PROFUNCTORS BETWEEN POSETS AND ALEXANDER DUALITY

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Abstract. We advocate profunctors $P \nrightarrow Q$ between posets as a more natural and fruitful notion than order preserving maps. We intoduce the *graph* and *ascent* of such profunctors. We apply this in commutative algebra where these give classes of Alexander dual square-free monomial ideals giving the full and natural generalized setting of isotonian ideals and letterplace ideals for posets. We study the poset of profunctors from $\mathbb{N}$ to $\mathbb{N}$. Such profunctors identify as order presering maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. For our applications when P and Q are infinite, we also introduce a topology on the set of profunctors $P \nrightarrow Q$, in particular on profunctors $\mathbb{N} \nrightarrow \mathbb{N}$.

Introduction

This article advocates for general posets P and Q the notion of profunctor $P \nrightarrow Q$ as more natural and well-behaved than the notion of isotone maps $P \to Q$ between posets, especially for applications in algebra. When Q is totally ordered, these notions are practically the same, but when Q is not totally ordered, profunctors seem to have a clear advantage for developing natural theory.

Let **Bool** be the two element boolean poset, often denoted by $\{0, 1\}$ or $\{\bot, \top\}$ but we shall use $\omega = \{d, u\}$ (down and up). Then a profunctor $P \nrightarrow Q$ is simply an isotone map $P \times Q^{\mathrm{op}} \to \omega$. If P and Q are sets (discrete posets), then this is simply a relation between P and Q. The notion of profunctor (also called distributor) may generally be defined between categories or between categories enriched in a monoidal category (like **Bool**), see [1], [3], or for a recent gentle introduction focusing on applications, [10, Section 4].

The opposite P^{op} of poset P, has the same elements but order relation reversed. The elements in the distributive lattice $\widehat{P}$ associated to P are pairs (I, F), called *cuts*, where I is an order ideal in P and F the complement filter. There is then a duality between $\widehat{P}$ and $\widehat{P^{\mathrm{op}}}$ sending (I, F) to $(F^{\mathrm{op}}, I^{\mathrm{op}})$. We call two such pairs dual or Alexander dual (as is common in combinatorial commutative algebra).

Denote by $\mathrm{Hom}_{pro}(P, Q)$ the profunctors $P \nrightarrow Q$. This is again a a partially ordered set and the opposite of this poset is $\mathrm{Hom}_{pro}(Q, P)$.

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The basic notions we introduce associated to a profunctor between posets $f : P \not\to Q$ are the notions of its *graph* Γf and its *ascent* Λf. These are dual notions in the sense that if $g : Q \not\to P$ is the dual profunctor, the graph of f equals the ascent of g. Let UP and UQ denote the underlying sets of P and Q. Given a poset ideal $\mathcal{I}$ in $\mathrm{Hom}_{pro}(P,Q)$ and $\mathcal{F}$ its complement filter, so $(\mathcal{I},\mathcal{F})$ is a cut for $\mathrm{Hom}_{pro}(P,Q)$. Let $\mathcal{F}_\Lambda$ be the *poset filter* in the Boolean lattice of all subsets of $UQ \times UP^{\mathrm{op}}$ generated by the ascents Λf for $f \in \mathcal{F}$. Let $\mathcal{I}_\Gamma$ be the *poset ideal* in this Boolean lattice generated by the the complemens of the grapsh Γf for $f \in \mathcal{I}$. Our main theorem states the following.

Theorem 3.5. (Preserving the cut) *Let P and Q be well-founded posets, and $(\mathcal{I},\mathcal{F})$ a cut for Hom$(P,\widehat{Q})$. Then $(\mathcal{I}_\Gamma,\mathcal{F}_\Lambda)$ is a cut for the Boolean lattice of subsets of $UQ \times UP^{op}$.*

This has alternative formulations in Theorem 3.6 asserting that two filters are Alexander dual, suitable to Stanley-Reisner theory, and in Theorem 3.7 asserting that this corrspondence respects the duality on profunctors. In Theorem 6.1 we give a version with conditions on P and Q ensuring that Γf and Λf are always finite sets, suitable for applications to monomial ideals, Section 7.

Although we develop a general theory here, our original motivation came from applications related to commutative algebra.

Applications to Stanley-Reisner theory. When P and Q are finite posets we get general constructions, Subsection 3.5, of Alexander dual squarefree monomial ideals, generalizing isotonian ideals and letterplace and co-letterplace ideals, [6],[9], [7], [13], and [12]. In particular, when Q is a chain these constructions have given very large classes of simplicial spheres, [4].

Applications to order preserving maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. The profunctors from $\mathbb{N}$ to $\mathbb{N}$ identify as order preserving maps from $\mathbb{N}$ to the distributive lattice $\widehat{\mathbb{N}}$, and the latter identifies as $\mathbb{N} \cup \{\infty\}$. The order preserving maps $f : \mathbb{N} \to \widehat{\mathbb{N}}$ are the topic of many our examples.

