Coequalisers under the lens

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Lenses encode protocols for synchronising systems. We continue the work begun in the ACT Adjoint School 2020 to study the properties of the category of small categories and (asymmetric delta) lenses. The forgetful functor from the category of lenses to the category of functors is already known to reflect monos and epis and preserve epis; we show that it preserves monos, and give a simpler proof that it preserves epis. Together this gives a complete characterisation of the monic and epic lenses in terms of elementary properties of their get functors.

Next, we initiate the study of coequalisers of lenses. We observe that not all parallel pairs of lenses have coequalisers, and that the forgetful functor from the category of lenses to the category of functors neither preserves nor reflects all coequalisers. However, some coequalisers are reflected; we study when this occurs, and then use what we learned to show that every epic lens is regular, and that discrete opfibrations have pushouts along monic lenses. Corollaries include that every monic lens is effective, every monic epic lens is an isomorphism, and the class of all epic lenses and the class of all monic lenses form an orthogonal factorisation system.

1 Introduction

A *bidirectional transformation* between two systems is a specification of when the joint state of the two systems should be regarded as consistent, together with a protocol for updating each system to restore consistency in response to a change in the other [11]. The study of bidirectional transformations goes back as far as 1981 with Bancilhon and Spyrato's work on the view-update problem for databases [2]. The view-update problem is about *asymmetric* bidirectional transformations; those where the state of one of the systems, called the *view*, is completely determined by that of the other, called the *source*. Bidirectional transformations also arise in many other contexts across computer science, such as when programming with complex data structures and when linking user interfaces to data models.

An *asymmetric state-based lens* is a mathematical encoding of an asymmetric bidirectional transformation in which the consistency restoration updates to the source are assumed to be dependent only on the old source state and the updated view state. If S is the set of source states and V is the set of view states, such a lens consists of a *get function* $S \to V$ and a *put function* $S \times V \to S$ which, ideally, satisfy certain laws. The earliest known account of asymmetric state-based lenses may be found in Oles' PhD thesis [16, Chapter VI], where they are called *extensions* of *store shapes*; they are a key ingredient in his semantics for an imperative stack-based programming language with block-scoped variables because they capture the essential properties of a data store which changes shape as variables come into and go out of scope. All recent notions of lens, including the name *lens*, may be traced back to Pierce et al.'s independent discovery of asymmetric state based lenses [9], more than twenty years after Oles. Pierce et al. popularised the use of lenses and lens combinators for programming with complex data structures.

Diskin et al. highlighted the inadequacy of state-based lenses as a general mathematical model for bidirectional transformations [7], providing several examples of situations in which consistency restoration would benefit from knowing more about each change to the view than just the view's new state. In

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an *asymmetric delta lens*, their proposed alternative, systems are modelled as categories of states and transitions (deltas) rather than simply as sets of states, and the put operation takes as input specifically which transition occurred in the view rather than just the end state of that transition.

Application of category theory to the study of lenses has already proved fruitful. Johnson and Rosebrugh's research program [12, 13, 14] has enabled a unified treatment of symmetric and asymmetric delta lenses, with the perspective that a symmetric delta lens is an equivalence class of spans of asymmetric delta lenses. Ahman and Uustalu's observation that asymmetric delta lenses are compatible functor cofunctor pairs [1], and Clarke's generalisation of these lenses to the internal category theory setting [5], have enabled an abstract diagrammatic approach to proofs involving these lenses [6], in which we may profit from the already well-developed theory of functors and opfibrations. Yet, until the work of Chollet et al. at the ACT Adjoint School 2020 [4], little was known about the category of asymmetric delta lenses. Building on their work, this paper aims to further our understanding of this category.

Outline

Henceforth, we refer to asymmetric delta lenses simply as *lenses*, which we formally define in Section 2.

In Section 3, we prove the conjecture by Chollet et al. [4] that the forgetful functor from the category of lenses to the category of functors preserves monos. Together with their result that the forgetful functor also reflects monos, we deduce that the monic lenses are precisely the unique lenses on cosieves and that the subobjects (in the category of lenses) of a category are in bijection with out-degree-zero subcategories of that category. We also provide a simpler proof than the original one sketched by Steve Lack that the forgetful functor preserves epis.

In Section 4, we initiate the study of coequalisers of lenses. We begin with examples of how they aren't as well behaved as one might hope; specifically, not all parallel pairs of lenses have coequalisers, and the forgetful functor neither preserves nor reflects all coequalisers. We then prove a result (Theorem 4.5) about the coequalisers which are actually reflected by the forgetful functor.

In Section 5, we use Theorem 4.5 to show that the category of lenses has pushouts of discrete opfibrations along monos. We then show that every monic lens is effective. It follows that the classes of all monos, all effective monos, all regular monos, all strong monos and all extremal monos in the category of lenses coincide, and thus also that all lenses which are both monic and epic are isomorphisms.

In Section 6, we use Theorem 4.5 again to show that every epic lens is regular. It follows that the classes of all epis, all regular epis, all strong epis and all extremal epis in the category of lenses coincide. It also follows that the class of all epic lenses is left orthogonal to the class of all monic lenses. Together with other known results, this means that they form an orthogonal factorisation system.

2 Background

Notation

Application of functions (functors, lenses, etc.) is written by juxtaposing the function name with its argument. Application is right associative (unlike in Haskell), so an expression like FGx parses as $F(Gx)$ and not $(FG)x$. Parentheses are only used when needed or for clarity. Binary operators like $\circ$ have lower precedence than application, so an expression like $Fa \circ Fb$ parses as $(Fa) \circ (Fb)$ and not $F\big((a \circ F)b\big)$.

Let $\mathcal{C}at$ denote the category whose objects are small categories and whose morphisms are functors. Categories with boldface names $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. are always small. We write $|\mathbf{C}|$ for the set of objects of a small category $\mathbf{C}$, and, for all $X, Y \in |\mathbf{C}|$, we write $\mathbf{C}(X,Y)$ for the set of morphisms of $\mathbf{C}$ from X to Y.

For each $X \in |\mathbf{C}|$, we write $\mathbf{C}(X,*)$ for the set $\bigsqcup_{Y\in|\mathbf{C}|}\mathbf{C}(X,Y)$ of all morphisms in $\mathbf{C}$ out of X. We write $\operatorname{src} f$ and $\operatorname{tgt} f$ for, respectively, the source and target of a morphism f. We also write $f\colon X \to Y$ to say that $X,Y \in |\mathbf{C}|$ and $f \in \mathbf{C}(X,Y)$. The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted $g \circ f$.

The category with a single object 0 and no non-identity morphisms, also known as the *terminal category*, is denoted $\mathbf{1}$. The category with two objects 0 and 1 and a single non-identity morphism, namely $u\colon 0 \to 1$, also known as the *interval category*, is denoted $\mathbf{2}$. The category with two objects 0 and 1 and two non-identity morphisms, namely $v\colon 0 \to 1$ and $v^{-1}\colon 1 \to 0$, also known as the *free living isomorphism*, is denoted $\mathbf{I}$. We will abuse notation and identify objects and morphisms of a small category $\mathbf{C}$ with the corresponding functors $\mathbf{1} \to \mathbf{C}$ and $\mathbf{2} \to \mathbf{C}$ respectively.

