of linear monoids, the comonoid morphism, $g : B' \longrightarrow B$, is the cyclic mate of the monoid morphism, $f : A \longrightarrow A'$. This means that a linear monoid morphism is not restricted to being an isomorphism.

Given an idempotent, $e_A : A \longrightarrow A$, and a monoid, (A, m, u) in a monoidal category, e_A is **retractional** on the monoid if $e_A m = e_A m(e_A \otimes e_A)$. e_A is **sectional** on the monoid if $m(e_A \otimes e_A) = e_A m(e_A \otimes e_A)$ and $u e_A = u$.

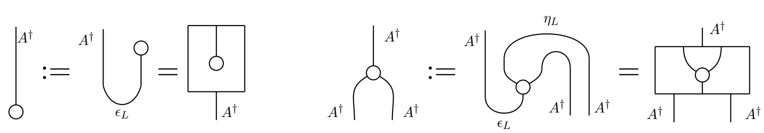
Lemma 4.9. *In a monoidal category, a split idempotent $e : A \longrightarrow A$ on a monoid (A, m, u), with splitting $A \xrightarrow{r} E \xrightarrow{s} A$, is sectional (respectively retractional) if and only if the section s (respectively the retraction r) is a monoid morphism for $(E, (s \otimes s)mr, ur)$.*

A binary idempotents, (u, v) is **sectional** (respectively **retractional**) on a linear monoid when $e_A = \mathsf{uv}$ and $e_B = \mathsf{vu}$ satisfies the conditions in the following table:

(u, v) **sectional** on $A \overset{\circ}{-\!\!\!\dashv} B$	(u, v) **retractional** on $A \overset{\circ}{-\!\!\!\dashv} B$
e_A preserves (A, m, u) sectionally	e_A preserves (A, m, u) retractionally
(e_A, e_B) preserves $(\eta_L, \epsilon_L) : A \dashv\!\!\vdash B$ sectionally	(e_A, e_B) preserves $(\eta_L, \epsilon_L) : A \dashv\!\!\vdash B$ retractionally
(e_B, e_A) preserves $(\eta_R, \epsilon_R) : B \dashv\!\!\vdash A$ retractionally	(e_B, e_A) preserves $(\eta_R, \epsilon_R) : B \dashv\!\!\vdash A$ sectionally

Splitting a sectional/retractional binary idempotent on a linear monoid gives a self-linear monoid on the splitting.

Definition 4.10. *A* **†-linear monoid***, $(A, \curlyvee, \circ) \overset{\dagger\circ}{-\!\!\!\dashv} (A^\dagger, \curlywedge, \circ)$, in a †-LDC is a linear monoid such that $(\eta_L, \epsilon_L) : A \dashv\!\!\vdash A^\dagger$, and $(\eta_R, \epsilon_R) : A^\dagger \dashv\!\!\vdash A$ are †-duals and:*

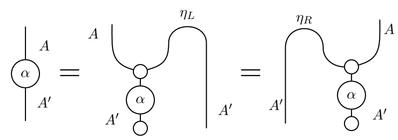


(4.10.1)

A **morphism** of †-linear monoids is a pair of maps $(f, f^\dagger)$ which are morphisms of underlying linear monoids. Similar to duals, splitting sectional/retractional binary idempotents on a linear monoid induces a self-linear monoid. In the presence of dagger, one gets a †-self-linear monoid (See Lemma C.8).

A †-linear monoid in a unitary category is equivalent to a †-FA under certain conditions:

Lemma 4.11. *In a compact LDC, a self-linear monoid, $A \overset{\circ}{\multimap\!\!\Vdash} A'$ with an isomorphism $\alpha : A \longrightarrow A'$ precisely corresponds to a Frobenius algebra under the linear equivalence, $\mathsf{Mx}_\downarrow$ if and only if the linear monoid satisfies the equation below. In a unitary category, any †-linear monoid $A \overset{\dagger\circ}{\multimap\!\!\Vdash} A^\dagger$ precisely corresponds to a †-Frobenius algebra under the same equivalence if and only if the †-linear monoid satisfies the below equation for the unitary structure isomorphism $\varphi_A : A \longrightarrow A^\dagger$.*

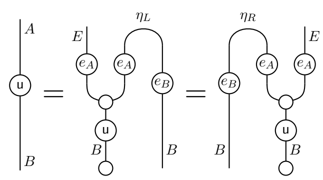


(4.11.1)

The equation in the previous Lemma should reminds us of involutive monoids [22, Theorem 5.28] in †-monoidal categories.

One can get Frobenius Algebras by splitting binary idempotents on linear monoids:

Lemma 4.12. *In an isomix category $\mathbb{X}$, let $E \overset{\bullet}{\multimap\!\!\Vdash} E'$ be a self-linear monoid in $\mathsf{Core}(\mathbb{X})$ given by splitting a coring sectional/retractional binary idempotent (u, v) on linear monoid $A \overset{\circ}{\multimap\!\!\Vdash} B$. Let $\alpha : E \longrightarrow E'$ be the isomorphism. Then, E is a Frobenius Algebra under the linear equivalence $\mathsf{Mx}_\downarrow$ if and only if the binary idempotent satisfies the following equation:*



(4.12.1)

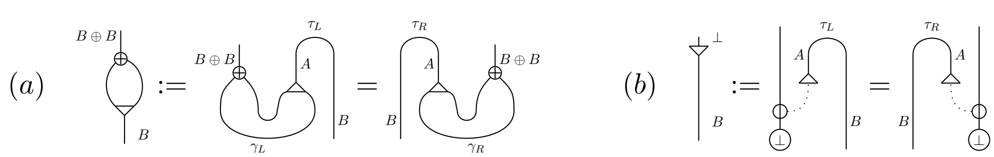
where $e_A = \mathsf{uv}$ and $e_B = \mathsf{vu}$.

In a †-isomix category splitting a sectional or retractional †-coring binary idempotent on a †-linear monoid gives a †-self-linear monoid on a pre-unitary object. If the binary idempotent satisfies equation 4.12.1, then, by using Lemma 4.11 and 4.12, one gets a †-Frobenius Algebra on the splitting.

4.3 Linear comonoid

The bialgebra law is a central ingredient of a complimentary system. The directionality of the linear distributors in an LDC forbids a bialgebraic interaction between two linear monoids. A linear monoid, however, can interact bialgebraically with a linear comonoid.

Definition 4.13. *A* **linear comonoid**, *$A \underset{\circ}{\multimap\!\!\Vdash} B$, in an LDC consists of a $\otimes$-comonoid, $(A, \curlywedge, \perp)$, and a left and a right dual, $(\tau_L, \gamma_L) : A \dashv\!\!\dashv B$, and $(\tau_R, \gamma_R) : B \dashv\!\!\dashv A$, such that:*



(4.13.1)

Note that while a linear monoid has a $\otimes$-monoid and an $\oplus$-comonoid, a linear comonoid has a $\otimes$-comonoid and an $\oplus$-monoid.

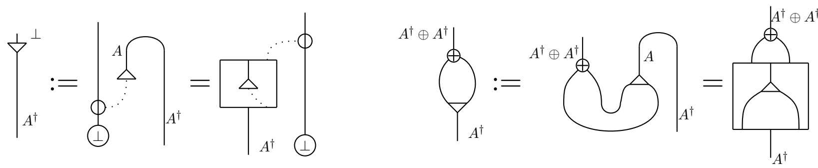
A **morphism** of linear comonoids, $(f, g) : (A \underset{\circ}{\multimap\!\!\Vdash} B) \longrightarrow (A' \underset{\circ}{\multimap\!\!\Vdash} B')$, consists of a pair of maps, $f : A \longrightarrow A'$ and $g : B' \longrightarrow B$, such that f is a comonoid morphism, and (f, g) and (g, f) are morphisms of the left and the right duals respectively.

In a monoidal category, an idempotent $e : A \longrightarrow A$ is **sectional** (respectively **retractional**) on a comonoid (A, d, k) if $ed = ed(e \otimes e)$ (respectively if $d(e \otimes e) = ed(e \otimes e)$, and $ek = k$). In an LDC, a binary idempotent (u, v) is **sectional** (respectively **retractional**) on a linear monoid when $e_A = \mathsf{uv}$ and $e_B = \mathsf{vu}$ satisfies the conditions in the table below.

(u, v) **sectional** on $A \underset{\circ}{-\!\!\dashv\!\vdash} B$	(u, v) **retractional** on $A \underset{\circ}{-\!\!\dashv\!\vdash} B$
e_A preserves (A, d, k) sectionally	e_A preserves (A, d, k) retractionally
(e_A, e_B) preserves $(\eta_L, \epsilon_L) : A \dashv\!\vdash B$ sectionally	(e_A, e_B) preserves $(\eta_L, \epsilon_L) : A \dashv\!\vdash B$ retractionally
(e_B, e_A) preserves $(\eta_R, \epsilon_R) : B \dashv\!\vdash A$ retractionally	(e_B, e_A) preserves $(\eta_R, \epsilon_R) : B \dashv\!\vdash A$ sectionally

Splitting a sectional or retractional binary idempotent on a linear comonoid gives a self-linear comonoid.

Definition 4.14. *A* **†-linear comonoid** *in a †-LDC is* $A \underset{\dagger\circ}{-\!\!\dashv\!\vdash} A^\dagger$ *is a linear comonoid,* $A \underset{\circ}{-\!\!\dashv\!\vdash} A^\dagger$ *such that* $(\tau_L, \gamma_L) : A \dashv\!\vdash A^\dagger$*, and* $(\tau_R, \gamma_R) : A^\dagger \dashv\!\vdash A$ *are †-duals, and:*



(4.14.1)

A **†-self-linear comonoid** *consists of an isomorphism* $\alpha : A \longrightarrow A^\dagger$ *such that* $\alpha\alpha^{-1\dagger} = \iota$*. A* **morphism of †-linear comonoids** *is a pair* $(f, f^\dagger)$ *such that* $(f, f^\dagger)$ *is a morphism of the underlying linear comonoids.*

In a †-LDC, splitting a †-binary idempotent on a †-linear comonoid gives a †-self-linear comonoid when the binary idempotent is either sectional or retractional. In the next section, we discuss linear bialgebras which are given by an interacting linear monoid and linear comonoid.

4.4 Linear bialgebras

All the results concerning bialgebras are necessarily set in symmetric LDCs and we shall assume that linear monoid and the linear comonoid are symmetric.

Definition 4.15. *A* **linear bialgebra,** $\frac{(a,b)}{(a',b')} : A \overset{\circ}{\underset{\triangledown}{-\!\!\dashv\!\vdash}} B$*, in an LDC consists of a linear monoid,* $(a, b) \colon A \overset{\circ}{-\!\!\dashv\!\vdash} B$ *and a linear comonoid,* $(a', b') \colon A \underset{\triangledown}{-\!\!\dashv\!\vdash} B$ *such that* $(A, \curlyvee, \circ, \curlywedge, \circ)$ *is a* $\otimes$*-bialgebra and* $(B, \curlyvee, \circ, \curlywedge, \circ)$ *is a* $\oplus$*-bialgebra. A* **morphism** *of linear bialgebras is a morphism both of the linear monoids and linear comonoids.*

A linear bialgebra is **commutative** if the $\oplus$-monoid and $\otimes$-monoid are commutative. A **self-linear bialgebra** is a linear bialgebra, in which there is an isomorphism $A \xrightarrow{\alpha} B$ (so essentially the algebra is on one object).

A binary idempotent on a linear bialgebra is **sectional (respectively retractional)** if its sectional (respectively retractional) on the linear monoid, and the linear comonoid. In an LDC, splitting a sectional or retractional binary idempotent on a linear bialgebra induces a self-linear bialgebra on the splitting.

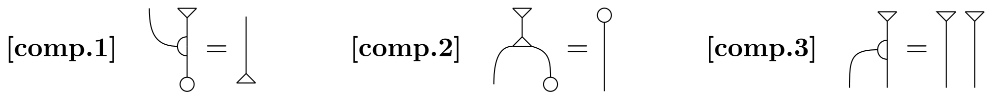
Definition 4.16. *A* **†-linear bialgebra,** $\frac{(a,b)}{(a',b')} : A \overset{\dagger\circ}{\underset{\triangledown}{-\!\!\dashv\!\vdash}} A^\dagger$*, is a linear bialgebra with a †-linear monoid, and a †-linear comonoid. A* **†-self-linear bialgebra** *is †-linear bialgebra which is also a self-linear bialgebra such that the isomorphism,* $\alpha : A \longrightarrow A^\dagger$*, satisfies* $\alpha\alpha^{-1\dagger} = \iota$*.*

Note that A is a weak preunitary object: if it was in the core, as well, it would be a preunitary object. In a †-LDC, splitting a †-binary idempotent on a †-linear bialgebra gives a †-self-linear bialgebra if the idempotent is either a sectional or retractional.

4.5 Complementary systems

In quantum mechanics, two quantum observables are complementary [19] if measuring one observable increases the uncertainty regarding the value of the other. Complementarity is a key feature distinguishing classical from quantum mechanics. In CQM, the complimentary principle is described using interacting commutative †-FAs. This section describes complementarity in isomix categories:

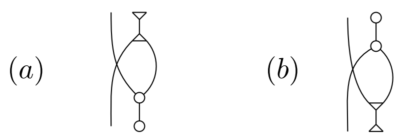
Definition 4.17. *A* **complementary system** *in an isomix category, $\mathbb{X}$, is a commutative and co-commutative self-linear bialgebra, $\frac{(a,b)}{(a',b')} : A \overset{\circ}{\underset{\triangledown}{\dashv}} A$ such that the following equations (with their 'op' symmetries) hold:*



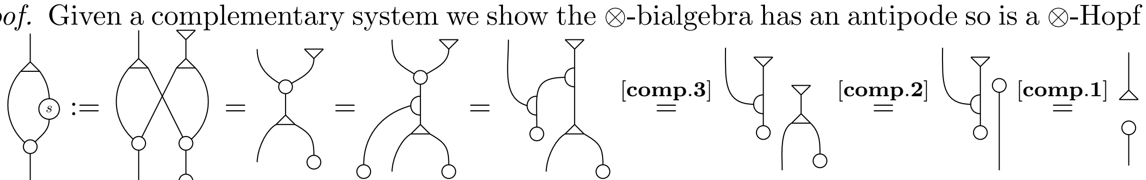
A **†-compementary system** *in a †-isomix category is a †-self-linear bialgebra which is also a complementary system.*

Notice that we are using the alternate presentation of linear monoids by actions and coactions (see Proposition 4.7). Thus, **[comp.1]** requires that the counit of the linear comonoid is dual to the counit via the linear monoid dual. while **[comp.2]** requires that the unit of the linear monoid is dual to the counit via dual of the linear comonoid. Finally, **[comp.3]** requires that the coaction map of the linear monoid duplicates the unit of †-linear comonoid. The 'co' symmetry of the equations are immediate from the commutativity and cocommutativity of the linear bialgebra. The 'op' symmetry of equations **[comp.1]**-**[comp.3]** hold automatically for a †-complimentary system.