Injective order preserving maps $f : \mathbb{N} \to \mathbb{N}$ form the so called increasing monoid, which has gained recent interest. In [15] Nagel and Römer show that ideals in the infinite polynomial ring invariant for the increasing monoid, have an essentially finite Grobner basis, thereby generalizing previous results for the symmetric group. In [11] Güntürkün and Snowden studies in depth the representation theory of the increasing monoid. Note that the injective order preserving maps $g : \mathbb{N} \to \mathbb{N}$ are in bijection with order preserving maps $f : \mathbb{N} \to \mathbb{N}$ by $g = f + \mathrm{id} - 1$. Order preserving maps $f : \mathbb{N} \to \mathbb{N}$ also occur in the definition of the bicylic semi-group [5], a basic notion in inverse semi-group theory. In [8] we apply the order preserving maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ to the duality theory of strongly stable ideals in the the infinite polynomial ring $k[x_i]_{i \in \mathbb{N}}$.

In order for the substantial parts of our theory, related to graphs and ascents, to work well we must have certain conditions on the posets P and Q. Our weakest condition is that they are *well-founded*. For our applications to polynomial rings, we work in the class of *natural* posets, Section 5. These are posets where all anti-chains are finite and for every x in the poset the downset $\{y \mid y \leq x\}$ is finite. This is a subclass, close to natural numbers, of *well partially ordered sets*.

Another feature we introduce is a topology on $\mathrm{Hom}_{pro}(P, Q)$, Section 4, in particular on $\mathrm{Hom}_{pro}(\mathbb{N}, \mathbb{N})$. This is needed for our applications to commutative algebra.

Note. We let $\mathbb{N} = \{1, 2, 3, \ldots\}$. We only use the ordered structure on this so we could equally well have used $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. Only in the last Section 7 do we, in a somewhat different setting, use the commutative monoid structure and then we explicitly write $\mathbb{N}_0$.

1. Basic notions for posets

We give basic notions and constructions concerning posets: filters, ideals, dualities, distributive lattices, simplicial complexes. We also recount the algebraic notions of Stanley-Reisner ideal and ring.

1.1. **Poset ideals and filters in P.** Let P be a partially ordered set. The *opposite poset* P^{op} has the same underlying set as P but with order relation $\leq_{\mathrm{op}}$ where $p \leq_{\mathrm{op}} q$ if $p \geq q$ in P.

A *poset ideal* I of P is a subset of P closed under taking smaller elements. A *poset filter* F in P is a subset closed under taking larger elements. If I and F are complements of each other, we call (I, F) a *cut* for P. Since each of I and F determine each other, we sometimes denote this as $(-, F)$ if we focus on F, and similarly with $(I, -)$.

Definition 1.1. The *Alexander dual* (or just *dual*) of the cut (I, F) is the cut $(F^{\mathrm{op}}, I^{\mathrm{op}})$ for P^{op}. The Alexander dual of the poset ideal I is the poset ideal $J = F^{\mathrm{op}}$ of P^{op}, and the Alexander dual of the filter F is the poset filter $G = I^{\mathrm{op}}$.

1.2. **The distributive lattice.** Denote by ω the ordered set $\{d < u\}$. The distributive lattice associated to P is $\widehat{P} = \mathrm{Hom}(P^{\mathrm{op}}, \omega)$. Given an $f \in \widehat{P}$, the elements p in P such that p^{op} maps to $u \in \omega$, constitute a poset ideal I in P. The complement filter F in P consists of those $p \in P$ such that p^{op} maps to d. We may thus identify an element f of $\widehat{P}$ with a cut (I, F) for P and these cuts are ordered by

$$(I, F) \leq (J, G) \text{ iff } I \subseteq J \text{ or equivalently } F \supseteq G.$$

$\widehat{P}$ has a unique maximal element, denoted ∞. It sends every p^{op} to u, and corresponds to $(I, F) = (P, \emptyset)$. In particular $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

1.3. **Poset ideals and filters in $\widehat{P}$.** Let $\mathcal{I}$ be a poset ideal of $\widehat{P}$, and $\mathcal{F}$ the complement filter of $\widehat{P}$. So $\mathcal{I}$ consists of cuts (I, F) closed under forming cuts with smaller I's, and $\mathcal{F}$ consists of cuts (I, F) closed under forming cuts with larger

I's (or equivalently smaller F's). Then $(\mathcal{I}, \mathcal{F})$ is a cut *for* $\widehat{P}$ (note terminology: (I, F) is a cut *in* $\widehat{P}$). Also $(\mathcal{I}, \mathcal{F})$ is a cut *in* $\widehat{\widehat{P}}$.

1.4. **Simplicial complexes and Stanley-Reisner rings.** Let A be a set. A simplicial complex X on A is a family of subsets of A closed under taking smaller subsets, i.e. if $I \in X$ and $J \subseteq I$, then $J \in X$.