If (1) is a pushout square in $\mathcal{C}at$ and $F'\colon \mathbf{A} \to \mathbf{E}$ and $G'\colon \mathbf{B} \to \mathbf{E}$ are functors for which $F' \circ S = G' \circ T$, then we write $[F', G']$ for the functor $\mathbf{C} \to \mathbf{E}$ induced from F' and G' by the universal property of the pushout. Similarly, if (1) is a pullback square in $\mathcal{C}at$ and $S'\colon \mathbf{E} \to \mathbf{A}$ and $T'\colon \mathbf{E} \to \mathbf{B}$ are functors for which $F \circ S' = G \circ T'$, then we write $\langle S', T'\rangle$ for the functor $\mathbf{E} \to \mathbf{D}$ induced from S' and T' by the universal property of the pullback. By our abuse of notation, if $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$ are such that $FA = GB$, then $\langle A, B\rangle$ is the object of $\mathbf{D}$ selected by the functor $\mathbf{1} \to \mathbf{D}$ induced by the universal property of the pullback from the functors $\mathbf{1} \to \mathbf{A}$ and $\mathbf{1} \to \mathbf{B}$ that respectively select the objects A and B.

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{T} & \mathbf{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{C} \end{array} \tag{1}$$

Lenses and discrete opfibrations

First, we recall Diskin et al.'s definition of a (asymmetric delta) lens [7].

Definition 2.1. Given small categories $\mathbf{A}$ and $\mathbf{B}$, a *lens* $F\colon \mathbf{A} \to \mathbf{B}$ consists of

- a functor $F\colon \mathbf{A} \to \mathbf{B}$, called the *get functor* of F, and
- a function $F^A\colon \mathbf{B}(FA,*) \to \mathbf{A}(A,*)$ for each $A \in |\mathbf{A}|$, collectively known as the *put functions*,

such that

- *PutGet*: $FF^A b = b$ for all $A \in |\mathbf{A}|$ and all $b \in \mathbf{B}(FA,*)$,
- *PutId*: $F^A \operatorname{id}_{FA} = \operatorname{id}_A$ for all $A \in |\mathbf{A}|$, and
- *PutPut*: $F^A(b' \circ b) = F^{A'}b' \circ F^A b$ for all $A \in |\mathbf{A}|$, $b \in \mathbf{B}(FA,*)$, $b' \in \mathbf{B}(FA',*)$, where $A' = \operatorname{tgt} F^A b$.

There is a category $\mathcal{L}ens$ whose objects are small categories and whose morphisms are lenses. The composite $G \circ F$ of lenses $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ has get functor which is the composite of the get functors of G and F, and has $(G \circ F)^A c = F^A G^{FA} c$ for all $A \in |\mathbf{A}|$ and all $c \in \mathbf{C}(GFA,*)$. There is also an identity-on-objects forgetful functor $\mathcal{U}\colon \mathcal{L}ens \to \mathcal{C}at$ that sends a lens to its get functor.

Definition 2.2. A functor $F\colon \mathbf{A} \to \mathbf{B}$ is a *discrete opfibration* if, for each $A \in |\mathbf{A}|$ and each $b \in \mathbf{B}(FA,*)$, there is a unique $a \in \mathbf{A}(A,*)$ such that $Fa = b$.

Remark 2.3. If $F\colon \mathbf{A} \to \mathbf{B}$ is a discrete opfibration, then there is a unique lens mapped by $\mathcal{U}$ to F. We will abuse notation and also use the name F to refer to this unique lens above F.

We also recall Johnson and Roseburgh's "pullback" of a cospan of lenses [12], which we will refer to as their *fake pullback* (not to be confused with Street's fake pullback [18]).

Definition 2.4. The *fake pullback* of a cospan $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$ in $\mathcal{L}ens$ is a span $\mathbf{A} \xleftarrow{\overline{G}} \mathbf{D} \xrightarrow{\overline{F}} \mathbf{B}$ in $\mathcal{L}ens$ where

- the get functors of $\overline{F}$ and $\overline{G}$ form a pullback square

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{\mathcal{U}\overline{F}} & \mathbf{B} \\ {\scriptstyle \mathcal{U}\overline{G}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{U}G} \\ \mathbf{A} & \xrightarrow[\mathcal{U}F]{} & \mathbf{C} \end{array}$$

  in $\mathcal{C}at$ (this determines them up to isomorphism), and

- for each $D \in |\mathbf{D}|$, each $a \in \mathbf{A}(\overline{G}D, *)$, and each $b \in \mathbf{B}(\overline{F}D, *)$,

$$\overline{F}^D b = \langle F^{\overline{G}D} Gb,\, b \rangle \qquad \text{and} \qquad \overline{G}^D a = \langle a,\, G^{\overline{F}D} Fa \rangle.$$

When $F = G$, the lenses $\overline{F}, \overline{G} \colon \mathbf{D} \to \mathbf{A}$ are also called the *fake kernel pair* of F.

3 Characterising monic and epic lenses

Monic lenses

We will study the monos in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$, expressed as follows.

Theorem 3.1. *The functor $\mathcal{U}$ preserves and reflects monos.*

Reflection was proved and preservation conjectured by Chollet et al. [4]. Recalling that a morphism is monic if and only if it has a kernel pair with both morphisms equal, we may prove preservation.

Proof that $\mathcal{U}$ preserves monos. Let $M \colon \mathbf{A} \to \mathbf{B}$ be a monic lens, and let $P_1, P_2 \colon \mathbf{Ker}\,\mathcal{U}M \to \mathbf{A}$ be its fake kernel pair in $\mathcal{L}ens$. As M is monic and $M \circ P_1 = M \circ P_2$, actually $P_1 = P_2$, and so $\mathcal{U}P_1 = \mathcal{U}P_2$. But $\mathcal{U}P_1$ and $\mathcal{U}P_2$ are the (real) kernel pair of $\mathcal{U}M$ in $\mathcal{C}at$. Hence $\mathcal{U}M$ is a monic functor. □

Chollet et al. [4] also showed that a lens' get functor is monic if and only if it is a cosieve.

Definition 3.2. A *cosieve* is an injective-on-objects discrete opfibration.

Corollary 3.3. *The functor $\mathcal{U}$ restricts to a bijection between monic lenses and cosieves.*

Proof. A cosieve is a discrete opfibration, so there is a unique lens above it; by reflection, this lens is monic. Conversely, the get functor of a monic lens is, by preservation, monic, and so is a cosieve. □

The above result says that monic lenses and cosieves are essentially the same thing; we continue to use the term cosieve for functors when we wish to distinguish these from monic lenses.

Lens images and factorisation

The images of the object and morphism maps of a functor don't always form a subcategory of the functor's target category. This is why the image of a functor is usually defined to be the smallest subcategory of its target containing the images of its object and morphism maps. The image of a lens is defined to be the image of its get functor. Images of lenses are better behaved than those of functors.

Proposition 3.4. *A lens' image consists only of the images of its get functor's object and morphism maps.*

Actually, we can say more.

Definition 3.5. A subcategory $\mathbf{B}$ of $\mathbf{A}$ is *out-degree-zero* if for all $B \in |\mathbf{B}|$ and all $a \in \mathbf{A}(B,*)$, $a \in \mathbf{B}(B,*)$.

Proposition 3.6. *The image of a lens is an out-degree-zero subcategory of its target.*

For each small category $\mathbf{C}$, if two lenses with target $\mathbf{C}$ are isomorphic in the slice category $\mathcal{L}ens/\mathbf{C}$ then they have the same image. In particular, the image of a subobject of $\mathbf{C}$ is well defined. This allows the following characterisation of subobjects in $\mathcal{L}ens$.