Lemma 4.18. *If $A \overset{\circ}{\underset{\triangledown}{\dashv}} A$ is a complementary system in an isomix category, then A is $\otimes$-bialgebra with antipode given by (a) and $\oplus$-bialgebra with antipode given by (b):*



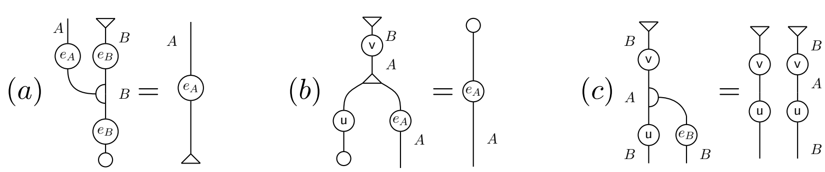
Proof. Given a complementary system we show the $\otimes$-bialgebra has an antipode so is a $\otimes$-Hopf algebra:



Similarly, the $\oplus$-bialgebra has an antipode using the 'op' versions of **[comp.1]**-**[comp.3]**. □

A †-complimentary system in a unitary category corresponds to the usual notion of interacting commutative †-FAs [11] when its linear monoid and the linear comonoid satisfy condition 4.11.1. Splitting binary idempotents on a linear bialgebra produces a complimentary systems under the following conditions:

Lemma 4.19. *In an isomix category, a self linear bialgebra given by splitting a coring binary idempotent* (u, v) *on a commutative and a cocommutative linear bialgebra $A \overset{\circ}{\underset{\triangledown}{\dashv}} B$ is a complimentary system if and only if the binary idempotent satisfies the following conditions (and their 'op' symmetric forms):*



(4.19.1)

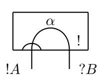
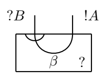
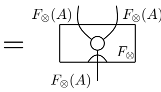
where $e_A = \mathsf{uv}$, *and* $e_B = \mathsf{vu}$.

In the next section, we provide an example of the previous Lemma using exponential modalities.

5 Exponential modalities

An LDC is said to have exponential modalities if it is equipped with a linear comonad $((!, ?), (\epsilon, \eta), (\delta, \mu))$ [3]. The linearity of the functors in a $(!, ?)$-LDC means that $(!, \delta, \epsilon)$ is monoidal comonad while $(?, \mu, \eta)$ is a comonoidal monad, and $(!(A), \Delta_A, \multimap_A)$ is a natural cocommutative comonoid while $(?(A), \nabla_A, \Upsilon_A)$ is a natural commutative monoid. A †-$(!, ?)$-LDC is a $(!, ?)$-LDC in which all the functors and natural transformations are †-linear (see [6]).

In a $(!, ?)$-LDC, any dual, $(\alpha, \beta) : A \dashv\!\vdash B$, induces a dual, $(\alpha_!, \beta_?) : !A \dashv\!\vdash ?B$ (see the below diagrams), on the exponential modalities using the linearity of $(!, ?)$. This means that, any dual induces a linear comonoid, $(\alpha_!, \beta_?) : !A \underset{\circ}{\dashv\!\vdash} ?B$, where the comonoid structure is given by the modality.

$\alpha_! :=$  $= m_\top(!\alpha)\nu_\otimes$ $\qquad \beta_? :=$  $= \nu_\oplus(?\epsilon)n_\perp$ $\qquad m_{F_\otimes} :=$  $= m_\otimes F_\otimes(m)$

Any linear functor $(F_\otimes, F_\oplus)$ applied to a linear monoid $(\alpha, \beta) : A \overset{\circ}{\dashv\!\vdash} B$ always produces a linear monoid $(\alpha_F, \beta_F) : F_\otimes(A) \overset{\boxed{\circ}}{\dashv\!\vdash} F_\oplus(B)$ with multplication m_F as shown in the above diagram. This simple observation when applied to the exponential modalities has a striking effect:

Lemma 5.1. *In any $(!, ?)$-LDC any linear monoid $(a, b) : A \overset{\circ}{\dashv\!\vdash} B$ and an arbitrary dual $(a', b') : A \dashv\!\vdash B$ gives a linear bialgebra $\frac{(a_!, b_!)}{(a'_!, b'_?)} : !A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv\!\vdash}} ?B$ using the natural cocommutative comonoid $(!A, \Delta_A, \multimap)$.*

The bialgebra structure results from the naturality of Δ and $\multimap$ over the functorially induced monoid structure.

A $(!, ?)$-LDC has **free exponential modalities** if, for any object A, $(!A, \Delta_A, \multimap_A)$ is a cofree cocommutative comonoid, and $(?(A), \nabla_A, \Upsilon_A)$ is a free commutative monoid [23]. An example of a †-LDC with free (†-)exponential modalites is finiteness matrices over the complex numbers, $\mathsf{FMat}(\mathbb{C})$. Moreover, $\mathsf{FMat}(\mathbb{C})$, is a †-isomix category and gives a key example of a MUC as discussed in [6] (although exponentials are not discussed). The universal property of free exponential modalities in a $(!, ?)$-LDC implies the following:

Lemma 5.2. *If (f, g) is a morphism of duals then, the unique map $(f^\flat, g^\sharp)$ induced by the universal property of the free exponential is a morphism of linear comonoids.*

This is illustrated by the commuting diagram (a), below.

$$(a)\quad \begin{array}{ccc} (x, y) : X \underset{\bullet}{\dashv\!\vdash} Y & & \\ \Big\downarrow{\scriptstyle (f^\flat, g^\sharp)} & \searrow{\scriptstyle (f,g)} & \\ (a_!, b_?) : !A \underset{\circ}{\dashv\!\vdash} ?B & \xrightarrow[(\epsilon,\eta)]{} & (a, b) : A \dashv\!\vdash B \end{array}$$

$$(b)\quad \begin{array}{ccc} \frac{(x,y)}{(x',y')} : X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\vdash}} Y & & \\ \Big\downarrow{\scriptstyle (f^\flat, g^\sharp)} & \searrow{\scriptstyle (f,g)} & \\ \frac{(a_!, b_?)}{(a'_!, b'_?)} : !A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv\!\vdash}} ?B & \xrightarrow[(\epsilon,\eta)]{} & (a, b) : A \overset{\circ}{\dashv\!\vdash} B\ ;\ (a', b') : A \dashv\!\vdash B \end{array}$$

The results discussed so far can be combined to give, as shown in diagram (b) above, the more complicated observation:

Proposition 5.3. *In a (!,?)-LDC with free exponential modalities, let $\frac{(x,y)}{(x',y')} : X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\vdash}} Y$ be a linear bialgebra, $(a, b) : A \overset{\circ}{\dashv\!\vdash} B$ a linear monoid, and $(a', b') : A \dashv\!\vdash B$ a dual, then*

$$(f^\flat, g^\sharp) : \left(\frac{(x, y)}{(x', y')} : X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\vdash}} Y \right) \longrightarrow \left(\frac{(a_!, b_?)}{(a'_!, b'_?)} : !A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv\!\vdash}} ?B \right)$$

is a morphism of bialgebras, whenever $f : (X, \blacktriangledown, \bullet) \longrightarrow (A, \triangledown, \circ)$ is a morphism of monoids, and (f, g) is a morphism of both duals:

$$(f, g) : ((x, y) : X \overset{\circ}{\dashv\!\vdash} Y) \longrightarrow ((a, b) : A \overset{\circ}{\dashv\!\vdash} B) \quad \text{and} \quad (f, g) : ((x', y') : X \dashv\!\vdash Y) \longrightarrow ((a', b') : A \dashv\!\vdash B)$$

Corollary 5.4. *In a (!,?)-LDC with free exponential modalities, if* $A \overset{\bullet}{\underset{\blacktriangledown}{\dashv\vdash}} B$ *is a linear bialgebra then* $(1^\flat, 1^\sharp) : (A \overset{\bullet}{\underset{\blacktriangledown}{\dashv\vdash}} B) \longrightarrow (!A \overset{\boxdot}{\underset{\triangledown}{\dashv\vdash}} ?A)$ *is a morphism of bialgebras, making* $A \overset{\bullet}{\underset{\blacktriangledown}{\dashv\vdash}} B$ *a retract of* $!A \overset{\boxdot}{\underset{\triangledown}{\dashv\vdash}} ?A$.

The corollary shows that every self-linear bialgebra in an (!,?)-LDC, with free exponential modalities, induces a sectional binary idempotent on the induced linear bialgebra on the exponential modalities:

$$!A \underset{1^\flat}{\overset{\epsilon}{\rightleftarrows}} A \simeq B \underset{1^\sharp}{\overset{\eta}{\rightleftarrows}} ?B$$

Combining Corollary 5.4, and Lemma 4.19, we get:

Theorem 5.5. *In an* $(!, ?)$*-isomix category with free exponential modalities, every complimentary system arises as a splitting of a sectional binary idempotent on the free exponential modalities.*

The above results extend directly to †-linear bialgebras in †-LDCs with free exponential modalities due to the †-linearity of $(!, ?)$, (η, ϵ), (Δ, ∇), and $(\curlywedge, \curlyvee)$.

6 Conclusion

Bohr's principle of complementarity [19] states that, due to the wave and particle nature of matter, physical properties occur in complimentary pairs. In the formulation of measurements in a MUC, a measurement on A induces a measurement on $A^\dagger$, and vice versa. A measurement transfers the structures of A and $A^\dagger$ – and the interactions between these – onto a single compact object. Our main result displays a complementary system as the result of a measurement of a †-linear bialgebra in which two distinct dual structures have been "compacted" into one structure. This provides an interesting perspective on Bohr's principle.

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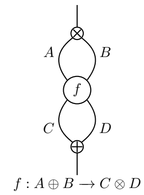
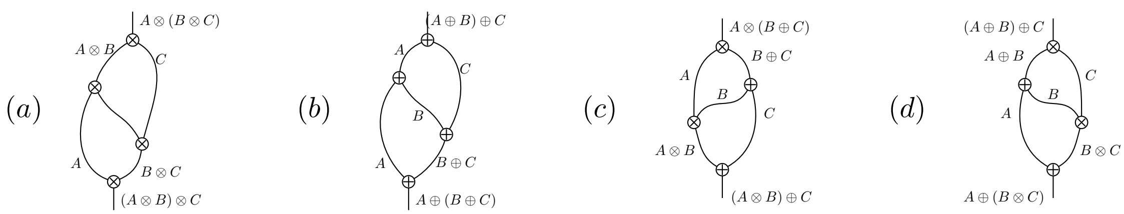
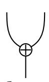
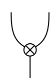
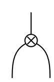
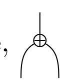
A Preliminaries

This section provides a review of the graphical calculus for LDCs. For a more detailed exposition, see [5, 8]. The following are the generators of LDC circuits: wires represent objects and circles represent maps. The input wires of a map are tensored (with $\otimes$), and the output wires are "par"ed (with $\oplus$). The following diagram represents a map $f : A \otimes B \longrightarrow C \oplus D$.



The $\otimes$-associator, the $\oplus$-associator, the left linear distributor, and the right linear distributors are, respectively, drawn as follows:



 is the $\oplus$-introduction rule,  is $\otimes$-introduction rule,  is the $\otimes$-elimination rule,  is the $\oplus$-elimination rule.

The unitors are drawn as follows:

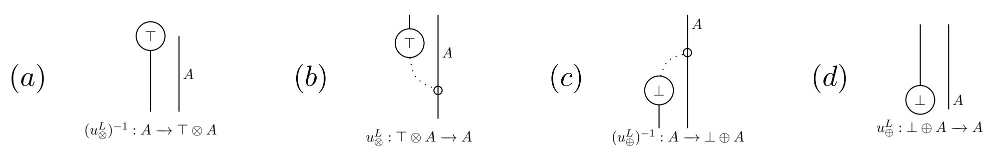
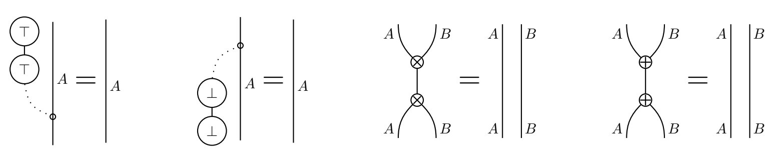
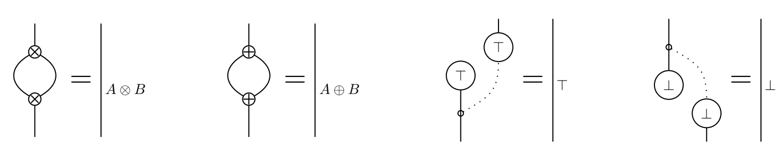


Diagram (a) is called the left $\top$-introduction, (b) is called the left $\top$-elimination, (c) is the left $\bot$-introduction, and (d) is the left $\bot$-elimination. The unit $\top$ is introduced, and the counit $\bot$ is eliminated using the thinning links which are shown using dotted wires in the diagrams.

