Commutative Monads for Probabilistic Programming Languages (Extended Abstract)

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Abstract

A long-standing open problem in the semantics of programming languages supporting probabilistic choice is to find a commutative monad for probability on the category DCPO. In this paper we present three such monads and a general construction for finding even more. We show how to use these monads to provide a sound and adequate denotational semantics for the Probabilistic FixPoint Calculus (PFPC) – a call-by-value simply-typed lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice. We also show that in the special case of continuous dcpo’s, all three monads coincide with the valuations monad of Jones, and we fully characterise the induced Eilenberg-Moore categories by showing that they are all isomorphic to the category of continuous Keigelspitzen of Keimel and Plotkin.

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I. INTRODUCTION

Probabilistic methods now are a staple of computation. The initial discovery of randomized algorithms [2] was quickly followed by the definition of Probabilistic Turing machines and related complexity classes [3]. There followed advances in a number of areas, including, e.g., process calculi, probabilistic model checking and verification [4]–[9], right through to the recent development of statistical probabilistic programming languages (cf. [7]–[9]), not to mention the crucial role probability plays in quantum programming languages [10], [11].

Domain theory, a staple of denotational semantics, has struggled to keep up with these advances. Domain theory encompasses two broad classes of objects: directed complete partial orders (dcpo’s), based on an order-theoretic view of computation, and the smaller class of (continuous) domains, those dcpo’s that also come equipped with a notion of approximation. However, adding probabilistic choice to the domain-theoretic approach has been a challenge. The canonical model of (sub)probability measures in domain theory is the family of valuations – certain maps from the lattice of open subsets of a dcpo to the unit interval. It is well-known that the valuations form a monad \( \mathcal{V} \) on DCPO (the category of dcpo’s and Scott-continuous functions) and on DOM (the full subcategory of DCPO consisting of domains) [12], [13].

In fact, the monad \( \mathcal{V} \) on DOM is commutative [13], which is important for two reasons: (1) its commutativity is equivalent to the Fubini Theorem [13], a cornerstone of integration theory and (2) computationally, commutativity of a monad together with adequacy can be used to establish contextual equivalences for effectful programs. However, in order to do so, one typically needs a Cartesian closed category for the semantic model, and DOM is not closed; in fact, despite repeated attempts, it remains unknown whether there is any Cartesian closed category of domains on which \( \mathcal{V} \) is an endofunctor; this is the well-known Jung-Tix Problem [14]. On the other hand, it also is unknown if the monad \( \mathcal{V} \) is commutative on the larger Cartesian closed category DCPO. In this paper, we offer a solution to this conundrum.

A. Our Contributions

We use topological methods to construct a commutative valuations monad \( \mathcal{M} \) on DCPO, as follows: it is straightforward to show the family \( SD \) of simple valuations on a dcpo \( D \) can be equipped with the structure of a commutative monad, but \( SD \) is not a dcpo, in general. So, we complete \( SD \) by taking the smallest subdcpo \( MD \subset SD \) that contains \( SD \). This defines the object-mapping of a monad \( \mathcal{M} \) on DCPO. The unit, multiplication and strength of the monad \( \mathcal{M} \) at \( D \) are given by the restrictions of the same operations of \( \mathcal{M} \) to \( MD \). Topological arguments then imply that \( \mathcal{M} \) is a commutative valuations monad on DCPO.

In fact, there are several completions of \( SD \) that give rise to commutative valuations monads on DCPO. These completions are determined by so-called \( \kappa \)-categories, introduced by Keimel and Lawson [15]. This observation allows us to define two additional commutative valuations monads, \( \mathcal{W} \) and \( \mathcal{P} \), on DCPO simply by specifying their corresponding \( \kappa \)-categories. Finally, while we have identified three such \( \kappa \)-categories, there likely are more that meet our requirements, each of which would define yet another commutative monad of valuations on DCPO containing \( S \).

With this background, we now summarise our main results.
Commulative monads: A \( \kappa \)-category is a full subcategory of the category \( T_\kappa \) of \( T_0 \)-spaces satisfying properties that imply it determines a completion of each \( T_0 \)-space among the objects of the \( \kappa \)-category. For example, each \( \kappa \)-category defines a completion of a poset endowed with its Scott topology, among the objects in the \( \kappa \)-category. In particular, each \( \kappa \)-category determines a completion of the family \( S \) when considered as a subset of \( VD \), for each dcpo \( D \).