Corollary 3.7. *For each small category* $\mathbf{C}$*, the function from the set of subobjects of* $\mathbf{C}$ *to the set of out-degree-zero subcategories of* $\mathbf{C}$ *that sends a subobject to its image is a bijection.*

Proof. An out-degree-zero subcategory inclusion functor is a cosieve and thus a monic lens. The inverse function sends each out-degree-zero subcategory to the equivalence class of its inclusion lens. □

Recall that a morphism $e\colon A \to B$ is *left orthogonal* to a morphism $m\colon C \to D$ if, for all pairs of morphisms $f\colon A \to C$ and $g\colon B \to D$ such that $g \circ e = m \circ f$, there is a unique morphism $h\colon B \to C$, called the *diagonal filler*, such that $f = h \circ e$ and $g = m \circ h$. Also recall that classes $\mathscr{E}$ and $\mathscr{M}$ of morphisms form an *orthogonal factorisation system* if $\mathscr{E}$ is the class of all morphisms that are left orthogonal to all morphisms in $\mathscr{M}$, and every morphism f factors as $f = m \circ e$ for some $e \in \mathscr{E}$ and some $m \in \mathscr{M}$.

Remark 3.8. Johnson and Roseburgh observed that the class of surjective-on-objects lenses and the class of injective-on-objects-and-morphisms lenses form an orthogonal factorisation system on $\mathcal{L}ens$ where a lens factors as its corestriction onto its image followed by the inclusion of its image into its target [15].

From the previous subsection, we already know that the right class of this factorisation system is the class of all monic lenses. In the next subsection, we will see that its left class is the class of all epic lenses. In Section 6, we will deduce the orthogonality without needing to construct diagonal fillers.

Epic lenses

We may also study the epis in $\mathcal{L}ens$ via their relation to those in $\mathcal{C}at$.

Theorem 3.9. *The functor* $\mathcal{U}$ *preserves and reflects epis.*

Again, reflection was proved and preservation conjectured by Chollet et al. [4]. Steve Lack was the first to sketch a proof of the preservation of epis; we present a new, simpler proof below. First, we recall some preliminary results about epic functors and epic lenses.

Proposition 3.10. *Every epic functor is surjective on objects. Every functor which is surjective both on objects and on morphisms is epic.*

Recall that not all epic functors are surjective on morphisms.

Example 3.11. Let $J\colon \mathbf{2} \to \mathbf{I}$ be the functor that sends the non-identity morphism u of the interval category $\mathbf{2}$ to the morphism v of the free living isomorphism $\mathbf{I}$. Then J is epic because any two functors out of $\mathbf{I}$ which agree on v must also agree on v^{-1}. However, the morphism v^{-1} is not in the image of J.

Remark 3.12. If the get functor of a lens is surjective on objects, then it is also surjective on morphisms by Proposition 3.6, and so it is an epic functor by Proposition 3.10.

Proof that $\mathcal{U}$ *preserves epis.* Let $F\colon \mathbf{A} \to \mathbf{B}$ be a lens, and suppose that $\mathcal{U}F$ is not an epic functor. By Remark 3.8, F has a factorisation $F = M \circ \overline{F}$ with $\overline{\mathbf{A}} = \operatorname{tgt}\overline{F}$ the image of F. Let $J_1, J_2\colon \mathbf{B} \to \mathbf{Coker}\,\mathcal{U}M$ be the cokernel pair of $\mathcal{U}M$ in $\mathcal{C}at$. It is well known that cosieves are pushout stable (more details in Proposition 5.1). As $\mathcal{U}M$ is a cosieve (Corollary 3.3), so are J_1 and J_2. Thus J_1 and J_2 are uniquely lenses, and $J_1 \circ M = J_2 \circ M$ in $\mathcal{L}ens$. By Remark 3.12, $\mathcal{U}F$ is not surjective on objects, so there is a $B \in |\mathbf{B}| \setminus |\overline{\mathbf{A}}|$, and $J_1 B \neq J_2 B$. As $J_1 \circ F = J_2 \circ F$, but $J_1 \neq J_2$, F is not an epic lens. □

Corollary 3.13. *Epic lenses are surjective on objects and morphisms.*

Proof. By Theorem 3.9 and Proposition 3.10, epic lenses are surjective on objects. By Proposition 3.6, the image of a lens is a full subcategory, and thus epic lenses are also surjective on morphisms. □

4 Coequalisers of lenses

Given morphisms $f_1, f_2 \colon A \to B$, we say that a morphism $e \colon B \to C$ *coforks* f_1 and f_2 if $e \circ f_1 = e \circ f_2$. Some authors would use the verb coequalise where we use the verb cofork. Unlike those authors, we say that e *coequalises* f_1 and f_2 only when e is universal among coforks of f_1 and f_2.

Non-existence, non-preservation and non-reflection of coequalisers

Recall that $\mathcal{C}at$ has all coequalisers. Shortly, we will construct several counterexamples to the well-behavedness of coequalisers in $\mathcal{L}ens$, at least with respect to those in $\mathcal{C}at$. To do this, we will use the following proposition, which gives necessary conditions for a cofork of lenses to be a coequaliser.

Proposition 4.1. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses with coequaliser $E \colon \mathbf{B} \to \mathbf{C}$ in $\mathcal{L}ens$. Then*

(1) for each cofork $G \colon \mathbf{B} \to \mathbf{D}$ of F_1 and F_2, $G^B d = E^B E G^B d$ for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$; and

(2) in particular, E is the unique lens above $\mathcal{U}E$ that coforks F_1 and F_2.

Proof. For (1), if $G \colon \mathbf{B} \to \mathbf{D}$ coforks F_1 and F_2, then there is a lens $H \colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$, and so $G^B d = (H \circ E)^B d = E^B H^{EB} d = E^B E E^B H^{EB} d = E^B E (H \circ E)^B d = E^B E G^B d$. For (2), if $G \colon \mathbf{B} \to \mathbf{C}$ is a lens above $\mathcal{U}E$ that coforks F_1 and F_2, then $G^B c = E^B E G^B c = E^B G G^B c = E^B c$ for each $B \in |\mathbf{B}|$ and each $c \in \mathbf{C}(EB, *)$, and so $G = E$. □

The first example shows that $\mathcal{L}ens$ doesn't have all coequalisers, and that $\mathcal{U}$ doesn't reflect them.

Example 4.2. Let $\mathbf{A}$ and $\mathbf{B}$ be the preordered sets generated respectively by the graphs

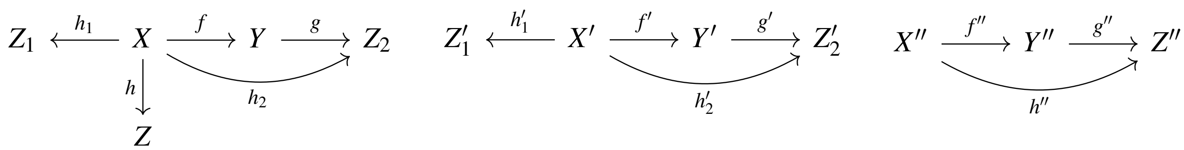
$$\begin{array}{ccccc} Y_1 & \xleftarrow{f_1} & X & \xrightarrow{f_2} & Y_2 \\ & & \downarrow{\scriptstyle f} & & \\ & & Y & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} Y_1' & \xleftarrow{f_1'} & X' & \xrightarrow{f_2'} & Y_2' \end{array} .$$

Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y_1 to Y_1', Y_2 to Y_2', and such that $F_1 Y = Y_1'$, ${F_1}^X f_1' = f_1$, $F_2 Y = Y_2'$, and ${F_2}^X f_2' = f_2$. Let $G \colon \mathbf{B} \to \mathbf{2}$ be the unique functor that sends X' to 0, and both Y_1' and Y_2' to 1; G coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. There are only two lens structures on G that cofork F_1 and F_2 in $\mathcal{L}ens$; one is determined by ${G_1}^{X'} u = f_1'$ and the other by ${G_2}^{X'} u = f_2'$. By Proposition 4.1, neither G_1 nor G_2 coequalises F_1 and F_2. Thus $\mathcal{U}$ doesn't reflect the coequaliser G of $\mathcal{U}F_1$ and $\mathcal{U}F_2$.