The following are a set of circuit equalities (which when oriented become reduction rewrite rules):

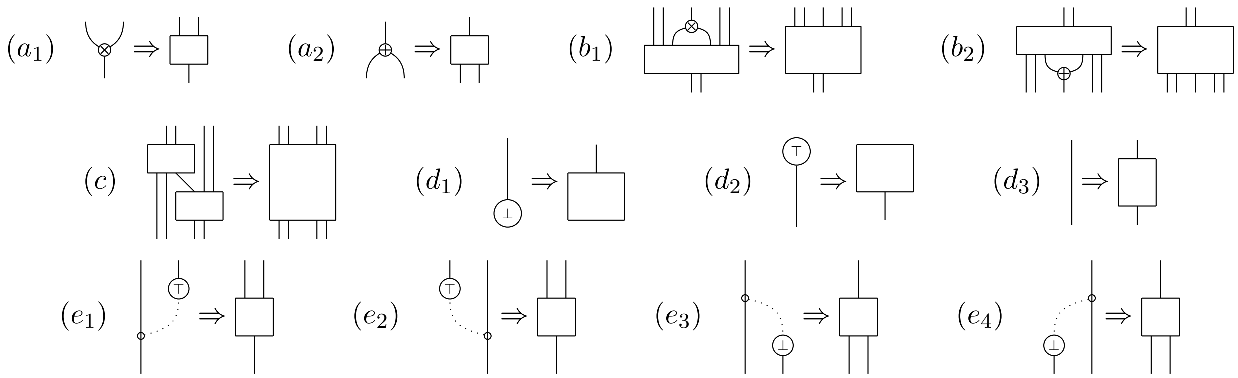


The following are also circuit equalities (and when oriented become expansion rules:)



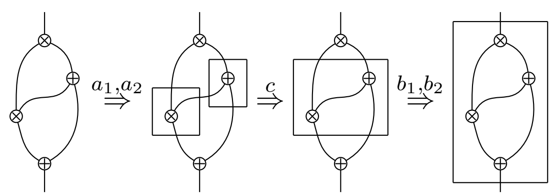
As in linear logic, not all circuit diagrams constructed from these basic components represent a valid LDC circuit. In his seminal paper on linear logic, [18], Girard introduced a criterion for the correctness of his representation of proofs using proof nets based on switching links. A valid proof structure must be connected and acyclic for all the switching link choices. Using this correctness criterion has the disadvantage of requiring exponential time in the number of switching links. Danos and Regnier [15] improved this situation significantly by providing an algorithm for correctness which takes linear time (see [20]) on the size of the circuit. To verify the validity of the circuit diagrams of LDCs, Blute et.al. [5], provided a boxing algorithm which was based on Danos and Regnier's more efficient algorithm which we now describe.

In order to verify that an LDC circuit is valid, circuit components are "boxed" using the following rules:



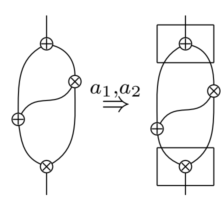
Double lines refer to multiple number of wires. $\otimes$-introduction and $\oplus$-elimination are boxed in (a_1) and (a_2) respectively. In (b_1), it is shown how a box 'eats' the $\otimes$-elimination: in (b_2) the dual rule shows a $\oplus$-introduction being eaten. (c) shows how boxes can be amalgamated when they are connected by a single wire. In (e_1)-(e_4), it is shown how the thinning links can be boxed. By progressively enclosing the components of the circuit in boxes using these rules, if we end up with a single box (or a wire), precisely when the circuit is valid.

As an example, we verify the validity of the left linear distributor:

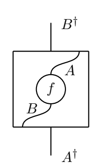


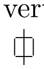
In the first step the $\otimes$-introduction and $\oplus$-elimination are boxed. In the second step the boxes are amalgamated along the single wire joining them. In the third step, the box absorbs the $\otimes$-elimination and $\oplus$-introduction.

In contrast we now show that the reverse of the linear distributor is invalid as the boxing process gets stuck:

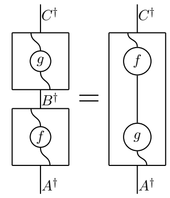


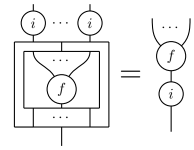
Suppose $\mathbb{X}$ is a †-LDC and $f : A \to B \in \mathbb{X}$. Then, the map $f^\dagger : B^\dagger \to A^\dagger$ is graphically depicted as follows:

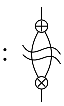


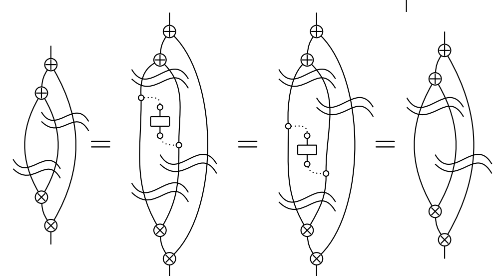
The rectangle is a functor box for the †-functor. Notice how we use vertical mirroring to express the contravariance of the †-functor. By the functoriality of $(\_)^\dagger$, we have:  = |.

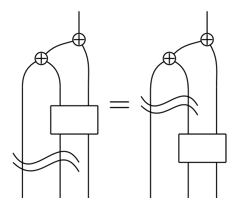
These contravariant functor boxes compose... contravariantly. Given maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$:



Dagger boxes interact with involution map $A \xrightarrow{\iota} A^{\dagger\dagger}$ as follows: 

Suppose U is in the Core($\mathbb{X}$) and $U \otimes A \xrightarrow{\mathsf{mx}_{U,A}} U \oplus B$. We represent $\mathsf{mx}_{U,A}^{-1}$ as follows: 

Observe that the mx^{-1} maps can be slid past each other: 

And, indeed,  slides over components in circuits: 

It is important to note that one may not have a legal LDC circuit inside a †-box. This complicates the correctness criterion. However, the required correctness criterion is discussed in [24].

B Compaction

Proof of Lemma 3.3 is as follows:

Lemma B.1. *In any category the following are equivalent:*

(i) $(\mathsf{u}, \mathsf{v}) : A \longrightarrow B$ *is a binary idempotent which splits.*

(ii) A pair of idempotents $e : A \longrightarrow A$, *and* $d : B \longrightarrow B$ *which split through isomorphic objects.*

Proof.

$(i) \Rightarrow (ii)$**:** Suppose that uv splits as $A \xrightarrow{r} A' \xrightarrow{s} A$ (so $rs = \mathsf{uv}$ and $sr = 1_{A'}$)and vu splits as $B \xrightarrow{p} B' \xrightarrow{q} B$ (so $pq = \mathsf{vu}$ and $qp = 1_{B'}$) then we obtain two maps $\alpha := s\mathsf{u}p : A' \longrightarrow B'$ and $\beta := q\mathsf{v}r : B' \longrightarrow A'$ which are inverse to each other:

$$\alpha\beta := (s\mathsf{u}p)(q\mathsf{v}r) = s\mathsf{u}(pq)\mathsf{v}r = s\mathsf{uvuv}r = s\mathsf{uv}r = srsr = 1_{A'}$$
$$\beta\alpha := (q\mathsf{v}r)(s\mathsf{u}p) = q\mathsf{v}(rs)\mathsf{u}p = q\mathsf{vuvu}p = q\mathsf{vu}p = qpqp = 1_{B'}.$$

$(ii) \Rightarrow (i)$**:** Suppose $e_A : A \longrightarrow A$ and $e_B : B \longrightarrow B$ are idempotents which split, respectively, as $A \xrightarrow{r} A' \xrightarrow{s} A$ (so $rs = e_A$ and $sr = 1_{A'}$) and $B \xrightarrow{p} B' \xrightarrow{q} B$ (so $pq = e_B$ and $qp = 1_{B'}$), so that

$\gamma : A' \longrightarrow B'$ is an isomorphism then we obtain two maps $\mathsf{u} := r\gamma q : A \longrightarrow B$ and $\mathsf{v} := p\gamma^{-1}s : B \longrightarrow A$. We observe:

$$\mathsf{uvu} := r\gamma q p\gamma^{-1} s r\gamma q = r\gamma\gamma^{-1} s r\gamma q = rsr\gamma q = r\gamma q = \mathsf{u}$$
$$\mathsf{vuv} := p\gamma^{-1} s r\gamma q p\gamma^{-1} s = p\gamma^{-1}\gamma q p\gamma^{-1} s = pqp\gamma^{-1} s = p\gamma^{-1} s = \mathsf{v}.$$

□

Proof of Lemma 3.5 is as follows:

Lemma B.2. *In a †-LDC with a †-binary idempotent $\dagger(\mathsf{u}, v) : A \longrightarrow A^\dagger$:*

(i) $e_{A^\dagger} := \mathsf{vu} = (\mathsf{uv})^\dagger =: (e_A)^\dagger$;

(ii) *if* $\dagger(\mathsf{u},\mathsf{v})$ *splits with* $e_A = A \xrightarrow{r} E \xrightarrow{s} A$ *then* E *is a weak preunitary object.*

Proof.

(i) $e_{A^\dagger} = \mathsf{vu} = \mathsf{v}\iota\mathsf{u}^\dagger = \mathsf{v}^\dagger\mathsf{u}^\dagger = (\mathsf{uv})^\dagger = e_A^\dagger$.

(ii) Suppose (u,v) is a †-binary idempotent, and uv splits as $\mathsf{uv} = A \xrightarrow{r} U \xrightarrow{s} A$. This means vu splits as $\mathsf{vu} = A^\dagger \xrightarrow{s^\dagger} U^\dagger \xrightarrow{r^\dagger} A^\dagger$. This yields an isomorphism $\alpha = s\mathsf{u}s^\dagger : E \longrightarrow E^\dagger$ satisfying:

$$\alpha(\alpha^{-1})^\dagger = s\mathsf{u}s^\dagger(r^\dagger\mathsf{v}r)^\dagger = s\mathsf{u}s^\dagger r^\dagger\mathsf{v}^\dagger r^{\dagger\dagger} = s\mathsf{uvuv}^\dagger r^{\dagger\dagger} = s\mathsf{uv}^\dagger r^{\dagger\dagger} = s\mathsf{uv}\iota r^{\dagger\dagger} = s\mathsf{uv}r\iota = srsr\iota = \iota.$$

□

Proof of Lemma 3.7 is as follows:

Lemma B.3. *In a mix category:*

(i) An idempotent splits through the core if and only if it is coring;

(ii) If (u,v) is a binary idempotent then uv *is coring if and only if* vu *is coring.*

Proof.

(i) Let $A \xrightarrow{e} A$ be a coring idempotent which splits as $A \xrightarrow{r} U \xrightarrow{s} A$. Define $\mathsf{mx}'_{U,X} := U \oplus X \xrightarrow{s \oplus 1} A \oplus X \xrightarrow{\kappa_R^X} A \otimes X \xrightarrow{r \otimes 1} U \otimes 1$, then U is in the core because $\mathsf{mx}'_{U,X} = \mathsf{mx}^{-1}_{U,X}$. Conversely the inverse of the mixor for U and an object X defines the κ_X^-.

(ii) The splitting of $u\mathsf{v}$ is isomorphic to the splitting of vu and the core includes isomorphism.

□

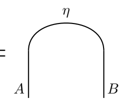
C Complimentary systems

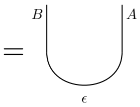
C.1 Duals

Definition C.1. *A* **dual** $(\eta, \epsilon) : A \dashv\!\!\dashv B$ *in a linearly distributive cateogry (LDC) consists of:*

- *unit map* $\eta : \top \longrightarrow A \oplus B$
- *counit map* $\epsilon : B \otimes A \longrightarrow \bot$ *such that the following diagrams commute:*

The unit map unit and the counit maps are drawn in string diagrams as a cap and a cup respectively:

$\eta :=$ 

and $\epsilon :=$ 

An LDC in which every object has a chosen left and right linear dual is called a **$*$-autonomous category**.

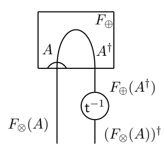
Lemma C.2. *†-linear functors preserve †-linear duals.*

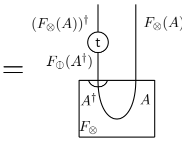
Proof. Let $\mathbb{X}$, and $\mathbb{Y}$ be †-LDCs. Let $(F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ be a †-linear functor. Then, for each object, we have maps:

$$\mathsf{s}_A : F_\otimes(A^\dagger) \xrightarrow{\simeq} (F_\oplus(A))^\dagger \qquad \mathsf{t}_A : (F_\otimes(A))^\dagger \xrightarrow{\simeq} F_\oplus(A^\dagger)$$

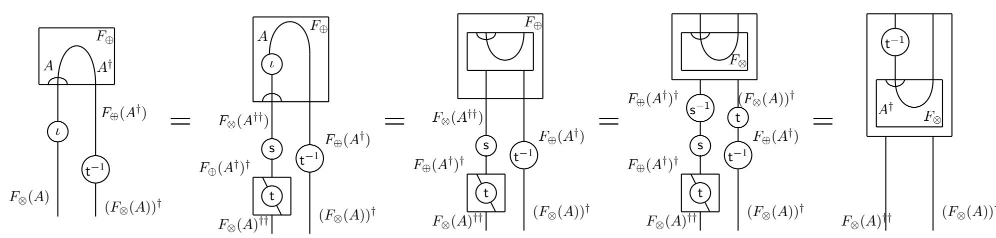
such that $(\mathsf{s}, \mathsf{t}) : (F_\otimes((-)^\dagger), (F_\oplus(-)^\dagger)) \Rightarrow ((F_\oplus(-))^\dagger, F_\otimes((-)^\dagger))$ is a linear natural isomorphism. We must prove that if $A \overset{\dagger\circ}{\dashv\!\!\vdash} A^\dagger$ is a right †-linear dual, then $F_\otimes(A)$ has a right †-dual.

Suppose $A \overset{\dagger\circ}{\dashv\!\!\vdash} A^\dagger$ is a right †-linear dual. Since, linear functors preserve linear duals, $F_\otimes(A) \overset{\circ}{\dashv\!\!\vdash} F_\otimes(A)^\dagger$ is also a linear dual with the cap and the cup given by:

$\top \xrightarrow{\eta'} F_\otimes(A) \oplus F_\otimes(A)^\dagger :=$ 

$F_\otimes(A)^\dagger \otimes F_\otimes(A) \xrightarrow{\epsilon'} \bot :=$ 

To show that it is a right †-dual:



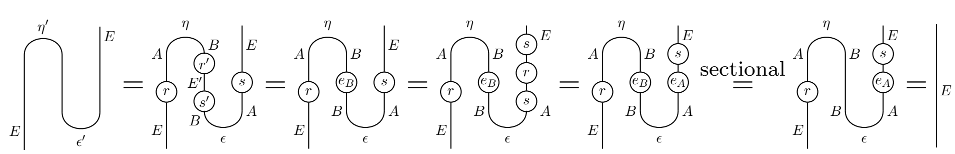
$\square$

The following is proof of Lemma 4.3:

Lemma C.3. *In an LDC, a binary idempotent,* (u, v) *on a dual* $(\eta, \epsilon) : A \dashv\!\!\vdash B$*, with splitting* $A \xrightarrow{r} E \xrightarrow{s} A$*, and* $B \xrightarrow{r'} E' \xrightarrow{s'} B$ *is sectional (respectively retractional) if and only if the section* (s, r') *(respectively the retraction* (r, s')*) is a morphism for* $(\eta(r \oplus r'), (s' \otimes s)\epsilon) : E \dashv\!\!\vdash E'$.