Then $\widehat{A}$ identifies as the Boolean poset on A, consisting of all subsets of A. A cut $(\mathcal{I}, \mathcal{F})$ for $\widehat{A}$ corresponds precisely to a simplicial complex X. The elements I in X give the cuts (I, I^c) in $\mathcal{I}$.

Denote by $k[x_A]$ the polynomial ring in the variables x_a for $a \in A$. When A is finite, to the simplicial complex X corresponding to the cut $(\mathcal{I}, \mathcal{F})$, we associate a monomial ideal I_X in $k[x_A]$, the Stanley-Reisner ideal of X. It is generated by monomials $x_I = \prod_{i \in I} x_i$ for $(I, F) \in \mathcal{F}$. These are the subset I of A such that I is *not* in the simplicial complex X. The monomials in the Alexander dual Stanley-Reisner ideal I_Y are then precisely those monomials which have non-trivial common divisor with every monomial in I_X.

2. Profunctors between posets

We introduce profunctors $P \to\!\!\!\!+ Q$ between posets. Such a profunctor has a dual $Q \to\!\!\!\!+ P$ and we investigate this correspondence. For an introduction to profunctors, see [10, Chap.4]. See also [3, Section 7] and [1] where they are called distributors.

2.1. **Duality on profunctors.** A profunctor $P \to\!\!\!\!+ Q$ is simply a poset homomorphism $P \to \widehat{Q}$. By the adjunction

$$\begin{aligned} \mathrm{Hom}(P, \widehat{Q}) &= \mathrm{Hom}(P, \mathrm{Hom}(Q^{\mathrm{op}}, \omega)) \\ &= \mathrm{Hom}(P \times Q^{\mathrm{op}}, \omega) \\ &= \mathrm{Hom}((Q \times P^{\mathrm{op}})^{\mathrm{op}}, \omega) = (Q \times P^{\mathrm{op}})\widehat{\ }. \end{aligned}$$

Thus a profunctor is equivalently an isotone map $P \times Q^{\mathrm{op}} \to \omega$ and this is often taken as the definition. It is also equivalently an element of the distributive lattice $(Q \times P^{\mathrm{op}})\widehat{\ }$, and so corresponds to a cut (I, F) for $Q \times P^{\mathrm{op}}$. (Our convention differs somewhat from [10], since there a profunctor $P \to\!\!\!\!+ Q$ corresponds to an isotone map $P^{\mathrm{op}} \times Q \to \omega$.)

In particular if $Q = B$ and $P = A$ are sets, this is simply a subset of $B \times A^{\mathrm{op}}$ or a relation between the sets A and B. Since $\mathrm{Hom}(P, \widehat{Q})$ identifies as $(Q \times P^{\mathrm{op}})\widehat{\ }$, the following is seen to be natural, by taking opposites.

Lemma 2.1. *Let P, Q be posets. There is a a natural isomorphism of posets*

$$Hom(P, \widehat{Q})^{op} \overset{D}{\cong} Hom(Q, \widehat{P}).$$

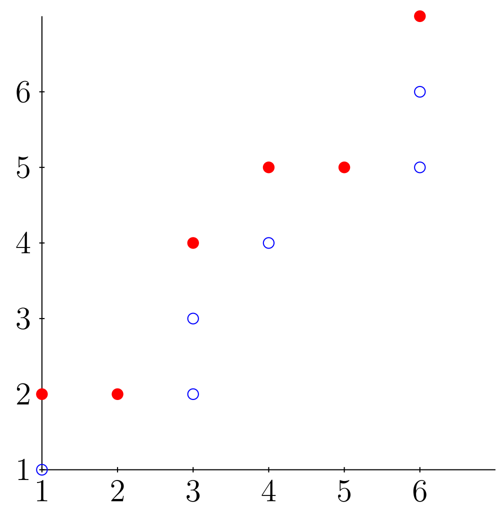

FIGURE 1

Example 2.2. Let $P = Q = \mathbb{N} = \{1, 2, 3, \cdots\}$ and consider an isotone map $f : \mathbb{N} \to \widehat{\mathbb{N}}$ with values given by

$$2, 2, 4, 5, 5, 7, \cdots .$$

In Figure 1 the graph of f are marked with red (black) discs $\bullet$. We fill in with blue circles $\circ$ to make a connected "snake", starting at position $(1, 1)$. The graph of the dual map $g = Df$ is given by the blue circles by considering the vertical axis as the argument for g. The values of g are

$$1, 3, 3, 4, 6, 6, \cdots .$$

Observe that for a map $f : \mathbb{N} \to \widehat{\mathbb{N}}$ then $f(1) \geq 2$ iff the dual map $g = Df$ has $g(1) = 1$. Hence there are no selfdual maps f.

The isotone map f corresponds to the cut (I, F) for $\mathbb{N} \times \mathbb{N}^{\text{op}}$ (where $\mathbb{N}$ corresponds to the y-axis and $\mathbb{N}^{\text{op}}$ to the (reversed) x-axis) where the filter F is given by filling in red discs vertically above those in the graph, see Figure 2, and the ideal I is given by filling in blue circles to the right of those which are present in Figure 1.