Actually F_1 and F_2 don't have a coequaliser in $\mathcal{L}ens$. Assume that $E \colon \mathbf{B} \to \mathbf{C}$ is such a coequaliser. Then $E f_1' = E F_1 f = E F_2 f = E f_2'$. As G_1 coforks F_1 and F_2, there is a lens $H \colon \mathbf{C} \to \mathbf{2}$ such that $G_1 = H \circ E$. As $HEX' = G_1 X' \neq G_1 Y_1' = HEY_1'$, we must have $EX' \neq EY_1'$. Hence EX' and EY_1' are distinct objects of the image of E, and $\mathrm{id}_{EX'}$, $E f_1'$ and $\mathrm{id}_{EY_1'}$ are distinct morphisms of the image of E. As E is a coequaliser, it is epi, and so, by Corollary 3.13, its image is all of $\mathbf{C}$. Thus $\mathcal{U}H$ is an isomorphism in $\mathcal{C}at$, and so H is an isomorphism in $\mathcal{L}ens$. Hence G_1 also coequalises F_1 and F_2, which is a contradiction.

There are even parallel pairs of lenses for which the coequaliser of their get functors has a unique lens structure that coforks them, and yet doesn't coequalise them.

Example 4.3. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the preordered sets generated respectively by the graphs



Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be the unique lenses that both send X to X', Y to Y', Z_1 to Z'_1, Z_2 to Z'_2, and such that $F_1 Z = Z'_1$, $F_1{}^X h'_1 = h_1$ and $F_2 Z = Z'_2$. Let $E \colon \mathbf{B} \to \mathbf{C}$ be the unique lens that sends X' to X'', Y' to Y'', and both Z'_1 and Z'_2 to Z''. Then $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$, and E coforks F_1 and F_2 in $\mathcal{L}ens$. However, E doesn't coequalise F_1 and F_2 in $\mathcal{L}ens$. Indeed, if $G \colon \mathbf{B} \to \mathbf{2}$ is the unique lens that sends X' to 0, all of Y', Z'_1 and Z'_2 to 1, and for which $G^{X'} u = h'_1$, then $E^{X'} E G^{X'} u = E^{X'} E h'_1 = E^{X'} h'' = h'_2 \neq h'_1 = G^{X'} u$.

The final example shows that $\mathcal{U}$ doesn't preserve coequalisers. It also shows that there are parallel pairs of lenses for which the coequaliser of their get functors has no lens structure that coforks them.

Example 4.4. Let $\mathbf{A}$ be the preordered set generated by the graph

$$Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2$$

Let $I \colon \mathbf{A} \to \mathbf{A}$ denote the identity lens, and let $S \colon \mathbf{A} \to \mathbf{A}$ be the unique lens that maps X to X, Y_1 to Y_2 and Y_2 to Y_1. The coequaliser of $\mathcal{U}I$ and $\mathcal{U}S$ in $\mathcal{C}at$ is the unique functor $E \colon \mathbf{A} \to \mathbf{2}$ that sends X to 0 and both Y_1 and Y_2 to 1. Recall that $\mathbf{1}$ is terminal in $\mathcal{L}ens$ [4]. We claim that the coequaliser of I and S in $\mathcal{L}ens$ is the unique lens $! \colon \mathbf{A} \to \mathbf{1}$. Let $G \colon \mathbf{A} \to \mathbf{C}$ be a lens that coforks I and S in $\mathcal{L}ens$. Let $f = G f_1$. Then $f = G f_1 = G I f_1 = G S f_1 = G f_2$. As $G^X f \in \mathbf{A}(X, *)$, it is one of f_1, f_2 and id_X. If $G^X f = f_1$, then

$$f_1 = I^X f_1 = I^X G^X f = (G \circ I)^X f = (G \circ S)^X f = S^X G^X f = S^X f_1 = f_2,$$

which is a contradiction. We get a similar contradiction if $G^X f = f_2$. By elimination, $G^X f = \mathrm{id}_X$, and so $f = G G^X f = G \,\mathrm{id}_X = \mathrm{id}_{GX}$. The image of G thus consists of the object GX and the morphism id_{GX}. If $H \colon \mathbf{1} \to \mathbf{C}$ is a lens such that $G = H \circ !$, then H must send 0 to GX, and this uniquely determines H. As the image of any lens, in particular G, is an out-degree-zero subcategory of its target category, this definition of H does indeed give a lens, and $G = H \circ !$. Of course, the factorisation $G = H \circ !$ is really the image factorisation of G from Remark 3.8.

Coequalisers which are reflected

Although the counterexamples in the previous subsection suggest that coequalisers in $\mathcal{L}ens$ have little relation to those in $\mathcal{C}at$, we will see in Theorem 5.4 and Corollary 6.4 two classes of coequalisers in $\mathcal{L}ens$ which do lie over coequalisers in $\mathcal{C}at$. The following theorem, a partial converse to Proposition 4.1, reduces checking the coequaliser property in these cases to checking that equation (2) always holds.

Theorem 4.5. *Let $F_1, F_2 \colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E \colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Then E coequalises F_1 and F_2 in $\mathcal{L}ens$ if and only if for all lenses $G \colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2 in $\mathcal{L}ens$, all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB, *)$, we have*

$$G^B d = E^B E G^B d \tag{2}$$

In the proof of the following lemma and again, later, in the proof of Lemma 5.3, we use the induction principle for the equivalence relation $\simeq$ on a set S generated by a binary relation R on S, that is,

$$\forall P\, x_0\, y_0.\ \left[\begin{array}{l} x_0 \simeq y_0 \\ \quad\wedge \quad \forall x\, y.\ x\, R\, y \implies P(x,y) \\ \quad\wedge \quad \forall x.\ P(x,x) \\ \quad\wedge \quad \forall x\, y.\ [x \simeq y \wedge P(x,y)] \implies P(y,x) \\ \quad\wedge \quad \forall x\, y\, z.\ [x \simeq y \wedge P(x,y) \wedge y \simeq z \wedge P(y,z)] \implies P(x,z) \end{array}\right] \implies P(x_0,y_0). \tag{3}$$