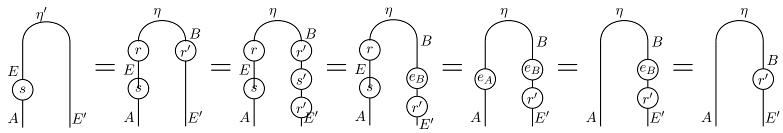
Proof. Suppose (u, v) is sectional binary idempotent of a dual $(\eta, \epsilon) : A \dashv\!\!\vdash B$ with the splitting $A \xrightarrow{r} E \xrightarrow{s}$ A, and $B \xrightarrow{r'} E' \xrightarrow{s'} B$. Let $e_A = \mathsf{uv}$, and $e_B = \mathsf{vu}$. We must how that $(\eta(r \oplus r'), (s' \otimes s)\epsilon) : E \dashv\!\!\vdash E'$ is dual and (s, r') is a dual homomorphism.

We first prove that $(\eta', \epsilon') : E \dashv\!\!\vdash E'$ satisfies the snake equations, where $\eta' = \eta(r \oplus r')$, $\epsilon' = (s' \otimes s)\epsilon$.

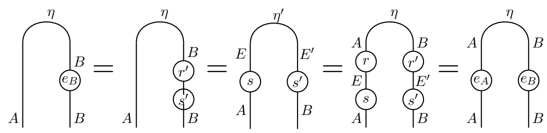


Similary, the other snake equation can be proven.

Next, we show that $(s, r') : ((\eta', \epsilon') : E \dashv\vdash E') \longrightarrow ((\eta, \epsilon) : A \dashv\vdash B)$ is a dual homomorphism:



For the converse assume that $(\eta' := \eta(r \oplus r'), \epsilon' := (s' \otimes s)\epsilon) : E \dashv\vdash E'$ is a dual and (s, r') is a dual homomorphism. We prove that (u, v) is sectional on $(\eta, \epsilon) : A \dashv\vdash B$:



Similary, one can prove the statement for retractional binary idempotents. □

Lemma C.4. *In an †-LDC, a †-binary idempotent, $\dagger(\mathsf{u}, \mathsf{v})$ on a †-dual $(\eta, \epsilon) : A \dashv\vdash B$, with splitting $A \xrightarrow{r} E \xrightarrow{s} A$, and $A^\dagger \xrightarrow{s^\dagger} E^\dagger \xrightarrow{r^\dagger} A^\dagger$ is sectional (respectively retractional) if and only if the section $(s, s^\dagger)$ (respectively the retraction $(r, r^\dagger)$) is a morphism for $(\eta(r \oplus s^\dagger), (r^\dagger \otimes s)\epsilon) : E \dashv\vdash E^\dagger$.*

Proof. Let (u, v) be a sectional †-binary idempotent on a †-dual $(\eta, \epsilon) : A \overset{\dagger}{\dashv\vdash} A^\dagger$. Let the †-binary idempotent split:

$$\mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A \qquad \mathsf{vu} = A^\dagger \xrightarrow{s^\dagger} E^\dagger \xrightarrow{r^\dagger} A^\dagger$$

It follows from Lemma C.3 that the splitting is self-dual. The self-dual is given as $(\eta', \epsilon') : E \dashv\vdash E^\dagger$, where $\eta' := \eta(r \oplus s^\dagger)$, and $\epsilon' := (r^\dagger \otimes s)\epsilon$. We must prove that the self-dual is also a †-dual i.e, equation 4.5.1 holds for $(\eta', \epsilon') : E \dashv\vdash E^\dagger$:

$$\begin{aligned}
\eta'(\iota \oplus 1) &= \eta(r \oplus s^\dagger)(\iota \oplus 1) = \eta(r\iota \oplus s^\dagger) \\
&\overset{nat.\ \iota}{=} \eta(\iota r^{\dagger\dagger} \oplus s^\dagger) = \eta(\iota \oplus 1)(r^{\dagger\dagger} \oplus s^\dagger) \\
&\overset{\dagger-dual}{=} (\epsilon)^\dagger \lambda_\oplus^{-1}((r^{\dagger\dagger} \oplus s^\dagger)) \overset{nat.\ \lambda_\oplus}{=} (\epsilon)^\dagger((r^\dagger \otimes s)^\dagger)\lambda_\oplus^{-1} \\
&= ((r^\dagger \otimes s)\epsilon)^\dagger \lambda_\oplus^{-1} = (\epsilon')^\dagger \lambda_\oplus^{-1}
\end{aligned}$$

The statement for a retractional †-binary idempotent is proven similarly.

The converse is straightforward. □

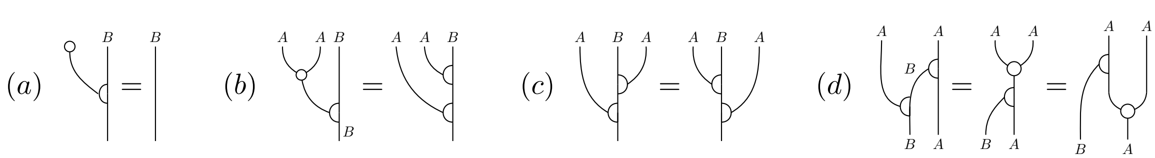
C.2 Linear monoids

Proof for Proposition 4.7:

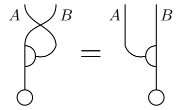
Proposition C.5. *A* **linear monoid**, *$A \overset{\circ}{\dashv\vdash} B$, in an LDC is equivalent to the following data:*

- *a monoid $(A, \ : A \otimes A \longrightarrow A, \ : \top \longrightarrow A)$*
- *a comonoid $(B, \ : B \longrightarrow B \oplus B, \ : B \longrightarrow \bot)$*
- *actions, $: A \otimes B \longrightarrow B$, $: B \otimes A \longrightarrow B$, and coactions $: A \longrightarrow B \oplus A$, $: A \longrightarrow A \oplus B$,*

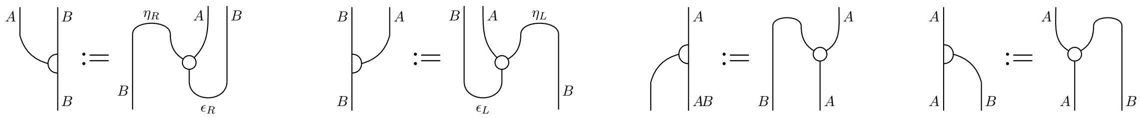
such that the following axioms (and their 'op' and 'co' symmetric forms) hold:


(C.5.1)

If the linear monoid is symmetric, then:

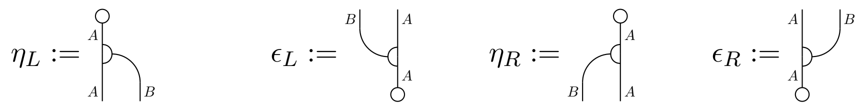


Proof. Suppose $A \overset{\circ}{\dashv} B$ is a linear monoid. Then, the comonoid on B is given by the left dual (or equivalently the right dual) of the monoid. The action and the coaction maps are defined as follows:



Equations (a)-(d) can be verified easily.

For the converse, the left and the right duals are defined as follows:

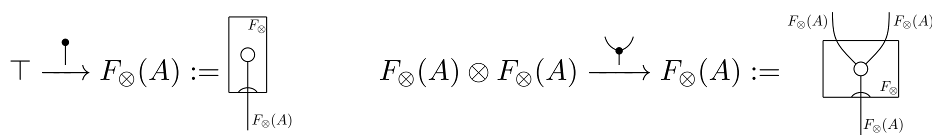


□

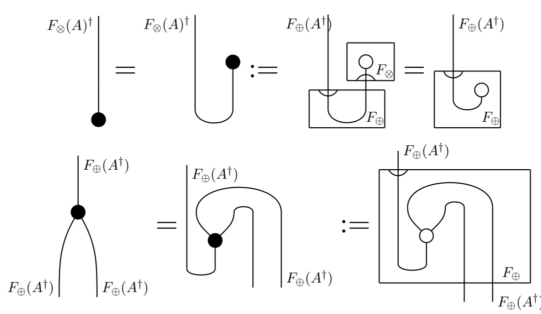
Lemma C.6. *†-linear functors preserve †-linear monoids.*

Proof. Suppose $(F_\otimes, F_\oplus) : \mathbb{X} \longrightarrow \mathbb{Y}$ is a †-linear functor, and $A \overset{\dagger\circ}{\dashv} A^\dagger$ is a dagger linear monoid in $\mathbb{X}$. Since, linear functors preserve linear monoids, $F_\otimes(A) \overset{\circ}{\dashv} F_\oplus(A^\dagger)$ is a linear monoid.

To prove that $F_\otimes(A)$ is a right †-linear monoid, define the multiplication, unit, comultiplication, counit, action and coaction maps are given as follows:

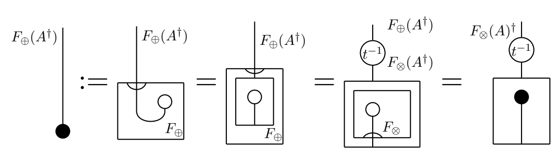


Define:

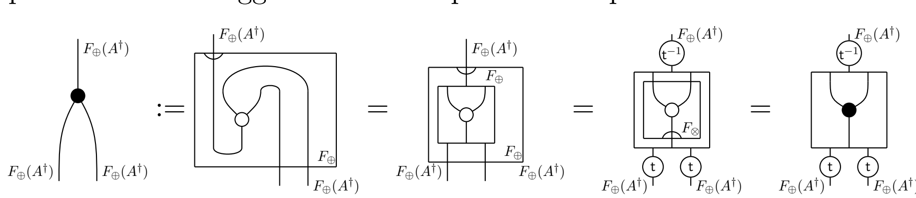


It remains to prove that:

- The counit is the dagger of the unit map:



- The comultiplication is the dagger of the multiplication map:



□

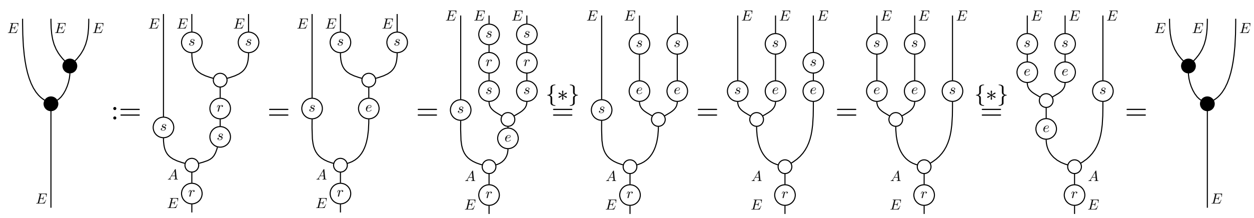
Lemma C.7. *In a monoidal category, a split idempotent $e : A \longrightarrow A$ on a monoid (A, m, u), with splitting $A \xrightarrow{r} E \xrightarrow{s} A$, is sectional (respectively retractional) if and only if the section s (respectively the retraction r) is a monoid morphism for $(E, (s \otimes s)mr, ur)$.*

Proof. Suppose $e : A \longrightarrow A$ is an idempotent with the splitting $A \xrightarrow{r} E \xrightarrow{s}$ and (A, m, u) is a monoid. Suppose e is sectional on A i.e, $(e \otimes e)m = (e \otimes e)me$, and $ue = u$.

We must prove that (E, m', u') is a monoid where $m' := (s \otimes s)mr$ and $u' := ur$. Unit law and associativity law are proven as follows.

$$(u' \otimes 1)m' = (ur \otimes 1)(s \otimes s)mr = (urs \otimes s)mr = (ue \otimes s)mr \overset{sectional}{=} (u \otimes s)mr = sr = 1$$

In the following string diagrams, we use black circle for (E, m', u'), and white circle (A, m, u).



$\{*\}$ is true because e is sectional on (A, m, u).

Finally, $s : E \longrightarrow A$ is a monoid homomorphism because:

$m's = (s \otimes s)mrs = (srs \otimes srs)me = (s \otimes s)(e \otimes e)me = (s \otimes s)(e \otimes e)m = (s \otimes s)m.$

For **the converse** assume that (E, m', u') is a monoid where $m' = (s \otimes s)mr$, and $u' = ur$, and s is a monoid homomorphism. Then, $e : A \longrightarrow A$ is sectional on (A, m, u) because:

$$ue = urs = u's = u$$
$$(e \otimes e)m = (r \otimes r)(s \otimes s)m = (r \otimes r)m's = (r \otimes r)(s \otimes s)mrs = (e \otimes e)me$$

The statement is proven similarly when the idempotent is retractional on the monoid. □

Lemma C.8.

(i) In an LDC, let (u, v) be a split binary idempotent on a linear monoid $A \overset{\circ}{\dashv\!\!\dashv} B$ with monoid (A, m, u) and splitting $A \xrightarrow{r} E \xrightarrow{s} A$, and $B \xrightarrow{r'} E' \xrightarrow{s'} B$. The binary idempotent (u, v) is sectional (respectively retractional) if and only if the section (s, r') (respectively the retraction (r, s')) is a morphism of linear monoid for $E \overset{\bullet}{\dashv\!\!\dashv} E'$ with monoid $(E, (s \otimes s)mr, ur)$.

(ii) In an †-LDC, let $\dagger(\mathsf{u}, \mathsf{v})$ be a split †-binary idempotent on a †-linear monoid $A \overset{\circ}{\dashv\!\!\dashv} A^\dagger$ with monoid (A, m, u) and splitting $A \xrightarrow{r} E \xrightarrow{s} A$. The binary idempotent $\dagger(\mathsf{u}, \mathsf{v})$ is sectional (respectively retractional) if and only if the section $(s, s^\dagger)$ (respectively the retraction $(r, r^\dagger)$) is a morphism of †-linear monoid for $E \overset{\dagger\bullet}{\dashv\!\!\dashv} E^\dagger$ with monoid $(E, (s \otimes s)mr, ur)$.