2.2. **Profiles and co-profiles.**

Definition 2.3. The *profile* of an isotone map $f : P \to \widehat{Q}$ is the cut (I, F) for P where the *profile filter* F consists of all $p \in P$ such that $f(p) = \infty$, and the *profile ideal* $I = F^c$ is the complement ideal.

The co-profile of f is the cut (J, G) for Q where J is the union of all $f(p)$ (considered as a poset ideal of Q) for $p \in P$, and G is the complement of J.

These notions are dual to each other as the following shows.

Lemma 2.4. *Let f be an isotone map and $g = Df$ its dual map. The co-profile of f equals the profile of g.*

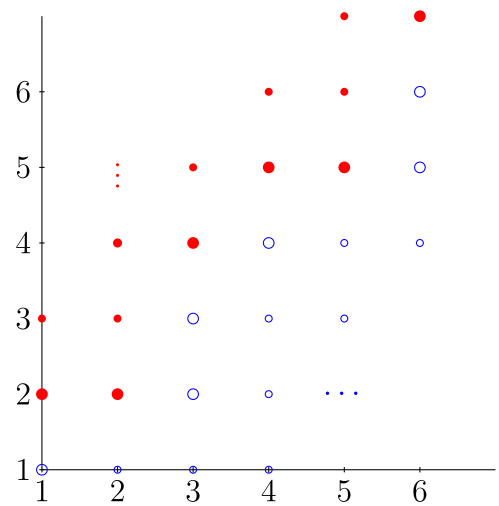

FIGURE 2

We identify the following subsets of $\mathrm{Hom}(P, \widehat{Q})$.

- $\mathrm{Hom}^L(P, \widehat{Q})$ are the maps f with finite profile ideal I. These maps are called *large*. Then $f(p) = \infty$ for all but a finite number of p's.
- $\mathrm{Hom}_S(P, \widehat{Q})$ are the f with finite co-profile ideal J. These maps are called *small*. Then there is a finite poset ideal J bounding the $f(p)$, i.e. all $f(p) \subseteq J$.
- $\mathrm{Hom}^u(P, \widehat{Q})$ are the f which are in neither the above, so both the profile ideal I and co-profile ideal J are infinite.

A consequence of Lemma 2.4 is the following.

Lemma 2.5. *D switches $Hom^L(P, \widehat{Q})$ and $Hom_S(Q, \widehat{P})$ and maps $Hom^u(P, \widehat{Q})$ to $Hom^u(Q, \widehat{P})$.*

Example 2.6. Consider isotone maps $f : \mathbb{N} \to \widehat{\mathbb{N}}$ where $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Such a map is large if $f(n) = \infty$ for some n. It is small if f is eventually constant, so $f(n) = c$ for all $n \geq n_0$. The maps in $\mathrm{Hom}^u(\mathbb{N}, \widehat{\mathbb{N}})$ are the maps $f : \mathbb{N} \to \mathbb{N}$ which are not bounded, so $\lim_{n \to \infty} f(n) = \infty$. We see the naturalness of considering $\mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}})$ instead of $\mathrm{Hom}(\mathbb{N}, \mathbb{N})$: The latter is not self-dual while the former is. $\mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}})$ has two countable dual "shores" $\mathrm{Hom}^L(\mathbb{N}, \widehat{\mathbb{N}})$ and $\mathrm{Hom}_S(\mathbb{N}, \widehat{\mathbb{N}})$ and then between them an uncountable "ocean" $\mathrm{Hom}^u(\mathbb{N}, \widehat{\mathbb{N}})$.

2.3. **Adjunctions.** Given an isotone map $f : P \to Q$ it induces a pull-back map

$$f^* : \widehat{Q} \to \widehat{P}, \quad (J, G) \mapsto (f^{-1}(J), f^{-1}(G)).$$

This map has a left adjoint

$$f^! : \widehat{P} \to \widehat{Q}, \quad (I, F) \mapsto (f(I)^{\downarrow}, -)$$

where $f(I)^{\downarrow}$ is the smallest poset ideal containing $f(I)$. There is also a right adjoint of f^*:

$$f^{\mathsf{i}} : \widehat{P} \to \widehat{Q}, \quad (I, F) \mapsto (-, f(F)^{\uparrow})$$

where $f(F)^{\uparrow}$ is the smallest poset filter in $\widehat{Q}$ containing $f(F)$. All these maps are functorial in P and Q.

3. The graph, the ascent, and Alexander duality

We define the two significant notions of this article, the graph and ascent of a profunctor $P \to\!\!\!\!\!+ Q$, or equivalently the right and left boundaries of the cut (I, F) corresponding to this profunctor. Then we state several versions of the main theorem of this article, Theorem 3.5 on preserving cuts.