Lemma 4.6. *Let $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2 in $\mathcal{L}ens$, and let $H\colon \mathbf{C} \to \mathbf{D}$ be the unique functor such that $\mathcal{U}G = H \circ \mathcal{U}E$. Then there is a unique lens structure on H that, for all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB,*)$, satisfies the equation*

$$H^{EB}d = EG^{B}d. \tag{4}$$

Proof. For each $C \in |\mathbf{C}|$, as $\mathcal{U}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Hence, we may define H^C using equation (4), so long as, for all $B_1, B_2 \in |\mathbf{B}|$, if $EB_1 = EB_2$ then, for all $d \in \mathbf{D}(EB_1,*)$, we have $EG^{B_1}d = EG^{B_2}d$. Let $\simeq$ be the smallest equivalence relation on $|\mathbf{B}|$ such that $F_1A \simeq F_2A$ for all $A \in |\mathbf{A}|$. As $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$, we have [3, Proposition 4.1], for all $B_1, B_2 \in |\mathbf{B}|$, that $EB_1 = EB_2$ if and only if $B_1 \simeq B_2$. We proceed using the induction principle in equation (3). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\simeq$ hold as $=$ is an equivalence relation. For the remaining one, for all $A \in |\mathbf{A}|$ and all $d \in \mathbf{D}(F_1A,*)$, we have

$$EG^{F_1A}d = EF_1{F_1}^{A}G^{F_1A}d = (E \circ F_1)(G \circ F_1)^{A}d = (E \circ F_2)(G \circ F_2)^{A}d = EF_2{F_2}^{A}G^{F_2A}d = EG^{F_2A}d.$$

Define H^C using equation (4). It remains to check that the lens laws hold for H. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H^C \mathrm{id}_{HC} = EG^B \mathrm{id}_{GB} = E\,\mathrm{id}_B = \mathrm{id}_C$; hence PutId holds. For all $C \in |\mathbf{C}|$, all $d \in \mathbf{D}(HC,*)$ and all $d' \in \mathbf{D}(\mathrm{tgt}\, d,*)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and

$$H^C(d' \circ d) = EG^B(d' \circ d) = E\big(G^{B'}d' \circ G^B d\big) = EG^{B'}d' \circ EG^B d = H^{C'}d' \circ H^C d,$$

where $B' = \mathrm{tgt}\, G^B d$ and $C' = EB'$; hence PutPut holds. Finally, for all $C \in |\mathbf{C}|$ and all $d \in \mathbf{D}(HC,*)$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $HH^C d = HEG^B d = GG^B d = d$; hence PutGet holds. □

Proof of Theorem 4.5. We proved the only if direction in Proposition 4.1. For the if direction, suppose, for all lenses $G\colon \mathbf{B} \to \mathbf{D}$ that cofork F_1 and F_2, that equation (2) always holds. We must show that E is the universal cofork of F_1 and F_2 in $\mathcal{L}ens$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be another cofork of F_1 and F_2 in $\mathcal{L}ens$. Suppose that there is a lens $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. Then $\mathcal{U}G = \mathcal{U}H \circ \mathcal{U}E$, and so $\mathcal{U}H$ is the unique functor that composes with $\mathcal{U}E$ to give $\mathcal{U}G$. Let $C \in |\mathbf{C}|$ and $d \in \mathbf{D}(HC,*)$. As $\mathcal{U}E$ is epic, there is a $B \in |\mathbf{B}|$ such that $EB = C$. Then $H^C d = EE^B H^C d = E(H \circ E)^B d = EG^B d$. Hence H is uniquely determined. Now let $H\colon \mathbf{C} \to \mathbf{D}$ be the lens defined as in Lemma 4.6. For all $B \in |\mathbf{B}|$ and all $d \in \mathbf{D}(GB,*)$, we have $G^B d = E^B EG^B d = E^B H^{EB} d = (H \circ E)^B d$, and so $G = H \circ E$. □

Corollary 4.7. *Let $F_1, F_2\colon \mathbf{A} \to \mathbf{B}$ be lenses. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a cofork of F_1 and F_2 in $\mathcal{L}ens$, and suppose that $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$. If $\mathcal{U}E$ is a discrete opfibration then E coequalises F_1 and F_2.*

Proof. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2, let $B \in |\mathbf{B}|$ and let $d \in \mathbf{D}(GB,*)$. Then $G^B d$ and $E^B EG^B d$ are both elements of $\mathbf{B}(B,*)$ which are sent by E to the same morphism $EG^B d$ of $\mathbf{C}$. If $\mathcal{U}E$ is a discrete opfibration, then $EG^B d$ has a unique lift to $\mathbf{B}(B,*)$, and so $G^B d$ and $E^B EG^B d$ must be equal. □

5 Pushouts of discrete opfibrations along monos

In the proof that $\mathcal{U}$ preserves epis (Theorem 3.9), we used the well-known result that cosieves are pushout stable to explain why the pushout in $\mathcal{C}at$ of the get functors of a span of monic lenses lifts uniquely to a commutative square in $\mathcal{L}ens$; this lifted square is actually a pushout square in $\mathcal{L}ens$. In this section, we will show, more generally, that $\mathcal{L}ens$ has pushouts of discrete opfibrations along monics, and that $\mathcal{U}$ creates these pushouts. In what follows, we use square brackets for equivalence classes of elements.

Fritsch and Latch [10, Proposition 5.2] explicitly construct the pushout in $\mathcal{C}at$ of a functor along a full monic functor. Specialising to when the full monic functor is a cosieve, and recalling that the image of a cosieve is out-degree-zero, we obtain the following simplification of Fritsch and Latch's construction.

Proposition 5.1. *Let $F\colon \mathbf{A} \to \mathbf{C}$ be a functor and $J\colon \mathbf{A} \to \mathbf{B}$ be a cosieve. Then*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{J} & \mathbf{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\ \mathbf{C} & \xrightarrow[\overline{J}]{} & \mathbf{D} \end{array}$$

is a pushout square in $\mathcal{C}at$ and $\overline{J}$ is a cosieve, where $\mathbf{D}$, $\overline{F}$ and $\overline{J}$ are defined as follows:

- Object set:
$$|\mathbf{D}| = |\mathbf{C}| \sqcup \big(|\mathbf{B}| \setminus |\mathbf{A}|\big)$$

- Hom-sets: *for all $C_1, C_2 \in |\mathbf{C}|$ and all $B_1, B_2 \in |\mathbf{B}| \setminus |\mathbf{A}|$,*
$$\mathbf{D}(C_1,C_2) = \mathbf{C}(C_1,C_2) \qquad \mathbf{D}(C_1,B_2) = \emptyset$$
$$\mathbf{D}(B_1,B_2) = \mathbf{B}(B_1,B_2) \qquad \mathbf{D}(B_1,C_2) = \Big(\coprod_{A\in|\mathbf{A}|} \mathbf{C}(FA,C_2) \times \mathbf{B}(B_1,A)\Big)/\sim$$
where $\sim$ is the equivalence relation on $\coprod_{A\in|\mathbf{A}|} \mathbf{C}(FA,C_2) \times \mathbf{B}(B_1,A)$ generated by $(c, a\circ b) \sim (c\circ Fa, b)$ for all $A_1,A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B_1,A_1)$, all $a \in \mathbf{A}(A_1,A_2)$ and all $c \in \mathbf{C}(FA_2,C_2)$.