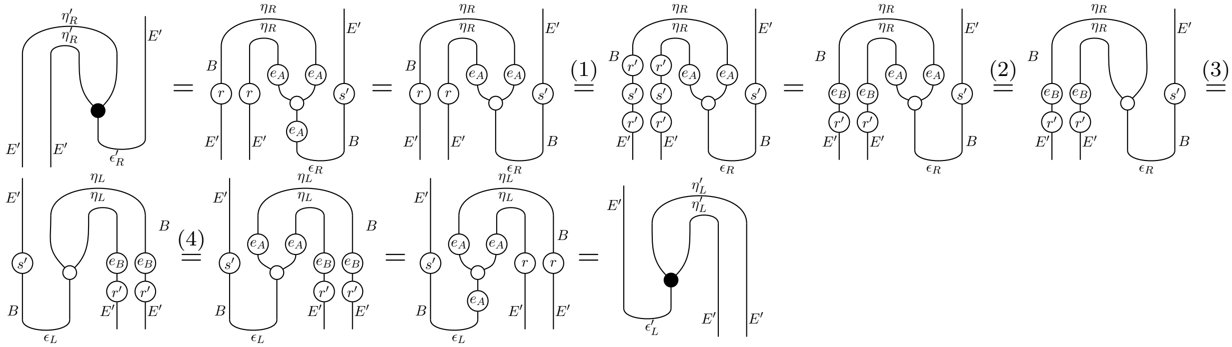
Proof.

(i) Suppose (u, v) is sectional on $A \overset{\circ}{\dashv\!\!\dashv} B$. Let,

$$e_A = \mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A \qquad e_B = \mathsf{vu} = B \xrightarrow{r'} E' \xrightarrow{s'} B$$

be a splitting of (u, v). From Lemma C.3, and Lemma C.7 we know that (A, m', u') is a monoid where $m' = (s \otimes s)mr$ and $u' = ur$, and $(\eta_L(r \otimes r'), (s' \otimes s)\epsilon_L) : E \dashv\!\!\dashv E'$, and $(\eta_R(r' \otimes r), (s \otimes s')\epsilon_R) : E' \dashv\!\!\dashv E$ are the left and the right duals respectively.

To prove that the above data is a linear monoid we must show that equation 4.6.1 holds:



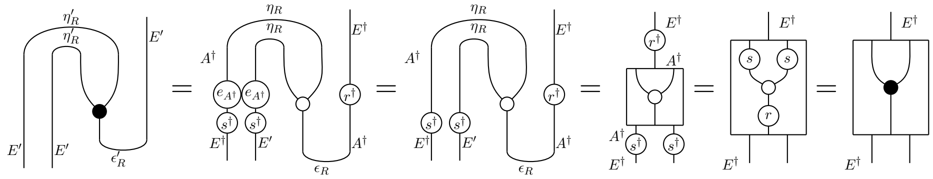
were (1) is true because e_A is section on (A, m, u), (2) because (e_B, e_A) is retractional on $(\eta_R, \epsilon_R) : B \dashv\!\!\dashv A$, (3) holds because A is a linear monoid, and (4) holds because (e_A, e_B) is sectional on $(\eta_L, \epsilon_L) : A \dashv\!\!\dashv B$.

The converse of the statement is straightforward from Statement (i) of Lemma C.3, and Lemma C.7.

(ii) Suppose $\dagger(\mathsf{u}, \mathsf{v})$ is a sectional $\dagger$-binary idempotent on a $\dagger$-linear monoid $A \overset{\dagger\circ}{-\!\!\shortmid\!\shortmid} A^\dagger$. Moreover, $\dagger(\mathsf{u}, \mathsf{v})$ splits as follows:

$$e_A = \mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A \qquad\qquad e_{A^\dagger} = \mathsf{vu} = A^\dagger \xrightarrow{s^\dagger} E' \xrightarrow{r^\dagger} A^\dagger$$

We must show that A is a $\dagger$-linear monoid. From statement (i) of this Lemma, we know that $E \overset{\circ}{-\!\!\shortmid\!\shortmid} E'$ is a linear monoid. It is also a $\dagger$-linear monoid because:

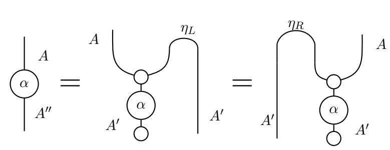


The statements are proven similarly for retractional binary idempotent. The converse is straightforward. □

The following Lemma provides equivalent conditions for a linear monoid to correspond to a Frobenius Algebra as in Lemma 4.7

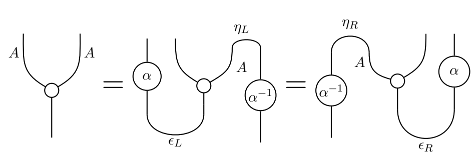
Lemma C.9. *In an LDC, the following statements (and their 'op' symmetries) are equivalent for a self-linear monoid $A \overset{\circ}{-\!\!\shortmid\!\shortmid} A'$ with the isomorphism $\alpha : A \longrightarrow A'$.*

(a) The isomorphism α coincides with the following maps:



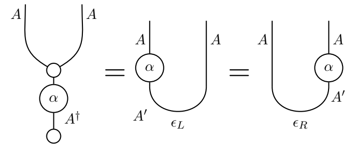
(C.9.1)

(b) The coaction maps given by the comultiplication and the linear duality coincides with the comultiplication:



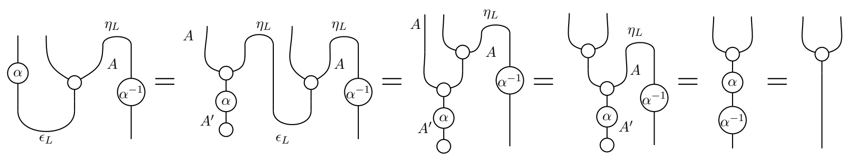
(C.9.2)

(c) *The self-dual cup and cap coincide with the cups and the caps of the duals given in the linear monoid:*



(C.9.3)

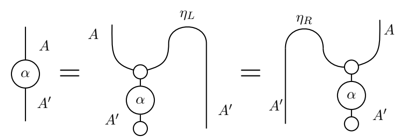
Proof. For $(a) \Rightarrow (b)$, substitute α in (b) using C.9.1.



$(b) \Rightarrow (c)$, and $(c) \Rightarrow (a)$ are strightforward. ☐

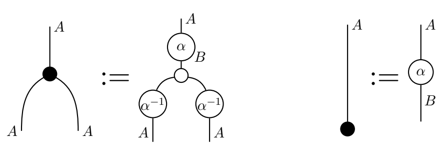
Proof of Lemma 4.11 is given by the following Lemma and its Corollary:

Lemma C.10. *In a compact LDC, a self-linear monoid, $A \overset{\circ}{-\!\!\!\shortmid\!\vdash} A'$ with an isomorphism $\alpha : A \longrightarrow A'$ precisely corresponds to a Frobenius algebra under the linear equivalence, $\mathsf{Mx}_\downarrow$ if and only if the linear monoid satisfies the equation below.*

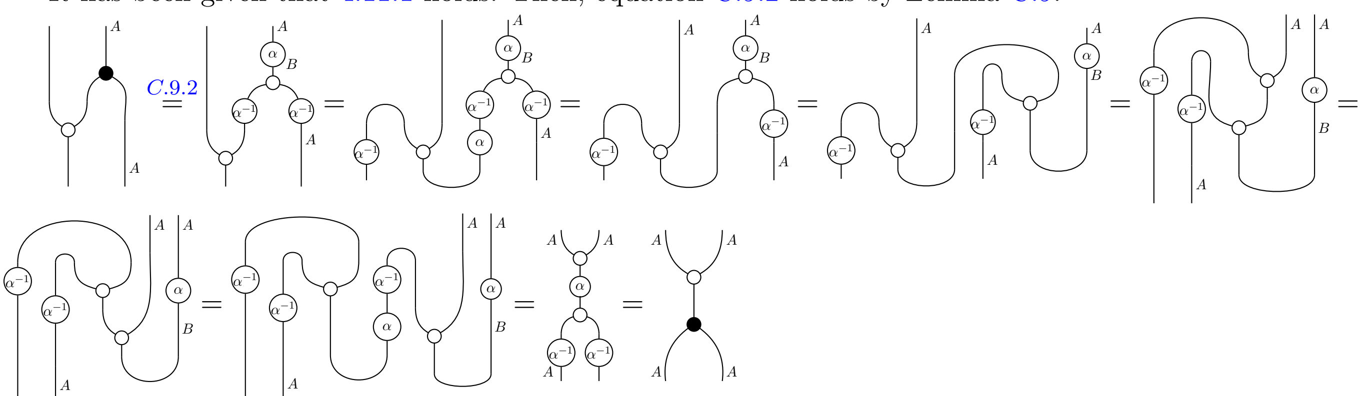


(C.10.1)

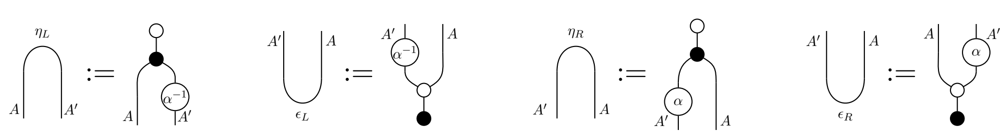
Proof. Let $(\mathbb{X}, \otimes, \oplus)$ be a compact-LDC. We know that there exists a linear equivalence, $\mathsf{Mx}_\downarrow : (X, \otimes, \oplus) \longrightarrow (\mathbb{X}, \otimes, \otimes)$. Let $A \overset{\circ}{-\!\!\!\shortmid\!\vdash} A'$ be a self-linear monoid in $\mathbb{X}$ with isomorphism $\alpha : A \longrightarrow A'$. The Frobenius Algebra $(A, \curlyvee, \varphi, \curlywedge, \downarrow)$ in $(\mathbb{X}, \otimes, \otimes)$ is defined as follows: the monoid is same as that of the linear monoid, the comonoid is given the by the following data.



It has been given that 4.11.1 holds. Then, equation C.9.2 holds by Lemma C.9.

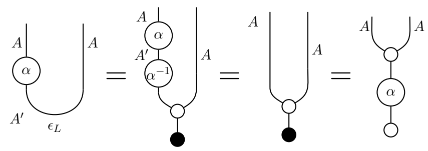


Now assume that there exists a Frobenius Algebra $(A, \curlyvee, \varphi, \curlywedge, \downarrow)$ in $(\mathbb{X}, \otimes, \otimes)$. The Frobenius Algebra induces a linear monoid, $A \overset{\circ}{-\!\!\!\shortmid\!\vdash} A'$ in $(\mathbb{X}, \otimes, \oplus)$. The linear monoid contains the monoid $(A, \curlyvee, \varphi)$ and the duals are given as follows:



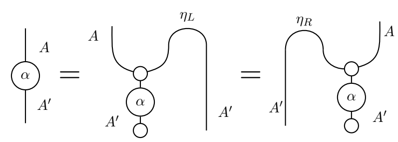
It is easy to verify using the Frobenius Law that the duals that the left and the right duals are cyclic for the monoid.

For the converse assume that a self-linear monoid $A \xrightarrow{\circ}\!\!\dashv A'$ corresponds to a Frobenius Algebra $(A, \curlyvee, \circ, \curlywedge, \bullet)$ under the linear equivalence $\mathsf{Mx}_{\downarrow}$. We prove that statement (iii) of Lemma 4.7 which is an equivalent form of the equation given in the current Lemma.



This proves the converse. □

Corollary C.11. *In a unitary category, any †-linear monoid $A \xrightarrow{\dagger\circ}\!\!\dashv A^\dagger$ precisely corresponds to a †-Frobenius algebra under the same equivalence if and only if the †-linear monoid satisfies the below equation for the unitary structure isomorphism $\varphi_A : A \longrightarrow A^\dagger$:*



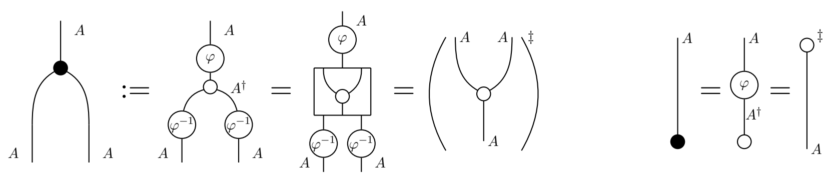
(C.11.1)

Proof. Suppose $A \xrightarrow{\dagger\circ}\!\!\dashv A^\dagger$ is a self-†-linear monoid in a unitary category $(\mathbb{X}, \otimes, \oplus)$. There exists a †-linear equivalence $\mathsf{Mx}_{\downarrow} : (\mathbb{X}, \otimes, \oplus) \longrightarrow (\mathbb{X}, \otimes, \otimes)$ with the dagger on $(\mathbb{X}, \otimes, \otimes)$ given as follows:

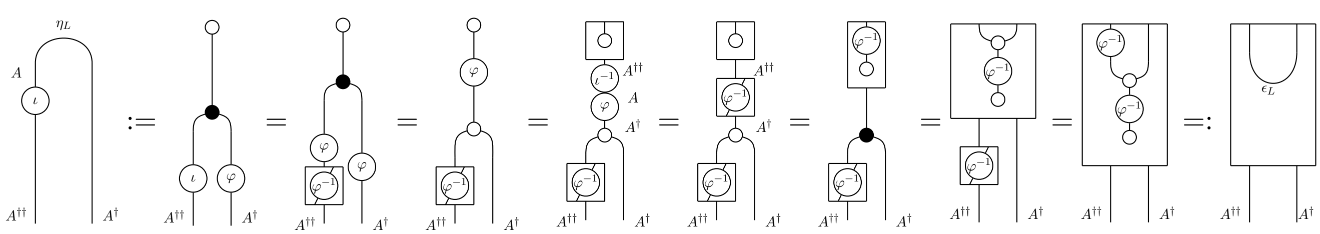
$$f^\ddagger = B \xrightarrow{\varphi_B} B^\dagger \xrightarrow{f^\dagger} A^\dagger \xrightarrow{\varphi_A^{-1}} A$$

The dagger ‡ is stationary on object, that is, $A^\ddagger = A$.