3.1. **The graph and ascent.** Recall that for a poset P then UP denotes the underlying set, considered as a discrete poset.

Definition 3.1. Let $f : P \to \widehat{Q}$ be isotone. Its *ascent* is

$$\Lambda f = \{(q, p) \,|\, q \in f(p) \text{ but } q \notin f(p') \text{ for } p' < p\} \subseteq UQ \times UP^{\mathrm{op}}. \tag{1}$$

Its *graph* is

$$\Gamma f = \{(q, p) \,|\, q \text{ minimal in the complement } f(p)^c\} \subseteq UQ \times UP^{\mathrm{op}}. \tag{2}$$

Note that Λf and Γf are disjoint.

Example 3.2. For an isotone map $f : \mathbb{N} \to \widehat{\mathbb{N}}$, see Figure 1, then Γf are the red (or black) discs, and Λf are the blue circles.

3.2. **Left and right boundaries.** We have seen that $\mathrm{Hom}(P, \widehat{Q})$, the profunctors from P to Q, identify as $(Q \times P^{\mathrm{op}})\widehat{\ }$. So an isotone map $f : P \to \widehat{Q}$ corresponds to a cut (I, F) for $Q \times P^{\mathrm{op}}$ where

$$I = \{(q, p) \,|\, q \in f(p)\}, \quad F = \{(q, p) \,|\, q \notin f(p)\}. \tag{3}$$

Definition 3.3. If (I, F) is a cut for $Q \times P^{\mathrm{op}}$ its *left and right boundaries* are respectively

$$\begin{aligned} \Lambda I \quad &= \{(q, p) \in I \,|\, (q, p') \notin I \text{ for } p' < p\} \subseteq UQ \times UP^{\mathrm{op}} \\ \Gamma F \quad &= \{(q, p) \in F^{\mathrm{op}} \,|\, (p, q') \notin F^{\mathrm{op}} \text{ for } q' < q\} \subseteq UQ \times UP^{\mathrm{op}} \end{aligned}$$

Corollary 3.4. *Given dual maps*

$$f : P \to \widehat{Q}, \quad g = Df : Q \to \widehat{P}.$$

Let (I, F) be the cut in $Q \times P^{op}$ associated to f, and (J, G) the cut in $P \times Q^{op}$ associated to g.

a. $(J, G) = (F^{op}, I^{op})$.
b. $\Gamma G = \Lambda I, \quad \Lambda J = \Gamma F$.
c. $\Gamma g = \Lambda f, \quad \Lambda g = \Gamma f$.

3.3. **Extending Λ and Γ to the next level.** We have looked at cuts (I, F) for $Q \times P^{\text{op}}$. Proceeding to the next level, we look at cuts $(\mathcal{I}, \mathcal{F})$ for $\text{Hom}(P, \widehat{Q}) = (Q \times P^{\text{op}})\widehat{}$. Elements in this latter set are cuts (I, F) for $Q \times P^{\text{op}}$ partially ordered by $(I, F) \leq (I', F')$ if $I \subseteq I'$. Thus the ideal $\mathcal{I}$ consists of cuts (I, F) closed under taking cuts with smaller I's. Similarly the poset filter $\mathcal{F}$ is closed under taking larger I's (and so smaller $F = I^c$'s).

We have a map

$$\Lambda : U\text{Hom}(P, \widehat{Q}) \to (UQ \times UP^{\text{op}})\widehat{}, \quad f \mapsto (\Lambda f, -) \tag{4}$$

and a map

$$\Gamma : U\text{Hom}(P, \widehat{Q}) \to (UP \times UQ^{\text{op}})\widehat{}, \quad f \mapsto (-, \Gamma f). \tag{5}$$

The map Λ induces an isotone map of posets:

$$\Lambda^{\mathsf{i}} : \text{Hom}(P, \widehat{Q})\widehat{} \to (UQ \times UP^{\text{op}})\widehat{\widehat{}}, \quad (\mathcal{I}, \mathcal{F}) \mapsto (-, \Lambda(\mathcal{F})^{\uparrow}).$$

For the cut $(\mathcal{I}, \mathcal{F})$, denote by $(-, \mathcal{F}_\Lambda)$ its image. Then $\mathcal{F}_\Lambda$ is the filter in the Boolean lattice $(UQ \times UP^{\text{op}})\widehat{}$ generated by cuts $(\Lambda f, -)$ where $f \in \mathcal{F}$.

The map Γ induces an isotone map of posets:

$$\Gamma^{!} : \text{Hom}(P, \widehat{Q})\widehat{} \to (UQ \times UP^{\text{op}})\widehat{\widehat{}}, \quad (\mathcal{I}, \mathcal{F}) \mapsto (\Gamma(\mathcal{I})^{\downarrow}, -).$$

Denote by $(\mathcal{I}_\Gamma, -)$ the image of $(\mathcal{I}, \mathcal{F})$. Then $\mathcal{I}_\Gamma$ is the ideal in the Boolean lattice $(UQ \times UP^{\text{op}})\widehat{}$ generated by cuts $(-, \Gamma f)$ for $f \in \mathcal{I}$.