- Composition: *for all $B_1,B_2,B_3 \in |\mathbf{B}| \setminus |\mathbf{A}|$, all $A \in |\mathbf{A}|$, all $C_1,C_2,C_3 \in |\mathbf{C}|$, all $b_1 \in \mathbf{D}(B_1,B_2)$, all $b_2 \in \mathbf{D}(B_2,B_3)$, all $a \in \mathbf{D}(B_2,A)$, all $c \in \mathbf{D}(FA,C_2)$, all $c_1 \in \mathbf{D}(C_1,C_2)$ and all $c_2 \in \mathbf{D}(C_2,C_3)$,*
$$b_2 \circ_{\mathbf{D}} b_1 = b_2 \circ_{\mathbf{B}} b_1 \qquad [(c,a)] \circ_{\mathbf{D}} b_1 = [(c, a\circ_{\mathbf{B}} b_1)]$$
$$c_2 \circ_{\mathbf{D}} c_1 = c_2 \circ_{\mathbf{C}} c_1 \qquad c_2 \circ_{\mathbf{D}} [(c,a)] = [(c_2 \circ_{\mathbf{C}} c, a)]$$

- Identity morphisms: *same as in $\mathbf{B}$ and $\mathbf{C}$.*

- Injections: *the functor $\overline{J}\colon \mathbf{C} \to \mathbf{D}$ is the obvious inclusion of $\mathbf{C}$ as a full subcategory of $\mathbf{D}$; the functor $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ is defined, for all $B, B' \in |\mathbf{B}| \setminus |\mathbf{A}|$, all $A, A' \in |\mathbf{A}|$, all $b \in \mathbf{B}(B,B')$, all $b' \in \mathbf{B}(B,A)$ and all $a \in \mathbf{B}(A,A')$, as follows:*
$$\overline{F}B = B \qquad \qquad \overline{F}A = FA$$
$$\overline{F}b = b \qquad \overline{F}b' = [(\mathrm{id}_{FA}, b')] \qquad \overline{F}a = Fa$$

Theorem 5.2. *The pushout in $\mathcal{C}at$ of a discrete opfibration along a cosieve is a discrete opfibration.*

Lemma 5.3. *Let $F\colon \mathbf{A} \to \mathbf{C}$ be a discrete opfibration, let $J\colon \mathbf{A} \to \mathbf{B}$ be a cosieve, let $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and let $C \in |\mathbf{C}|$. Then, for all $A_1, A_2 \in \mathbf{A}$, all $b_1 \in \mathbf{B}(B, A_1)$, all $b_2 \in \mathbf{B}(B, A_2)$, all $c_1 \in \mathbf{C}(FA_1, C)$ and all $c_2 \in \mathbf{C}(FA_2, C)$, if $(c_1, b_1) \sim (c_2, b_2)$ then $F^{A_1}c_1 \circ b_1 = F^{A_2}c_2 \circ b_2$.*

Proof. We proceed by induction, using the induction principle for $\sim$ in equation (3). The proof obligations from the reflexivity, symmetry and transitivity axioms for $\sim$ hold because $=$ is an equivalence relation. For the remaining proof obligation, for all $A_1, A_2 \in |\mathbf{A}|$, all $b \in \mathbf{B}(B, A_1)$, all $a \in \mathbf{A}(A_1, A_2)$ and all $c \in \mathbf{C}(FA_2, C)$, we have $F^{A_1}Fa = a$ as F is a discrete opfibration, and so

$$F^{A_2}c \circ (a \circ b) = F^{A_2}c \circ F^{A_1}Fa \circ b = F^{A_1}(c \circ Fa) \circ b. \qquad \square$$

Proof of Theorem 5.2. Using the notation of Proposition 5.1, suppose that F is a discrete opfibration. We must show that $\overline{F}$ is also a discrete opfibration. Let $B \in |\mathbf{B}|$ and $d \in \mathbf{D}(\overline{F}B, *)$.

Suppose that $B \in |\mathbf{A}|$. Then $\overline{F}B = FB$, and $d \in \mathbf{C}(FB, *)$. As F is a discrete opfibration, there is a unique $a \in \mathbf{A}(B, *)$ such that $d = Fa$. But $\mathbf{A}(B, *) = \mathbf{B}(B, *)$ as $\mathbf{A}$ is out-degree-zero in $\mathbf{B}$; also $\overline{F}a = Fa$ for each $a \in \mathbf{B}(B, *)$. Hence there is a unique $a \in \mathbf{B}(B, *)$ such that $d = \overline{F}a$.

Suppose that $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and $\operatorname{tgt} d \in |\mathbf{B}| \setminus |\mathbf{A}|$. Then $\overline{F}B = B$, $d \in \mathbf{B}(B, *)$ and $\overline{F}d = d$. As $\overline{F}$ is injective on the morphisms of $\mathbf{B}$ not in $\mathbf{A}$, d is the unique morphism in $\mathbf{B}(B, *)$ mapped by $\overline{F}$ to d.

Otherwise, $B \in |\mathbf{B}| \setminus |\mathbf{A}|$ and $\operatorname{tgt} d \in |\mathbf{C}|$. Then $\overline{F}B = B$, and $d = [(c_1, b_1)]$ for some $A_1 \in |\mathbf{A}|$, some $b_1 \in \mathbf{B}(B, A_1)$ and some $c_1 \in \mathbf{C}(FA_1, C)$, where $C = \operatorname{tgt} d$. For uniqueness of lifts, suppose that $b_2 \in \mathbf{B}(B, *)$ is such that $d = \overline{F}b_2$. Let $A_2 = \operatorname{tgt} b_2$. Then $A_2 \in |\mathbf{A}|$ as $\overline{F}A_2 = \operatorname{tgt} d = C$, and so $\overline{F}b_2 = [(\mathrm{id}_C, b_2)]$. As $d = \overline{F}b_2$, we have $(\mathrm{id}_C, b_2) \sim (c_1, b_1)$. By the lemma, $b_2 = F^{A_2}\mathrm{id}_C \circ b_2 = F^{A_1}c_1 \circ b_1$; this determines b_2. For existence of lifts, note that $\overline{F}(F^{A_1}c_1 \circ b_1) = [(\mathrm{id}_C, F^{A_1}c_1 \circ b_1)] = [(FF^{A_1}c_1, b_1)] = [(c_1, b_1)] = d$. $\square$

Theorem 5.4. *The functor $\mathcal{U}$ creates pushouts of monic lenses with discrete opfibrations.*

Proof. Using the notation of Proposition 5.1, suppose that F is a discrete opfibration. Then $\overline{F}$ is also a discrete opfibration (Theorem 5.2). Let $J_{\mathbf{B}}\colon \mathbf{B} \to \mathbf{B} \sqcup \mathbf{C}$ and $J_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{B} \sqcup \mathbf{C}$ be the coproduct injection functors. Coproduct injections in $\mathcal{C}at$ are always discrete opfibrations, as is the coproduct copairing of any two discrete opfibrations. Hence $J_{\mathbf{B}}$, $J_{\mathbf{C}}$ and $[\bar{J}, \overline{F}]$ are all discrete opfibrations. As the composite of two discrete opfibrations is a discrete opfibration, so is $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$. So far, we know that $[\bar{J}, \overline{F}]$ is the coequaliser in $\mathcal{C}at$ of $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$, all of these functors have canonical lens structures as they are discrete opfibrations, and $[\bar{J}, \overline{F}]$ coforks $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $[\bar{J}, \overline{F}]$ is a discrete opfibration, the conditions of Theorem 4.5 are satisfied, and so $[\bar{J}, \overline{F}]$ coequalises $J_{\mathbf{B}} \circ J$ and $J_{\mathbf{C}} \circ F$ in $\mathcal{L}ens$. As $\mathcal{U}$ creates coproducts [4], it follows that $\bar{J}$ and $\overline{F}$ exhibit $\mathbf{D}$ as the pushout of J and F in $\mathcal{L}ens$. $\square$

One might hope that the above result generalises to pushouts of two discrete opfibrations, or of arbitrary lenses along monics; this isn't the case. The following is an example of two discrete opfibrations whose pushout injection functors have no lens structures that give a commutative square of lenses.