By specifying an additional constraint on \( \kappa \)-categories, we can show the corresponding completions of \( S \) define cummative monads on DCPO. We identify three commulative monads concretely: \( \mathcal{M} \), \( \mathcal{W} \) and \( \mathcal{P} \), corresponding to the \( \kappa \)-categories of d-spaces, that of well-filtered spaces and that of sober spaces, respectively. As part of our construction, we also prove the most general Ufbin Theorem for dcpo’s yet available.

Eilenberg-Moore Algebras: All three of \( \mathcal{M} \), \( \mathcal{W} \) and \( \mathcal{P} \) restrict to monads on \( \text{DOM} \), where they coincide with \( \mathcal{V} \). We characterize their Eilenberg-Moore categories over \( \text{DOM} \) by showing they are isomorphic to the category of continuous Kegelspitzen and Scott-continuous linear maps [16]; this corrects an error in [13].

On the larger category \( \text{DCPO} \), we show the Eilenberg-Moore algebras of our monads \( \mathcal{M} \), \( \mathcal{W} \) and \( \mathcal{P} \) are Kegelspitzen. It is unknown if every Kegelspitze is an \( \mathcal{M} \)-algebra.

Semantics: We consider the Probabilistic FixPoint Calculus (PFPC) – a call-by-value simply-typed lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice. We show that each of the Kleisli categories of our three commulative monads is a sound and computationally adequate model of PFPC. Moreover, we show that adequacy holds in a strong sense, i.e., the interpretation of each term is a (potentially infinite) convex sum of the values it reduces to.

II. Commutative Monads for Probability

A. Domain-theoretic and Topological Preliminaries

A nonempty subset \( A \) of a partially ordered set (poset) \( D \) is directed if each pair of elements in \( A \) has an upper bound in \( A \). A directed-complete partial order, (dcpo, for short) is a poset in which every directed subset \( A \) has a supremum \( \text{sup} A \). For example, the unit interval \([0, 1]\) is a dcpo in the usual ordering. A function \( f : D \rightarrow E \) between two (posets) dcpo’s is Scott-continuous if it is monotone and preserves (existing) suprema of directed subsets. The category \( \text{DCPO} \) of dcpo’s and Scott-continuous functions is complete, cocomplete and cartesian closed [17]. A domain is a dcpo which comes equipped with some suitable additional structure for approximation (details omitted here). Domains and Scott-continuous maps form an important subcategory \( \text{DOM} \).

The Scott topology \( \sigma D \) on a dcpo \( D \) consists of the upper subsets \( U = \uparrow U = \{ x \in D \mid (\exists u \in U) u \leq x \} \) that are inaccessible by directed suprema: i.e., if \( A \subseteq D \) is directed and \( \text{sup} A \in U \), then \( A \cap U \neq \emptyset \). The space \((D, \sigma D)\) is also written as \( \Sigma D \). Scott-continuous functions between \( \text{DCPO} \)’s \( D \) and \( E \) are exactly the continuous functions between \( \Sigma D \) and \( \Sigma E \) [18, Proposition II.2.1].

A subset \( B \) of a dcpo \( D \) is a sub-dcpo if every directed subset \( A \subseteq B \) satisfies \( \text{sup} D \in B \). In this case, \( B \) is a dcpo in the induced order from \( D \). The \( d \)-topology on \( D \) is the topology whose closed subsets consist of sub-dcpo’s of \( D \). Open (closed) sets in the \( d \)-topology will be called \( d \)-open (\( d \)-closed). The \( d \)-closure of \( C \subseteq D \) is the topological closure of \( C \) with respect to the \( d \)-topology on \( D \), which is the intersection of all sub-dcpo’s of \( D \) containing \( C \).

The family of open sets of a topological space \( X \), denoted \( \mathcal{O} X \), is a complete lattice in the inclusion order. The specialization order \( \leq_s \) on \( X \) is defined as \( x \leq_s y \) if and only if \( x \) is in the closure of \( \{y\} \), for \( x, y \in X \). We write \( \Omega X \) to denote \( X \) equipped with the specialization order. It is well-known that \( X \) is \( T_0 \) if and only if \( \Omega X \) is a poset. A subset of \( X \) is called saturated if it is an upper set in \( \Omega X \). A space \( X \) is called a d-space or a monotone-convergence space if \( \Omega X \) is a dcpo and each open set of \( X \) is Scott open in \( \Omega X \). As an example, \( \Sigma D \) is always a d-space for each dcpo \( D \). The full subcategory of \( T_0 \) consisting of d-spaces is denoted by \( \mathcal{D} \). There is a functor \( \Sigma : \text{DCPO} \rightarrow \mathcal{D} \) that assigns the space \( \Sigma D \) to each dcpo \( D \), and the map \( f : \Sigma D \rightarrow \Sigma E \) to the Scott-continuous map \( f : D \rightarrow E \). Dually, the functor \( \Omega : \mathcal{D} \rightarrow \text{DCPO} \) assigns \( \Omega X \) to each d-space \( X \) and the map \( f : \Omega X \rightarrow \Omega Y \) to each continuous map \( f : X \rightarrow Y \). In fact, \( \Sigma \downarrow \Omega \), i.e., \( \Sigma \) is left adjoint to \( \Omega \) [19].