3.4. **Main theorem: Preserving the cut.** In order for the left and right boundaries of a cut (I, F) to give enough information we need to make sure that minimal elements of poset filters of P and Q exists. A poset P is *well-founded* if every subset of P has a minimal element. Equivalently, any descending chain of elements

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots$$

stabilizes, i.e. for some N we have $p_n = p_N$ for $n \geq N$.

The following theorem is a strong generalization of the results in several articles [6], [9], [7],[12], see the next Subsection 3.5 for more on this. The most significant tool in the argument is Zorn's lemma (which is equivalent to the axiom of choice). Note also that $\text{Hom}(P, \widehat{Q})$ is a distributive lattice and so has all joins (colimits) and meets (limits).

Theorem 3.5 (Preserving the cut). *Let P and Q be well-founded posets, and $(\mathcal{I}, \mathcal{F})$ a cut for Hom$(P, \widehat{Q})$. Then $(\mathcal{I}_\Gamma, \mathcal{F}_\Lambda)$ is a cut for the Boolean lattice $(UQ \times UP^{op})\widehat{}$. In other words, the maps $\Gamma^! = \Lambda^{\mathsf{i}}$.*

Before proving this we state two alternative formulations of this theorem.

Theorem 3.6. *Let P and Q be well-founded posets, and $(\mathcal{I}, \mathcal{F})$ a cut for Hom$(P, \widehat{Q})$. The filter $\mathcal{F}_\Lambda = \Lambda^{\mathsf{i}}(\mathcal{F})$ for $(UQ \times UP^{op})\widehat{}$ generated by all $(\Lambda f, -)$ for $f \in \mathcal{F}$, and the filter $(\Gamma^{op})^{\mathsf{i}}(\mathcal{I}^{op})$ for $(UP \times UQ^{op})\widehat{}$ generated by all $((\Gamma f)^{op}, -)$ for $f \in \mathcal{I}$,* *are Alexander dual filters. (This is the version applied in Stanley-Reisner theory, giving Alexander dual monomial ideals, see Subsection 3.5.)*

By Corollary 3.4 we also have a commutative diagram

$$
\begin{array}{ccc}
U\mathrm{Hom}(P,\widehat{Q}) & \xrightarrow{\Lambda} & \widehat{(UQ \times UP^{\mathrm{op}})} . \\
\downarrow D & & \downarrow D \\
U\mathrm{Hom}(Q,\widehat{P}) & \xrightarrow{\Gamma} & \widehat{(UP \times UQ^{\mathrm{op}})}
\end{array}
\tag{6}
$$

Below we write $\Lambda_{P,Q}$ and $\Gamma_{Q,P}$ for these maps.

Theorem 3.7. *Let P and Q be well-founded posets. The following diagram commutes:*

$$
\begin{array}{ccc}
\widehat{Hom(P,\widehat{Q})} & \xrightarrow{\Lambda^{¡}} & \widehat{\widehat{(UQ \times UP^{op})}} . \\
\downarrow D & & \downarrow D \\
\widehat{Hom(Q,\widehat{P})} & \xrightarrow{\Lambda^{¡}} & \widehat{\widehat{(UP \times UQ^{op})}}
\end{array}
$$

There is of course also a similar diagram for $\Gamma^{!} = \Lambda^{¡}$.

Proof of Theorem 3.5. We need to show that ideal $\mathcal{I}_\Gamma$ is the complement of the filter $\mathcal{F}_\Lambda$ in the Boolean lattice $\widehat{(UQ \times UP^{\mathrm{op}})}$.

Part I. We first show that for any $(I, F) \in \mathcal{I}$ and $(J, G) \in \mathcal{F}$ that

$$\Gamma F \cap \Lambda J \neq \emptyset.$$

This will show that $\mathcal{I}_\Gamma$ is contained in the complement of $\mathcal{F}_\Lambda$. **Part II.** Let $S \subseteq UQ \times UP^{\mathrm{op}}$ be such that $S \cap \Lambda J \neq \emptyset$ for *every* $(J, G) \in \mathcal{F}$. We show that $S \supseteq \Gamma F$ for *some* $(I, F) \in \mathcal{I}$. This gives that $\mathcal{I}_\Gamma$ contains the complement of $\mathcal{F}_\Lambda$. □

Example 3.8. Consider profunctors $\mathrm{Hom}([5], \widehat{[3]})$ where $\widehat{[3]} = [3] \cup \{\infty\}$. Let $\mathcal{I}$ be the ideal consisting of all f not taking the value ∞. The graph of such and f is shown in Figure 3. Such a path in the rectangle $[5] \times [3]$ is a *right path.* Let $\mathcal{F}$ be the complement filter, consisting of all g which take the value ∞ for some argument in $[5]$. The ascent of such a g is also shown in Figure 3. Such a path in the rectangle $[5] \times [3]$ is an *up path.* Theorem 3.5 above says that given any subset S of $[5] \times [3]$, exactly one of the following holds: i. S contains an up path, ii. the complement S^c contains a right path.