Example 5.5. Let $\mathbf{A}$ and $\mathbf{B}$ be the preordered sets generated respectively by the graphs

$$Y_1' \xleftarrow{f_1'} X' \xrightarrow{f_2'} Y_2' \qquad Y_1'' \xleftarrow{f_1''} X'' \xrightarrow{f_2''} Y_2'' \qquad \text{and} \qquad Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2 \ .$$

Let $F\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_1'' to Y_1, and both Y_2' and Y_2'' to Y_2. Let $G\colon \mathbf{A} \to \mathbf{B}$ be the unique functor that sends both X' and X'' to X, both Y_1' and Y_2'' to Y_1, and both Y_2' and Y_1'' to Y_2. Both F and G are discrete opfibrations. Their pushout in $\mathcal{C}at$ is $\mathbf{2}$; the pushout injections $\overline{F}, \overline{G}\colon \mathbf{B} \to \mathbf{2}$ are both the unique functor that sends X to 0, and both Y_1 and Y_2 to 1.

There are two different lens structures on this functor; one lifts the unique morphism u of $\mathbf{2}$ to f_1, the other lifts it to f_2. This gives four different combinations of lens structures on $\overline{F}$ and $\overline{G}$. Assume, for a contradiction, that one of these combinations satisfies $\overline{F}G = \overline{G}F$ in $\mathcal{L}ens$. As $G^{X'}\overline{F}^X u = F^{X'}\overline{G}^X u$, we must have $\overline{F}^X u = \overline{G}^X u$. If $\overline{F}^X u = f_1$, then $G^{X''}\overline{F}^X u = G^{X''} f_1 = f_2'$ and $F^{X''}\overline{G}^X u = F^{X''} f_1 = f_1' \neq f_2'$, which is a contradiction. If $\overline{F}^X u = f_2$, we obtain a similar contradiction.

Next is an example of a lens and a cosieve where the pushout of the get functor of the lens along the cosieve doesn't have a lens structure (incidentally this lens and cosieve don't have a pushout in $\mathcal{L}ens$).

Example 5.6. Let $\mathbf{B}$ and $\mathbf{D}$ be the preordered sets generated respectively by the graphs

$$\begin{array}{ccccc} X & \xleftarrow{\;f\;} & W & \xrightarrow{\;g\;} & Y \\ {\scriptstyle s}\downarrow & & \downarrow & & \downarrow{\scriptstyle t} \\ Z_2 & \longleftarrow & Z_1 & \longrightarrow & Z_3 \end{array} \qquad\text{and}\qquad \begin{array}{ccccc} X' & \xleftarrow{\;f'\;} & W' & \xrightarrow{\;g'\;} & Y' \\ & {\scriptstyle s'}\searrow & \downarrow & \swarrow{\scriptstyle t'} & \\ & & Z' & & \end{array} .$$

Let $\mathbf{A}$ be the out-degree-zero subcategory of $\mathbf{B}$ with the objects Z_1, Z_2 and Z_3, and let $J\colon \mathbf{A} \rightarrowtail \mathbf{B}$ the inclusion lens. As $\mathbf{1}$ is terminal in $\mathcal{L}ens$ [4], there is a unique lens $F\colon \mathbf{A} \to \mathbf{1}$. By Proposition 5.1, the pushout of $\mathcal{U}F$ along $\mathcal{U}J$ in $\mathcal{C}at$ is the unique functor $\overline{F}\colon \mathbf{B} \to \mathbf{D}$ that maps W to W', X to X', Y to Y', and all of Z_1, Z_2 and Z_3 to Z'. The functor $\overline{F}$ has no lens structure, otherwise we could derive the contradiction

$$s \circ f = \overline{F}^X s' \circ \overline{F}^W f' = \overline{F}^W (s' \circ f') = \overline{F}^W (t' \circ g') = \overline{F}^Y t' \circ \overline{F}^W g' = t \circ g.$$

Proposition 5.7. *Every monic lens is effective (i.e. is the equaliser of its cokernel pair).*

Proof. Let $M\colon \mathbf{A} \to \mathbf{B}$ be a monic lens, and let $J_1, J_2\colon \mathbf{B} \to \mathbf{Coker}\,M$ be its cokernel pair, which exists by Theorem 5.4. Based on Proposition 5.1, if $B \in |\mathbf{B}|$ is such that $J_1 B = J_2 B$, then $B \in |\mathbf{A}|$; and similarly for morphisms of $\mathbf{B}$. In particular, the image of any lens which forks J_1 and J_2 is contained in $\mathbf{A}$, and thus its corestriction to $\mathbf{A}$ is the unique comparison lens. □

Corollary 5.8. *In $\mathcal{L}ens$, the classes of all monos, effective monos, regular monos, strong monos and extremal monos coincide.*

Corollary 5.9. *Every lens that is both epic and monic is an isomorphism.*

6 Regular epic lenses

In this section, we show that all epis in $\mathcal{L}ens$ are regular. This gives us another class of coequalisers in $\mathcal{L}ens$, namely, the epic lenses. For contrast, recall that not all epis in $\mathcal{C}at$ are regular.

Example 6.1. In Example 3.11, we saw that the functor $J\colon \mathbf{2} \to \mathbf{I}$ is epic. It is, however, not a regular epi. Indeed, if J coforks $F_1, F_2\colon \mathbf{A} \to \mathbf{2}$, then $F_1 = F_2$ as J is monic, and so $\mathrm{id}_{\mathbf{2}}$ is the coequaliser of F_1 and F_2, but $\mathbf{2}$ and $\mathbf{I}$ aren't isomorphic.

Proposition 6.2. *The get functor of every epic lens is an effective epi in $\mathcal{C}at$.*

A functor $E\colon \mathbf{B} \to \mathbf{C}$ is *surjective on composable pairs* if for each composable pair (c, c') of $\mathbf{C}$, there is a composable pair (b, b') of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$; such functors are necessarily also surjective on objects and morphisms. If $E\colon \mathbf{B} \to \mathbf{C}$ is an epic lens, then $\mathcal{U}E$ is surjective on composable pairs; indeed, if (c, c') is a composable pair of $\mathbf{C}$, then there is a $B \in |\mathbf{B}|$ such that $EB = \operatorname{src} c$, and $(E^B c, E^{\operatorname{tgt} E^B c} c')$ is a composable pair above (c, c'). Hence it suffices to prove the following lemma.

Lemma 6.3. *All functors that are surjective on composable pairs are effective epis in* $\mathcal{C}at$.

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be a functor that is surjective on objects and composable pairs, and let its kernel pair be $F_1, F_2\colon\ \mathbf{Ker}\,E \to \mathbf{B}$. We must show that E coequalises F_1 and F_2. Let $G\colon \mathbf{B} \to \mathbf{D}$ cofork F_1 and F_2.

Suppose that there is a functor $H\colon \mathbf{C} \to \mathbf{D}$ such that $G = H \circ E$. As E is surjective on objects, for all $C \in |\mathbf{C}|$ there is a $B \in |\mathbf{B}|$ such that $EB = C$, and so $HC = HEB = GB$; this equation determines H on objects. As E is surjective on morphisms, a similar equation determines H on morphisms.

To define $H\colon \mathbf{C} \to \mathbf{D}$ with these equations, the values of GB and Gb should be independent of the choice of B above C and b above c. For all $C \in |\mathbf{C}|$ and all $B, B' \in |\mathbf{B}|$ such that $EB = EB' = C$, we have $GB = GF_1\langle B, B'\rangle = GF_2\langle B, B'\rangle = GB'$, where $\langle B, B'\rangle \in |\mathbf{Ker}\,E|$ comes from the pullback property; hence the object map of H is well defined. Its morphism map is similarly also well defined.