Now, we know that the linear monoid $A \xrightarrow{\circ}\!\!\dashv A^\dagger$ in $(\mathbb{X}, \otimes, \oplus)$ corresponds to a Frobenius Algebra $(A, \curlyvee, \circ, \curlywedge, \bullet)$ in $(\mathbb{X}, \otimes, \otimes)$. If A is a †-linear monoid, then $(A, \curlyvee, \circ, \curlywedge, \bullet)$ is a †-Frobenius Algebra because:



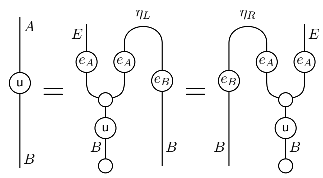
For the other way, the linear monoid $A \xrightarrow{\circ}\!\!\dashv A^\dagger$ given by the †-Frobenius Algebra $(A, \curlyvee, \circ, \curlywedge, \bullet)$ is a †-linear monoid because the duals of the linear monoid are †-duals:



The dual of the monoid is same as its dagger for $A \xrightarrow{\circ}\!\!\dashv A^\dagger$ because $(A, \curlyvee, \circ, \curlywedge, \bullet)$ is †-Frobenius. □

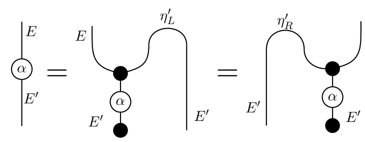
The proof of Lemma 4.12 is as follows:

Lemma C.12. *In an isomix category $\mathbb{X}$, let $E \overset{\bullet}{\multimap} E'$ be a self-linear monoid in $\mathsf{Core}(\mathbb{X})$ given by splitting a coring sectional/retractional binary idempotent (u, v) on linear monoid $A \overset{\circ}{\multimap} B$. Let $\alpha : E \to E'$ be the isomorphism. Then, E is a Frobenius Algebra under the linear equivalence $\mathsf{Mx}_{\downarrow}$ if and only if the binary idempotent satisfies the following equation where $e_A = \mathsf{uv}$ and $e_B = \mathsf{vu}$:*

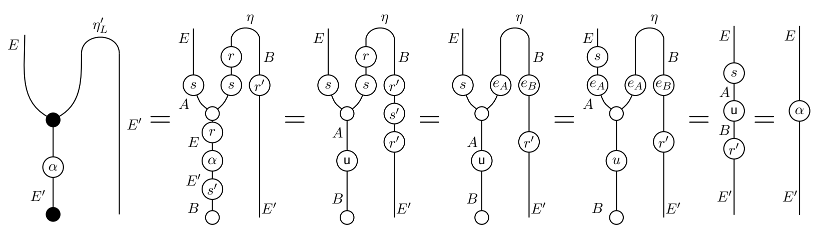


(C.12.1)

Proof. Let $E \overset{\bullet}{\multimap} E'$ be a self-linear monoid given by splitting a sectional or retractional binary idempotent on $A \overset{\circ}{\multimap} B$. Let the splitting of the binary idempotent be $e_A = \mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A$, and $e_B = \mathsf{vu} = B \xrightarrow{r'} E' \xrightarrow{s'} B$. Suppose u satisfies the given equation. If we show that the following equation holds, then by Lemma 4.11 E is a Frobenius algebra:



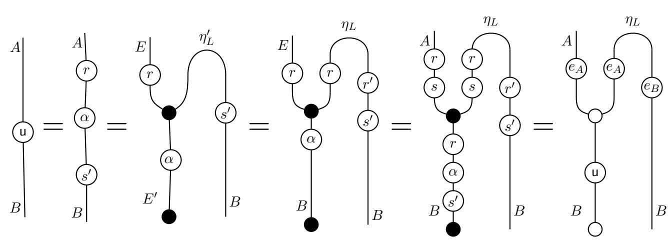
The proof is as follows:



For the converse assume that $E \overset{\bullet}{\multimap} E'$ is a self-linear monoid given by splitting a coring sectional/retractional binary idempotent (u, v) on linear monoid $A \overset{\circ}{\multimap} B$. The linear monoid $E \overset{\bullet}{\multimap} E'$ correspond precisely to a Frobenius Algebra $(E, \curlyvee, \circ, \curlywedge, \bullet)$ under the linear equivalence $\mathsf{Mx}_{\downarrow}$. We must prove that the binary idempotent satisfies the given equation. Let the binary idempotent be either sectional or retractional and split as follows:

$$A \underset{s}{\overset{r}{\rightleftarrows}} E \xrightarrow[\simeq]{\alpha} E' \underset{r'}{\overset{s'}{\rightleftarrows}} B$$

Then,



□

C.3 Linear comonoids

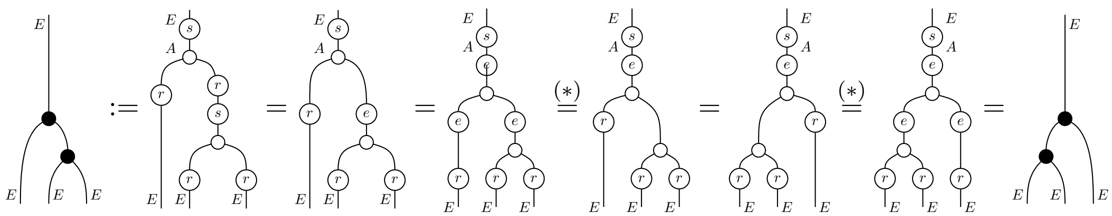
Lemma C.13. *In a monoidal category, a split idempotent $e : A \to A$ on a comonoid (A, d, k), with splitting $A \xrightarrow{r} E \xrightarrow{s} A$, is sectional (respectively retractional) if and only if the section s (respectively the retraction r) is a comonoid morphism for $(E, sd(r \otimes r), sk)$.*

Proof. Suppose $e : A \longrightarrow A$ is an idempotent with the splitting $A \xrightarrow{r} E \xrightarrow{s}$ and (A, d, k) is a monoid. Suppose e is sectional on A i.e, $d(e \otimes e) = ed(e \otimes e)$.

We must prove that (E, d', k') is a monoid where $d' := sd(r \otimes r)$ and $k' := sk$. Unit law and associativity law are proven as follows.

$$d'(k' \otimes 1) = sd(r \otimes r)(sk \otimes 1) = sd(rsk \otimes r) = sd(ek \otimes r)$$
$$= srsd(ek \otimes rsr) = sed(ek \otimes er) \overset{sectional}{=} sed(k \otimes r) = ser = srsr = 1$$

In the following string diagrams, we use black circle for (E, d', k'), and white circle for (A, d, k):



The steps labelled by $(*)$ are because e is sectional on (A, d, k).

Finally, $s : E \longrightarrow A$ is a comonoid homomorphism because:

$$d'(s \otimes s) = sd(r \otimes r)(s \otimes s) = sd(rs \otimes rs) = sd(e \otimes e) = srsd(e \otimes e) = sed(e \otimes e) = sed = sd$$

For the converse, assume that (E, d', k') is a comonoid where $d' = sd(r \otimes r)$, and $k' = sk$, and s is a monoid homomorphism, then, $e : A \longrightarrow A$ is sectional on (A, d, k) because:

$$ed = rsd = rd'(s \otimes s) = rsd(r \otimes r)(s \otimes s) = ed(e \otimes e)$$

The statement is proven similarly when the idempotent is retractional on the comonoid. ∎

In an LDC, splitting a binary idempotent on a linear comonoid induces a self-linear comonoid on the splitting if and only if the binary idempotent is sectional or retractional:

Lemma C.14. *In an LDC, let* (u, v) *be a split binary idempotent on a linear comonoid* $A \overset{}{\underset{\circ}{-\!\!\!\vdash}} B$ *with* $\otimes$*-comonoid* (A, d, e) *and splitting* $A \xrightarrow{r} E \xrightarrow{s} A$*, and* $B \xrightarrow{r'} E' \xrightarrow{s'} B$*. The binary idempotent* (u, v) *is sectional (respectively retractional) if and only if the section* (s, r') *(respectively the retraction* (r, s')*) is a morphism of linear comonoid for* $E \underset{\bullet}{-\!\!\!\vdash} E'$ *with* $\otimes$*-comonoid* $(E, sd(r \otimes r), se)$*.*

The proof is similar to that of Lemma C.8. The result directly extends to †-binary idempotents on †-linear comonoids which split to give †-self-linear comonoids.

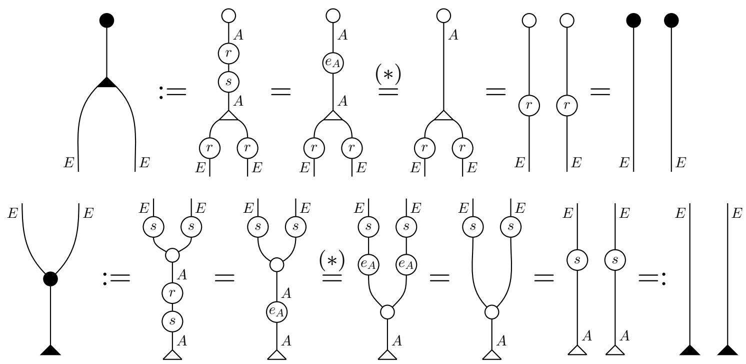
C.4 Linear bialgebra

In an LDC, splitting a binary idempotent on a linear bialgebra induces a self-linear bialgebra on the splitting if and only if the binary idempotent is sectional or retractional:

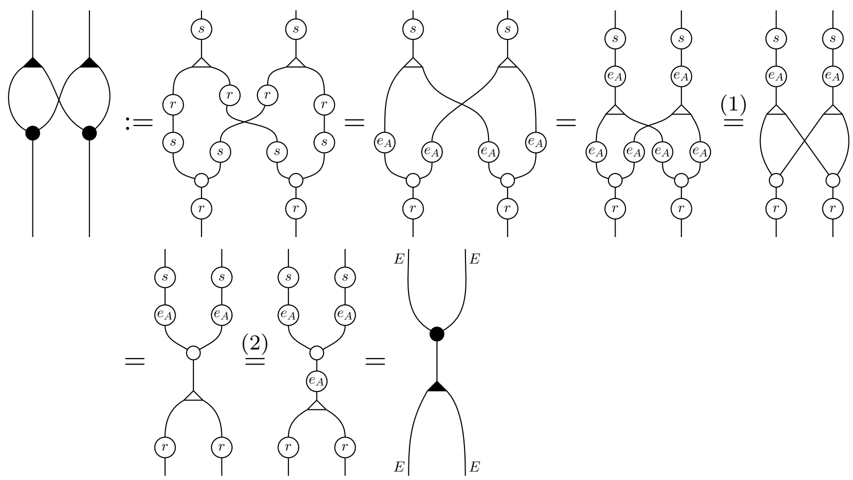
Lemma C.15. *In an LDC, let* (u, v) *be a split binary idempotent on a linear bialgebra* $A \overset{\circ}{\underset{\triangledown}{-\!\!\!\vdash}} B$ *with splitting* $A \xrightarrow{r} E \xrightarrow{s} A$*, and* $B \xrightarrow{r'} E' \xrightarrow{s'} B$ *is sectional (respectively retractional) if and only if the section* (s, r') *(respectively the retraction* (r, s')*) is a morphism of linear bialgebra for* $E \overset{\bullet}{\underset{\blacktriangledown}{-\!\!\!\vdash}} E'$ *with its linear monoid and linear comonoid given as in Statement* (i) *of Lemma C.8 and Lemma C.14 respectively.*

Proof. Let (u, v) be a sectional binary idempotent on a linear bialgebra $E \xrightarrow{\triangle}\!\!\!\dashv E'$. Let (u, v) split as follows: $e_A = \mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A$, and $e_B = \mathsf{vu} = B \xrightarrow{r'} E' \xrightarrow{s'} B$. Using Statement (i) of Lemma C.8, and Statement (i) of Lemma C.14, we know that E is both a linear monoid and a linear comonoid.

It remains to show that $E \xrightarrow{\bullet}\!\!\!\dashv E'$, and $E \xrightarrow[\bullet]{}\!\!\!\dashv E'$ give a $\otimes$-bialgebra on E, and a $\oplus$-bialgebra on E'. The proof that E is a $\otimes$-bialagebra is as follows:



Where the steps labelled $(*)$ are because e_A is sectional on the linear monoid.



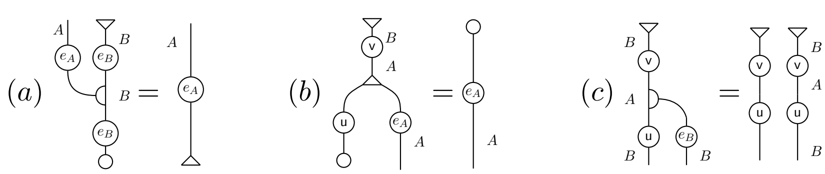
Where (1) is because e_A is sectional on the linear comonoid, and (2) is true because e_A is sectional on the linear monoid respectively. Similarly, E' is a $\oplus$-bialagebra. The converse is straightforward from Statement (i) of Lemma C.8 and C.14. $\square$

The above Lemma extends directly to $\dagger$-binary idempotents on $\dagger$-linear bialgebras.

C.5 Complimentary system

Proof for Lemma 4.19 is as follows:

Lemma C.16. *In an isomix category, a self linear bialgebra given by splitting a coring binary idempotent* (u, v) *on a commutative and a cocommutative linear bialgebra* $A \xrightarrow[\triangledown]{\circ}\!\!\!\dashv B$ *is a complimentary system if and only if the binary idempotent satisfies the following conditions (and 'op' symmetries):*

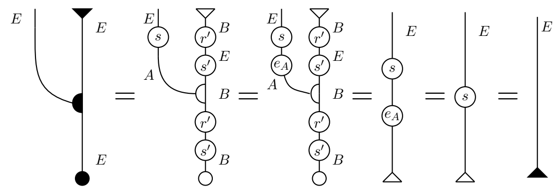


(C.16.1)

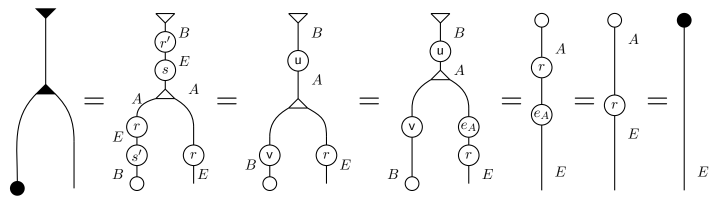
where $e_A = \mathsf{uv}$, *and* $e_B = \mathsf{vu}$.