3.5. **Applications: Finite posets and Stanley-Reisner ideals.** In the following for ease of notation we denote a polynomial ring $k[x_{UQ\times UP^{op}}]$ as $k[x_{Q\times P^{\mathrm{op}}}]$.

Definition 3.9. Let P and Q be *finite* posets. From the cut $(\mathcal{I}, \mathcal{F})$ for $\mathrm{Hom}(P, \widehat{Q})$, we get the cut $(\mathcal{I}_\Gamma, \mathcal{F}_\Lambda)$ for $\widehat{(UQ \times UP^{\mathrm{op}})}$. Since $UQ \times UP^{\mathrm{op}}$ is simply a set, this gives a Stanley-Reisner ideal in $k[x_{Q\times P^{\mathrm{op}}}]$. This is the Λ-*ideal* for $(\mathcal{I}, \mathcal{F})$ and we

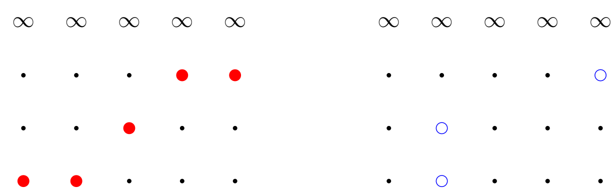

FIGURE 3. Profunctors $\mathrm{Hom}([5], \widehat{[3]})$. A graph to the left and an ascent to the right.

denote it as $L_\Lambda(\mathcal{F}; P, Q)$, or simply $L_\Lambda(\mathcal{F})$. As f varies in $\mathcal{F}$, it is generated by the Λf (or rather the squarefree monomials $\prod_{(q,p)\in\Lambda f} x_{q,p}$).

Similarly we get a cut $((\mathcal{F}_\Lambda)^{\mathrm{op}}, (\mathcal{I}_\Gamma)^{\mathrm{op}})$ for $\widehat{(UP \times UQ^{\mathrm{op}})}$, corresponding to a Stanley-Reisner ideal in $k[x_{P\times Q^{\mathrm{op}}}]$. This is the Γ-*ideal* of the cut $(\mathcal{I}, \mathcal{F})$ and is denoted $L_\Gamma(\mathcal{I}; P, Q)$, or simply $L_\Gamma(\mathcal{F})$. As f varies in $\mathcal{I}$, it is generated by the Γf (or rather the squarefree monomials $\prod_{(p,q)\in\Gamma f} x_{p,q}$).

By Theorem 3.6 above, the Λ-ideal and Γ-ideal are Alexander dual ideals.

Application 1. When $Q = [n]$, the chain on n elements, the Λ-ideals $L_\Lambda(\mathcal{F}; P, [n])$ are the letterplace ideals of [9] and are shown to be Cohen-Macaulay ideals. The Alexander dual Γ-ideals $L_\Gamma(\mathcal{I}; P, [n])$ are the co-letterplace ideals in loc.cit. and thus have linear resolutions. The above definition and results may thus be seen as a full generalization of the setting of [9].

Application 2. Again when $Q = [n]$ the Λ-ideals $L_\Lambda(\mathcal{F}; P, [n])$ define *simplicial balls* by the Stanley-Reisner correspondence, [4]. Furthermore there is a very simple description of the Stanley-Reisner ideal of their boundaries, which are *simplicial spheres*. This gives the construction of an enormous amount of simplicial spheres. In particular they generalize comprehensively both Bier spheres [2] and Gil Kalai's squeezed spheres [14].

4. TOPOLOGY ON $\mathrm{HOM}(P, \widehat{Q})$

We want to get a setting where P and Q may be infinite but the ascents Λf and graphs Γf are finite. Then we get Λ- and Γ-ideals in infinite-dimensional polynomial rings. This can be achieved if we put a suitable topology on $\mathrm{Hom}(P, \widehat{Q})$.

4.1. **Defining the topology.** We define a topology on $\mathrm{Hom}(P, \widehat{Q})$ by defining a basis of open subsets.

Definition 4.1. Let $\overline{f}$ be a function in $\mathrm{Hom}^L(P, \widehat{Q})$ and $\underline{f}$ a function in $\mathrm{Hom}_S(P, \widehat{Q})$. The set of functions f such that $\underline{f} \leq f \leq \overline{f}$ is denoted $U(\underline{f}, \overline{f})$.

The sets $U(\underline{f}, \overline{f})$ form the basis of a topology on $\mathrm{Hom}(P, \widehat{Q})$. Note that if P and Q are finite we get the discrete topology on $\mathrm{Hom}(P, \widehat{Q})$.