Define H with the above equations. By construction, $G = H \circ E$. We must show that H is a functor. For all $C \in |\mathbf{C}|$, there is a $B \in |\mathbf{B}|$ such that $EB = C$, and $H\,\mathrm{id}_C = G\,\mathrm{id}_B = \mathrm{id}_{GB} = \mathrm{id}_{HC}$; thus H preserves identities. For all composable pairs c and c' of $\mathbf{C}$, there is a composable pair b and b' of $\mathbf{B}$ such that $Eb = c$ and $Eb' = c'$, and $H(c' \circ c) = G(b' \circ b) = Gb' \circ Gb = Hc' \circ Hc$; thus H preserves composites. □

Corollary 6.4. *Every epic lens coequalises its fake kernel pair, and so is regular.*

Proof. Let $E\colon \mathbf{B} \to \mathbf{C}$ be an epic lens. Let $F_1, F_2\colon\ \mathbf{Ker}\,\mathcal{U}E \to \mathbf{B}$ be the fake kernel pair of E in $\mathcal{L}ens$. By Proposition 6.2, $\mathcal{U}E$ coequalises $\mathcal{U}F_1$ and $\mathcal{U}F_2$ in $\mathcal{C}at$. Let $G\colon \mathbf{B} \to \mathbf{D}$ be a lens that coforks F_1 and F_2, let $B \in |\mathbf{B}|$, let $d \in \mathbf{D}(GB, *)$, and let $C = EB$. Then $(G \circ F_1)^{\langle B,B\rangle} d = F_1{}^{\langle B,B\rangle} G^B d = \big(G^B d, E^B E G^B d\big)$, and similarly $(G \circ F_2)^{\langle B,B\rangle} d = \big(E^B E G^B d, G^B d\big)$. As G coforks F_1 and F_2, we have $G^B d = E^B E G^B d$. By Theorem 4.5, E coequalises F_1 and F_2 in $\mathcal{L}ens$. □

Corollary 6.5. *In $\mathcal{L}ens$, the classes of all epis, regular epis, strong epis and extremal epis coincide.*

Corollary 6.6. *In $\mathcal{L}ens$, the class of all morphisms that are left orthogonal to the class of all monos is the class of all epis.*

Proof. As $\mathcal{L}ens$ has equalisers [4], every morphism that is left orthogonal to the class of all monos is an epi. Conversely, we have already shown that every epi is a strong epi. □

Remark 6.7. As every lens factors as an epi followed by a mono (Remark 3.8), it follows that the class of all epis and the class of all monos is an orthogonal factorisation system on $\mathcal{L}ens$.

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References

[1] Danel Ahman & Tarmo Uustalu (2017): *Taking Updates Seriously*. In Eramo & Johnson [8], pp. 59–73. Available at `http://ceur-ws.org/Vol-1827/`.

[2] François Bancilhon & Nicolas Spyratos (1981): *Update Semantics of Relational Views*. ACM Transactions on Database Systems 6(4), pp. 557–575, doi:10.1145/319628.319634.

[3] Marek A. Bednarczyk, Andrzej M. Borzyszkowski & Wieslaw Pawlowski (1999): *Generalized congruences — Epimorphisms in Cat*. Theory and Applications of Categories 5(11), pp. 266–280. Available at `http://www.tac.mta.ca/tac/volumes/1999/n11/5-11abs.html`.

[4] Emma Chollet, Bryce Clarke, Michael Johnson, Maurine Songa, Vincent Wang & Gioele Zardini (2021): *Limits and colimits in a category of lenses*. Submitted to Applied Category Theory 2021.

[5] Bryce Clarke (2020): *Internal lenses as functors and cofunctors*. In John Baez & Bob Coecke, editors: Proceedings Applied Category Theory 2019, University of Oxford, UK, 15–19 July 2019, Electronic Proceedings in Theoretical Computer Science 323, pp. 183–195, doi:10.4204/EPTCS.323.13.

[6] Bryce Clarke (2021): *A diagrammatic approach to symmetric lenses*. In Spivak & Vicary [17], pp. 79–91, doi:10.4204/EPTCS.333.6.

[7] Zinovy Diskin, Yingfei Xiong & Krzysztof Czarnecki (2011): *From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case*. Journal of Object Technology 10, pp. 6:1–25, doi:10.5381/jot.2011.10.1.a6.

[8] Romina Eramo & Michael Johnson, editors (2017): *Proceedings of the 6th International Workshop on Bidirectional Transformations*. Available at `http://ceur-ws.org/Vol-1827/`.

[9] J. Nathan Foster, Michael B. Greenwald, Jonathan T. Moore, Benjamin C. Pierce & Alan Schmitt (2007): *Combinators for Bidirectional Tree Transformations: A Linguistic Approach to the View-Update Problem*. ACM Transactions on Programming Languages and Systems 29(3), doi:10.1145/1232420.1232424.

[10] Rudolf Fritsch & Dana May Latch (1981): *Homotopy Inverses for Nerve*. Mathematische Zeitschrift 177(2), pp. 147–179, doi:10.1007/BF01214196.

[11] Jeremy Gibbons & Perdita Stevens, editors (2018): *Bidirectional Transformations*. Lecture Notes in Computer Science 9715, Springer International Publishing, doi:10.1007/978-3-319-79108-1.

[12] Michael Johnson & Robert Rosebrugh (2015): *Spans of Delta Lenses*. In Alcino Cunha & Ekkart Kindler, editors: Proceedings of the 4th International Workshop on Bidirectional Transformations, pp. 1–15. Available at `http://ceur-ws.org/Vol-1396/`.

[13] Michael Johnson & Robert Rosebrugh (2016): *Unifying Set-Based, Delta-Based and Edit-Based Lenses*. In Anthony Anjorin & Jeremy Gibbons, editors: Proceedings of the 5th International Workshop on Bidirectional Transformations, pp. 1–13. Available at `http://ceur-ws.org/Vol-1571/`.

[14] Michael Johnson & Robert Rosebrugh (2017): *Universal Updates for Symmetric Lenses*. In Eramo & Johnson [8], pp. 39–53. Available at `http://ceur-ws.org/Vol-1827/`.

[15] Michael Johnson & Robert Rosebrugh (2021): *The more legs the merrier: A new composition for symmetric (multi-)lenses*. In Spivak & Vicary [17], pp. 92–107, doi:10.4204/EPTCS.333.7.

[16] Frank J. Oles (1982): *A Category-Theoretic Approach to the Semantics of Programming Languages*. Ph.D. thesis, Syracuse University. Available at https://www.cs.cmu.edu/afs/cs.cmu.edu/project/fox-19/member/jcr/www/FrankOlesThesis.pdf.

[17] David I. Spivak & Jamie Vicary, editors (2021): Proceedings of the 3rd Annual International *Applied Category Theory Conference 2020,* Cambridge, USA, 6–10th July 2020. *Electronic Proceedings in Theoretical Computer Science* 333, Open Publishing Association, doi:10.4204/EPTCS.333.

[18] Ross Street (2021): *Span composition using fake pullbacks*. *Theory and Applications of Categories* 36(4), pp. 102–117. Available at http://www.tac.mta.ca/tac/volumes/36/4/36-04abs.html.