Proof. Suppose $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$ is a self-linear bialgebra given by splitting a binary idempotent (u, v) on a commutative and a cocommutative linear bialgbera $A \overset{\circ}{\underset{\triangledown}{\dashv}} B$. Let the splitting of the bialgebra be given as follows: $\mathsf{uv} = A \xrightarrow{r} E \xrightarrow{s} A$, and $\mathsf{vu} = B \xrightarrow{r'} E' \xrightarrow{s'} B$. If (u, v) satisfy the given equations, then $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$ is a complementary system because:

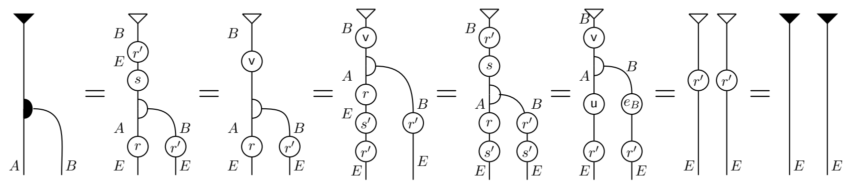
- [**comp.1**] holds for $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$.



- [**comp.2**] holds for $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$:

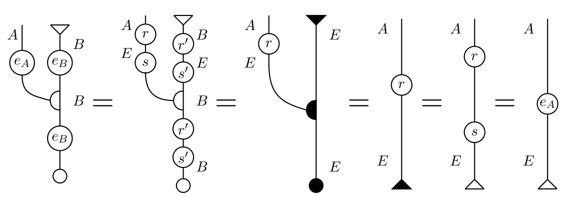


- [**comp.3**] holds for $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$:

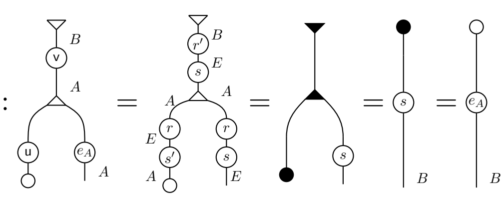


For the converse assume that, $E \overset{\bullet}{\underset{\blacktriangledown}{\dashv}} E$ is a complimentary system given by splitting a sectional or retractional binary idempotent (u, v) on a linear bialgebra $A \overset{\circ}{\underset{\triangledown}{\dashv}} B$. We show that $A \overset{\circ}{\underset{\triangledown}{\dashv}} B$ satisfies the equations, (a)-(c) given in the statement of the Lemma.

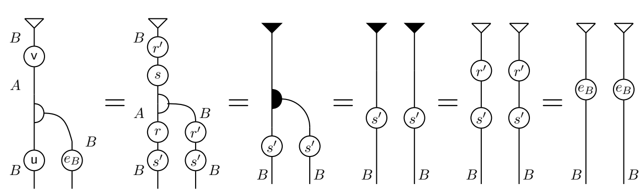
(a) :



(b) :



(c) :



□

D Exponential modalities

Definition D.1. *A* **†-$(!, ?)$-LDC** *is a $(!, ?)$-LDC and a †-LDC such that the following holds:*

(i) $(!,?)$ *is a* $\dagger$*-linear functor i.e., we have that:*

$$!(A^\dagger) \xrightarrow[\simeq]{\mathsf{s}} (?A)^\dagger \qquad\qquad (!A)^\dagger \xrightarrow[\simeq]{\mathsf{t}} ?(A^\dagger)$$

is a linear natural isomorphism such that

$$\begin{array}{ccc} !X & \xrightarrow{\iota} & (!X)^{\dagger\dagger} \\ {\scriptstyle !\iota}\downarrow & (a) & \downarrow{\scriptstyle (\mathsf{t}^{-1})^\dagger} \\ !(X^{\dagger\dagger}) & \xrightarrow[\mathsf{s}]{} & (?(X^\dagger))^\dagger \end{array} \qquad\qquad \begin{array}{ccc} ?X & \xrightarrow{\iota} & (?X)^{\dagger\dagger} \\ {\scriptstyle ?\iota}\downarrow & (b) & \downarrow{\scriptstyle \mathsf{s}^\dagger} \\ ?(X^{\dagger\dagger}) & \xrightarrow[\mathsf{t}^{-1}]{} & (!(X^\dagger))^\dagger \end{array} \tag{D.1.1}$$

(ii) The pair $\Delta_A : !A \to !A \otimes !A$*, and* $\nabla_A : ?A \oplus ?A \to ?A$ *is a* $\dagger$*-linear natural transformation. i.e., the following diagrams commute:*

$$\begin{array}{ccc} !(A^\dagger) & \xrightarrow{\Delta} & !(A^\dagger)\otimes !(A^\dagger) \\ & & \downarrow{\scriptstyle \mathsf{s}\otimes\mathsf{s}} \\ {\scriptstyle \mathsf{s}}\downarrow & (a) & (?A)^\dagger \otimes (?A)^\dagger \\ & & \downarrow{\scriptstyle \lambda_\otimes} \\ (?A)^\dagger & \xrightarrow[\nabla^\dagger]{} & (?A \oplus ?A)^\dagger \end{array} \qquad \begin{array}{ccc} (!A \otimes !A)^\dagger & \xrightarrow{\Delta^\dagger} & (!A)^\dagger \\ {\scriptstyle \lambda_\oplus^{-1}}\downarrow & & \\ (!A)^\dagger \oplus (!A)^\dagger & (b) & \downarrow{\scriptstyle \mathsf{t}} \\ {\scriptstyle \mathsf{t}\oplus\mathsf{t}}\downarrow & & \\ ?(A^\dagger) \oplus ?(A^\dagger) & \xrightarrow[\nabla]{} & ?(A^\dagger) \end{array} \tag{D.1.2}$$

Δ *is completely determined by* ∇ *and vice versa.*

(iii) The pair, $\Ydown_A : !A \to \top$ *and* $\Yup : \bot \to ?A$ *is a* $\dagger$*-linear natural transformation i.e., the following diagrams commute:*

$$\begin{array}{ccc} !(A^\dagger) & \xrightarrow{\Yup} & \top \\ {\scriptstyle \mathsf{s}}\downarrow & (a) & \downarrow{\scriptstyle \lambda_\top} \\ (?A)^\dagger & \xrightarrow[\Ydown^\dagger]{} & \bot^\dagger \end{array} \qquad\qquad \begin{array}{ccc} \top^\dagger & \xrightarrow{\Yup^\dagger} & (!A)^\dagger \\ {\scriptstyle \lambda_\bot^{-1}}\downarrow & (b) & \downarrow{\scriptstyle \mathsf{t}} \\ \bot & \xrightarrow[\Ydown]{} & (?A)^\dagger \end{array} \tag{D.1.3}$$

(iv) The pair $\delta : !A \to !!A$*, and* $\mu : ??A \to ?A$ *is a* $\dagger$*-linear natural transformation:*

$$\begin{array}{ccc} !(A^\dagger) & \xrightarrow{\delta} & !!(A^\dagger) \\ & & \downarrow{\scriptstyle !\mathsf{s}} \\ {\scriptstyle \mathsf{s}}\downarrow & (a) & !((?A)^\dagger) \\ & & \downarrow{\scriptstyle \mathsf{s}} \\ (?A)^\dagger & \xrightarrow[\mu^\dagger]{} & (??A)^\dagger \end{array} \qquad \begin{array}{ccc} (!!A)^\dagger & \xrightarrow{\delta^\dagger} & (!A)^(\dagger) \\ {\scriptstyle \mathsf{t}}\downarrow & & \\ ?((!A)^\dagger) & (b) & \downarrow{\scriptstyle \mathsf{t}} \\ {\scriptstyle ?\mathsf{t}}\downarrow & & \\ ??(A^\dagger) & \xrightarrow[\mu]{} & ?(A^\dagger) \end{array} \tag{D.1.4}$$

The pair $\epsilon_A : !A \to A$*, and* $\eta_A : A \to ?A$ *is a* $\dagger$*-linear natural transformation i.e., the following diagrams commute:*

$$\begin{array}{ccc} !(A^\dagger) & \xrightarrow{\epsilon} & A^\dagger \\ {\scriptstyle \mathsf{s}}\downarrow & (a) & \| \\ (?A)^\dagger & \xrightarrow[\eta^\dagger]{} & A^\dagger \end{array} \qquad\qquad \begin{array}{ccc} A^\dagger & \xrightarrow{\epsilon^\dagger} & (!A)^\dagger \\ \| & (b) & \downarrow{\scriptstyle \mathsf{t}} \\ A^\dagger & \xrightarrow[\eta]{} & ?(A^\dagger) \end{array} \tag{D.1.5}$$

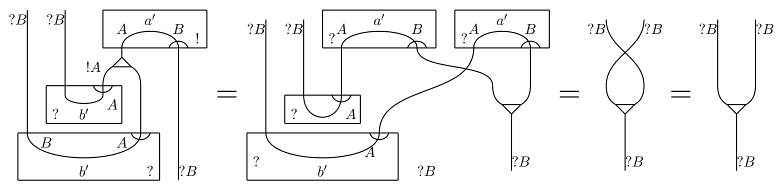
The follows Lemma gives the proof for Lemma 5.1:

Lemma D.2. *In any* $(!,?)$*-LDC any linear monoid* $(a,b) : A \overset{\circ}{\dashv\!\!\vdash} B$ *and an arbitrary dual* $(a',b') : A \dashv\!\!\vdash B$ *gives a linear bialgebra* $\frac{(a_!,b_!)}{(a'_!,b'_?)} : !A \overset{\boxdot}{\underset{\triangledown}{\dashv\!\!\vdash}} ?B$ *using the natural cocommutative comonoid* $(!A, \Delta_A, \Yup)$*.*

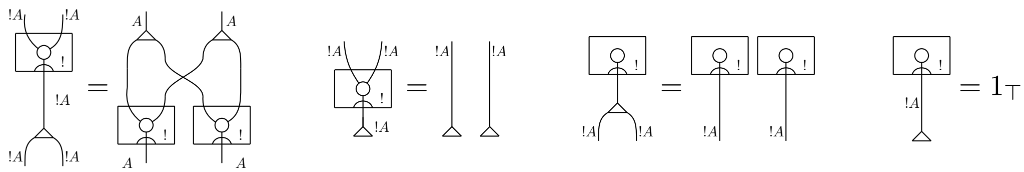
Proof. Given that $A \overset{\circ}{\dashv} B$ is a linear monoid, and $(a,b) : A \dashv\!\!\vdash B$ is a linear dual in a !-?-LDC.

Because, $(!,?)$ are linear functors, and linear functors preserve linear monoids, we know that $!A \overset{\boxed{\circ}}{\dashv} ?B$ is a linear monoid.

Similarly, by linearity of $(!,?)$, we have the dual $(a'_!, b'_?) :!A \dashv\!\!\vdash ?B$. The cocommutative $\otimes$-comonoid $(!A, \Delta_A, e_A)$ given by the modality ! together with the dual $(a'_!, b'_?) :!A \dashv\!\!\vdash ?B$ gives a linear comonoid $!A \underset{\triangledown}{\dashv} ?B$ Note that, $(!A, \Delta_A, e_A)$ is dual to the commutative comonoid $(?B, \nabla_B, u_B)$ given by the ? modality, using $(a', b') :!A \dashv\!\!\vdash ?B$:



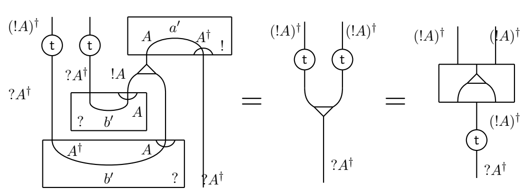
The linear monoid $!A \overset{\boxed{\circ}}{\dashv} ?B$ and the linear comonoid $!A \underset{\triangledown}{\dashv} ?B$ gives a $\otimes$-bialgebra on $!A$ and a $\oplus$-bialgbera on $?B$. The bialgebra rules are immediate by naturality of Δ, ∇, u, and e:



∎

Lemma D.3. *In any $(!,?)$-†-LDC any †-linear monoid $(a,b) : A \overset{\circ}{\dashv} A^\dagger$ and an arbitrary †-dual $(a',b') : A \dashv\!\!\vdash A^\dagger$ gives a †-linear bialgebra $!A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv}} ?A^\dagger$ with the †-linear monoid $(a_!, b_?) :!A \overset{\dagger\circ}{\dashv} ?A^\dagger$ and the †-linear comonoid $(a'_!, b'_?) :!A \underset{\dagger\triangle}{\dashv} ?A^\dagger$.*

Proof. Recall that there exists a natural isomorphism $\mathtt{t} : (!A)^\dagger \longrightarrow ?A^\dagger$ because $(!,?)$ is a †-linear functor. $(a!, b_?) :!A \overset{\dagger\circ}{\dashv} ?A^\dagger$ is a †-linear monoid because †-linear functors preserve †-linear monoids. Hence, $(a!, b_?) :!A \overset{\dagger\circ}{\dashv} ?A^\dagger$ is a †-linear monoid. $(a'_!, b'_?) :!A \underset{\dagger\triangle}{\dashv} ?A^\dagger$ is a †-linear comonoid because †-linear functors preserve †-duals, and (Δ, ∇) are †-linear tranformations i.e.,



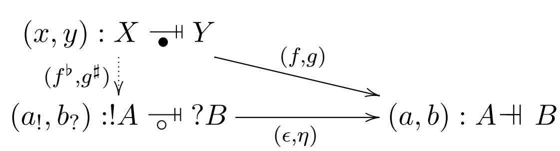
From Lemma 5.1, $!A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv}} ?A^\dagger$ is a linear bialgebra. Thereby, $!A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv}} ?A^\dagger$ is a †-linear bialgebra. ∎

The following is the proof for Lemma 5.2:

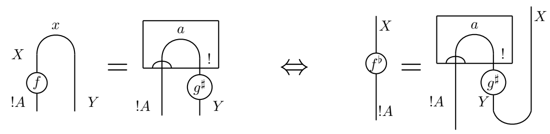
Lemma D.4. *In a $(!,?)$-LDC with free exponentials, if $(x,y)\colon X \underset{\bullet}{\dashv} Y$ is a linear comonoid and*

$$(f,g)\colon ((x,y)\colon X \dashv\!\!\vdash Y) \longrightarrow ((a,b)\colon A \dashv\!\!\vdash B)$$

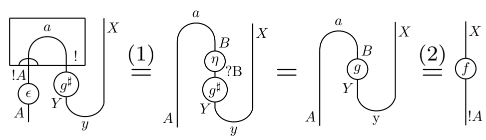
is a morphism of duals then, the unique map $(f^\flat, g^\sharp)$ induced by the universal property of the free exponential is a morphism of linear comonoids:



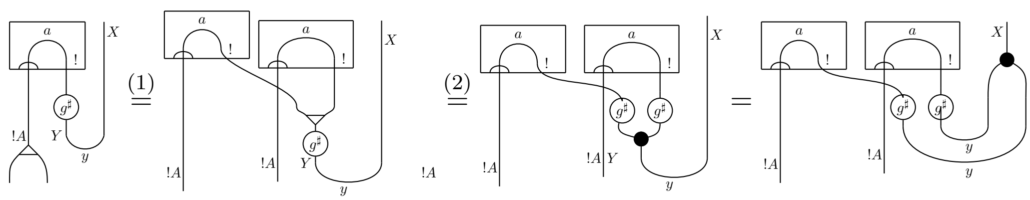
Proof. Given that (f, g) is a morphism of duals. We must show that $(f^\flat, g^\sharp)$ given by the universal property of $(!, ?)$ is a morphism of duals by showing the equation on the left, below, holds. However, the equation on the left holds if and only if the equation on the right holds:



Now, by the couniversal property of the ! functor, $f^\flat$ is a unique comonoid homomorpshim such that $f^\flat \epsilon = f$. To prove that the latter equation holds, it suffices to prove that:



Where (1) is holds because (ϵ, η) is a linear transformation, while (2) holds as f and g are mates.