Lemma 4.2. *The duality map D of Lemma 2.1 is a homeomorphism of topological spaces.*

5. Natural posets

We introduce the class of natural posets as the suitable generalization of natural numbers. We then have good criterions for when open poset ideals are closed or interior. In Section 6 for this class of posets we get a version of Theorem 3.5 which may be applied to construct Alexander dual ideals in infinite-dimensional polynomial rings.

5.1. **Well partially ordered sets.** A poset P is *well partially ordered* if the following two conditions holds:

i. Any descending chain in P stabilizes.
ii. Any antichain in P is finite.

Lemma 5.1. *Let P and Q be well partially ordered sets, and $f : P \to \widehat{Q}$ an isotone map.*

a. *If f is large, then Γf is finite.*
b. *If f is small, then Λf is finite.*

5.2. **Natural posets.** The class of well partially ordered sets is very large, it includes all well-ordered sets. For our purposes we need an extra condition so that our posets are more like natural numbers.

Definition 5.2. A poset P is *down finite* if for each p in P, the principal ideal $I(p)$ is finite. A *natural* poset (in analogy with natural numbers) is a poset which is well partially ordered and down finite.

Lemma 5.3. *Let Q be down finite. Then for any large $f \in Hom(P, \widehat{Q})$ the open set $U(\mathbf{0}, f)$ is also closed.*

5.3. **Criterions for poset ideals being interior or for being clopen.** Let

$$f_1 \leq f_2 \leq f_2 \leq \cdots$$

be a weakly increasing set of maps in $\mathrm{Hom}(P, \widehat{Q})$, write $\mathrm{colim} f_r$ for their join. If the f_r's are a decreasing sequence we write $\lim f_r$ for their meet.

Posets in our application will typically be natural posets, like finite posets, the natural numbers $\mathbb{N}$. The following seems to provide the best way to check whether a poset ideal of isotone maps is interior.

Proposition 5.4. *Let P and Q be natural posets and $\mathcal{I}$ an open poset ideal in $Hom(P, \widehat{Q})$.*

a. *$\mathcal{I}$ is closed iff it contains the colimit of any increasing sequence of isotone maps in $\mathcal{I}$.*
b. *$\mathcal{I}$ is interior iff it contains any <u>large</u> colimit of an increasing sequence of isotone maps in $\mathcal{I}$.*

There are analog statements for open poset filters and decreasing sequences of maps in $\mathcal{F}$.

Theorem 5.5. *A poset ideal $\mathcal{I}$ of $Hom(P, \widehat{Q})$ is clopen (closed and open) iff $\mathcal{I}$ is a finite union of basis open subsets $U(\mathbf{0}, f)$. Alternatively formulated, an open poset ideal $\mathcal{I}$ is clopen iff it is finitely generated.*

Definition 5.6. A poset ideal $\mathcal{I}$ and a filter $\mathcal{F}$ in $\mathrm{Hom}(P, \widehat{Q})$ form an *interior cut*, if $\mathcal{I}$ is an interior open set and $\mathcal{F}$ is the interior set which is the complement of the closure of $\mathcal{I}$. We denote this as $[\mathcal{I}, \mathcal{F}]$.

6. Natural posets and monomial ideals

Here we give the variant of Theorem 3.5 such that the Λ- and Γ-ideals are finitely generated ideals in a polynomial ring. This works when P and Q are natural posets, which we assume in this section.

We have the following variation of Theorem 3.5.

Theorem 6.1. *Let P and Q be natural posets and $[\mathcal{I}, \mathcal{F}]$ an interior cut for $Hom(P, \widehat{Q})$. Then $(\mathcal{I}_\Gamma, \mathcal{F}_\Lambda)$ is a finite type cut for the Boolean lattice $\widehat{(UQ \times UP^{op})}$.*

We may now define the Λ- and Γ-ideals for (infinite) natural posets P and Q. These ideals then live in infinite-dimensional polynomial rings. These ideals will be square-free monomial ideals.

Definition 6.2. Let $L_\Lambda(\mathcal{F})$ the ideal in $k[x_{Q \times P^{\mathrm{op}}}]$ generated by the monomials $\prod_{(q,p) \in I} x_{q,p}$ for $(I, -) \in \mathcal{F}_\Lambda$. Let $L_\Gamma(\mathcal{I})$ be the ideal in $k[x_{P \times Q^{\mathrm{op}}}]$ generated by the monomials $\prod_{(p,q) \in F^{\mathrm{op}}} x_{p,q}$ for $(F^{\mathrm{op}}, -) \in \mathcal{I}_\Gamma^{\mathrm{op}}$.

By the theorem above, these ideals are Alexander dual ideals.

7. Monomial ideals

Large maps in $\mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}})$ correspond bijectively to monomials in the infinite dimensional polynomial ring $k[x_\mathbb{N}] = k[x_1, \ldots, x_n, \ldots]$. Open poset ideals in $\mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}})$ correspond bijectively to strongly stable ideals in $k[x_\mathbb{N}]$. Applying Theorem 6.1 this enables one to define a duality on strongly stable ideals, [8].

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