Step (1) is uses the fact (Δ, ∇) is a linear transformation, while (2) holds as $g^\sharp$ is a morphism of monoids. Hence, $(f^\flat, g^\sharp)$ is a morphism of duals. □

The following is the proof for Proposition 5.3

Proposition D.5. *In a (!,?)-LDC with free exponential modalities, let* $\frac{(x,y)}{(x',y')} : X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\vdash}} Y$ *be a linear bialgebra,* $(a, b) : A \overset{\circ}{\dashv\!\vdash} B$ *a linear monoid, and* $(a', b') : A \dashv\!\vdash B$ *a dual, then*

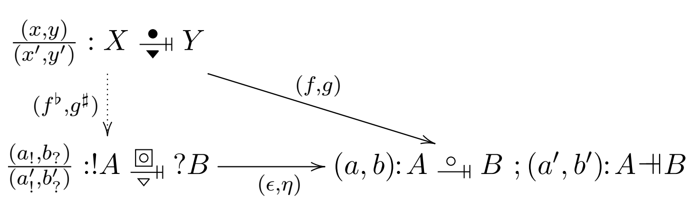
$$(f^\flat, g^\sharp) : \left(\frac{(x, y)}{(x', y')} : X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\vdash}} Y \right) \longrightarrow \left(\frac{(a_!, b_?)}{(a'_!, b'_?)} :!A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv\!\vdash}} ?B \right)$$

is a morphism of bialgebras, whenever $f : (X, \curlyvee_\bullet, \bullet) \longrightarrow (A, \curlyvee_\circ, \circ)$ *is a morphism of monoids, and* (f, g) *is a morphism of both duals*

$$(f, g) \colon ((x, y) \colon X \overset{\circ}{\dashv\!\vdash} Y) \longrightarrow ((a, b) \colon A \overset{\circ}{\dashv\!\vdash} B) \quad \text{and} \quad (f, g) \colon ((x', y') \colon X \dashv\!\vdash Y) \longrightarrow ((a', b') \colon A \dashv\!\vdash B)$$

We are given a linear monoid $(a, b) \colon A \overset{\circ}{\dashv\!\vdash} B$, and $(a', b') \colon A \dashv\!\vdash B$ is an arbitrary dual. Using Lemma 5.1, we know that $!A \overset{\boxed{\circ}}{\underset{\triangledown}{\dashv\!\vdash}} ?B$ is a linear bialgebra.

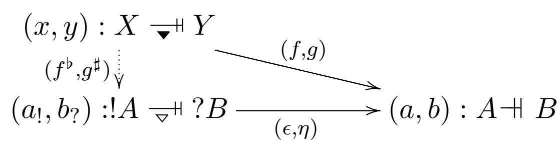
We are given that (f, g) is a morphism of linear monoids, and a morphism of duals as in the diagram below. We must prove that $(f^\flat, g^\sharp)$ is a morphism of bialgebras:



Since $(f, g) \colon ((x', y') : X \dashv\!\vdash Y) \longrightarrow ((a', b') \colon A \dashv\!\vdash B)$ is a morphism of duals it follows from Lemma 5.2 that $(f^\flat, g^\sharp) : ((x', y') : X \dashv\!\vdash Y) \longrightarrow ((a'_!, b'_!) :!A \dashv\!\vdash ?B)$ is a morphism of linear comonoids.

Given that (f,g) is a morphism of linear monoids. This means that $(f,g):((x,y):X \dashv\!\!\vdash Y) \to ((a,b):A \dashv\!\!\vdash B)$ is a morphism of duals, and $f:(X,\blacktriangledown,\bullet) \to (A,\curlyvee,\circ)$ is a morphism of monoids. We have to prove that $(f^\flat, g^\sharp)$ is a morphism of linear monoids.

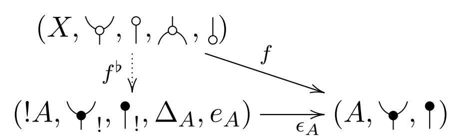
We know that (in a symmetric LDC) any dual with a $\otimes$-comonoid gives a linear comonoid. Hence, the dual $(x,y):X \dashv\!\!\vdash Y$ and the $\otimes$-comonoid $(X,\blacktriangle,\perp_\bullet)$ produces a linear comonoid $(x,y): X \overset{\blacktriangledown}{\dashv\!\!\vdash} Y$. Now, the dual $(a,b):A \dashv\!\!\vdash B$ induces a linear comonoid $(a_!, b_?) : !A \overset{\triangledown}{\dashv\!\!\vdash} ?B$ on the exponential modalities. This leads to the situation as illustrated in the following diagram:



Applying Lemma 5.2 to the above diagram, we have that $(f^\flat, g^\sharp)$ is a morphism of duals. Note that, the linear comonoids on X and $!A$ in the above diagram are different from the linear comonoids in the $X \overset{\bullet}{\underset{\blacktriangledown}{\dashv\!\!\vdash}} Y$ and $!A \overset{\circ}{\underset{\triangledown}{\dashv\!\!\vdash}} ?B$ bialgebras.

It remains to prove that $f^\flat:(X,\blacktriangledown,\bullet) \to (!A,\curlyvee_!,\circ_!)$ is a morphism of monoids. This is provided by the following:

Lemma D.6. *In* $(!,?)$*-LDC with free exponential modalities if* $(X,\curlyvee,\circ,\curlywedge,\perp_\circ)$ *is a* $\otimes$*-bialgebra, and* $f:X \to A$ *is a monoid morphism, then* $f^\flat : A \to !A$ *given by the couniversal property of the* $!$ *is a bialgebra homomorphism*

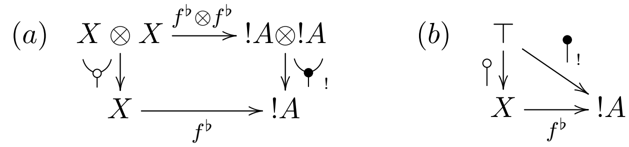


where the multiplication and the unit for $!A$ *is induced by linearity of* $(!,?)$*:*

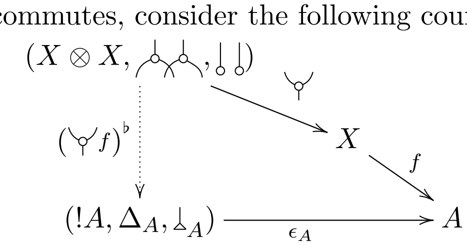
$$\blacktriangledown_! := !A \otimes !A \xrightarrow{m_\otimes} !(A\otimes A) \xrightarrow{!(\blacktriangledown)} !A \qquad \bullet_! := \top \xrightarrow{m_\top} !\top \xrightarrow{!(\bullet)} !A$$

Proof. Given that in a $(!,?)$-LDC with free exponential modalities, $(X,\curlyvee,\circ,\curlywedge,\perp_\circ)$ is a bialgebra, $(A,\blacktriangledown,\bullet)$ is a monoid, and $f:X\to A$ is a monoid morphism. The monoid on A induces a bialgebra on $(!A,\blacktriangledown_!,\bullet_!,\Delta_A,e_A)$. We must show that $f^\flat$ given by the couniversal property of $(!A,\Delta_A,\perp_A)$ is a bialgebra morphism, that is, $f^\flat$ is a monoid and a comonoid morphism.

From the given couniversal diagram, $f^\flat:(X,\blacktriangledown,\bullet) \to (!A,\Delta_A,e_A)$, is a comonoid morphism. We must prove that $f^\flat:(X,\curlyvee,\circ) \to (!A,\blacktriangledown_!,\bullet_!)$ is a monoid morphism, that is the following diagrams commute:



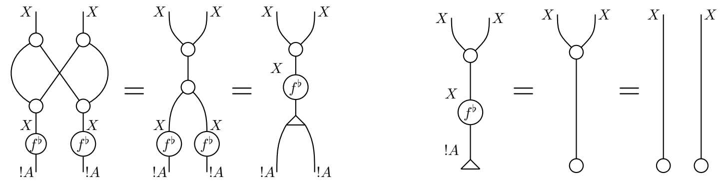
In order to prove that (a) commutes, consider the following couniversal diagram.



We will show that $(\curlyvee f)^\flat = \curlyvee f^\flat = (f^\flat \otimes f^\flat)\blacktriangledown_!$.

Proving that $(\curlyvee f)^\flat = \curlyvee f^\flat$:

It is immediate that $\curlyvee f^\flat \epsilon = \curlyvee f$. Moreover, $\curlyvee f^\flat$ is a comonoid morphism because $f^\flat : A \longrightarrow !A$ is a comonoid morphism, and X is a bialgebra:



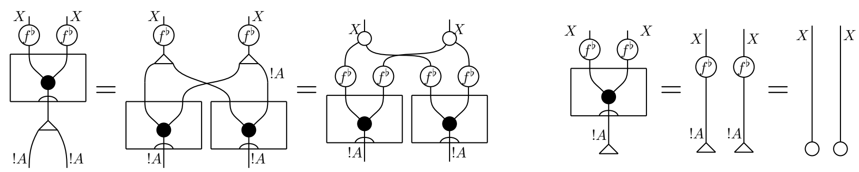
Hence, $\curlyvee f^\flat = (\curlyvee f)^\flat$.

Proving that $(\curlyvee f)^\flat = (f^\flat \otimes f^\flat)\curlyvee_{!}$:

$$X \otimes X \xrightarrow{f^\flat \otimes f^\flat} !A \otimes !A \xrightarrow{\curlyvee_!} !A \xrightarrow{\epsilon_A} A = (f^\flat \otimes f^\flat)(m_\otimes)\left(!\curlyvee\right)\epsilon_A$$

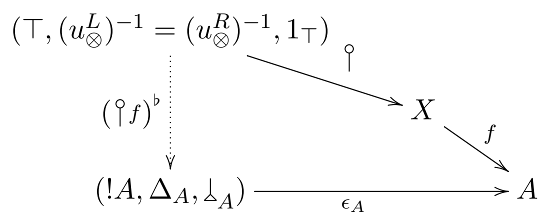
$$\overset{(*)}{=} (f^\flat \otimes f^\flat)(\epsilon_A \otimes \epsilon_A)\curlyvee = (f \otimes f)\curlyvee = \curlyvee f$$

where $(*)$ holds because ϵ is a monoidal transformation. Now, $(f^\flat \otimes f^\flat)\curlyvee_!$ is a comonoid homomorphism because $f^\flat$ is a comonoid homomorphism and Δ is a natural transformation:



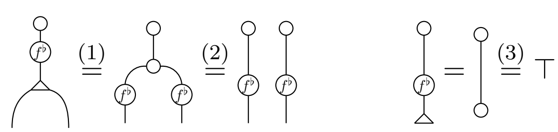
Hence, by the uniqueness of $(\curlyvee f)^\flat$, $\curlyvee f^\flat = (f^\flat \otimes f^\flat)\curlyvee_!$. This proves that $f^\flat$ preserves multiplication of X.

To prove that $f^\flat$ preserves the unit of X, that is, $\circ f^\flat = \bullet_!$, consider the couniversal diagram:



Proving that $(\circ f)^\flat = \circ f^\flat$:

It is straightforward that $(\circ f)^\flat \epsilon_A = \circ f^\flat \epsilon_A = \circ f$. Moreover, $\circ f^\flat$ is a comonoid homomorphism because:



Where (1) is because $f^\flat : X \longrightarrow !A$ is a comonoid morphism, and (2) and (3) are because X is a bialgebra.

Proving that $(\circ f)^\flat = \bullet_!$:

We have that $\bullet_! \epsilon_A = m_\top ! (\bullet)\, \epsilon_A = \bullet = \circ f$ because ϵ_A is a monoidal transformation and f is a monoid homomorphism. Moreover, $\bullet_!$ is a comonoid morphism due to the naturality of Δ and $\perp$.

By the uniqueness of $\circ^\flat$, we have that $(\circ f)^\flat \bullet_! = \circ f^\flat$. Thereby, $f^\flat$ preserves the unit of X. Thus, $f^\flat : (X, \curlyvee, \circ) \to (!A, \curlyvee_!, \bullet_!)$ is a monoid homomorphism. Since $f^\flat$ is a monoid and comonoid morphism, it is a bialgebra morphism. $\square$