A \( T_0 \) space \( X \) is called sober if every nonempty closed irreducible subset of \( X \) is the closure of some (unique) singleton set, where \( A \subseteq X \) is irreducible if \( A \subseteq B \cup C \) with \( B \) and \( C \) nonempty closed subsets implies \( A \subseteq B \) or \( A \subseteq C \). The category of sober spaces and continuous functions is denoted by \( \text{SOB} \). Sober spaces are d-spaces, hence \( \text{SOB} \subseteq \mathcal{D} \) [15].

B. A Commutative Monad for Probability

To begin, a subprobability valuation on a topological space \( X \) is a Scott-continuous function \( \nu : \mathcal{O}X \rightarrow [0, 1] \) that is strict \( (\nu(\emptyset) = 0) \), and modular \( (\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)) \). The set of subprobability valuations on \( X \) is denoted \( \mathcal{V}X \). The stochastic order on \( \mathcal{V}X \) is defined pointwise: \( \nu_1 \leq \nu_2 \) if and only if \( \nu_1(U) \leq \nu_2(U) \) for all \( U \in \mathcal{O}X \). \( \forall X \) is a pointed dcpo in the stochastic order, with least element given by the constantly zero valuation \( 0_X \) and where the supremum of a directed family \( \{\nu_i\}_{i \in I} \) is \( \text{sup}_{i \in I} \nu_i = \bigcup_{i \in I} \nu_i(U) \).

The canonical examples of subprobability valuations are the Dirac valuations \( \delta_x \) for \( x \in X \), defined by \( \delta_x(U) = 1 \) if \( x \in U \) and \( \delta_x(U) = 0 \) otherwise. \( \forall X \) enjoys a convex structure: if \( \nu_i \in \mathcal{V}X \) and \( r_i \geq 0 \), with \( \sum_{i=1}^n r_i = 1 \), then the convex sum

\( \nu = \sum_{i=1}^n r_i \nu_i \)
Theorem 4. Any MDV also is in VX. The simple valuations on D are those of the form \( \sum_{i=1}^{n} r_i \delta_{x_i} \), where \( x_i \in X \), \( r_i > 0, i = 1, \ldots, n \) and \( \sum_{i=1}^{n} r_i \leq 1 \). The set of simple valuations on X is denoted by SX. Clearly, SX \( \subseteq VX \). Unlike VX, SX is not directed-complete in the stochastic order in general.

Given \( \nu \in VX \) and \( f : X \to [0,1] \) continuous, we can define the integral of \( f \) against \( \nu \) by the Choquet formula

\[
\int_{x \in X} f(x) d\nu \overset{\text{def}}{=} \int_0^1 \nu(f^{-1}((t,1])) dt,
\]

where the right side is a Riemann integral of the bounded antitone function \( \lambda t. \nu(f^{-1}((t,1])) \). If no confusion occurs, we simply write \( \int_{x \in X} f(x) d\nu \) as \( \int f d\nu \). Basic properties of this integral can be found in [13].

For a dcpo \( D, VD \) is defined as \( V(D, \sigma D) \). Using Manes’ description of monads (Kleisli triples) [20], Jones proved in her PhD thesis [13] that \( V \) is a monad on DCPO:

- The unit of \( V \) at \( D \) is \( \eta_D^V : D \to VD \) \( x \mapsto \delta_x \);
- The Kleisli extension \( f^\uparrow \) of a Scott-continuous map \( f : D \to VE \) maps \( \nu \in VD \) to \( f^\uparrow(\nu) \in VE \) by

\[
f^\uparrow(\nu) \overset{\text{def}}{=} \lambda U \in \sigma E. \int_{x \in D} f(x)(U)\,d\nu.
\]

Then the multiplication \( \mu^\uparrow_D : \forall VD \to VD \) is given by \( \text{id}_{\forall VD} \); it maps \( \varpi \in \forall VD \) to \( \lambda U \in \sigma D. \int_{\nu \in VD} \nu(U)\,d\varpi \in VD \). Thus, \( V \) defines an endofunctor on DCPO that sends a dcpo \( D \) to \( VD \), and a Scott-continuous map \( h : D \to E \) to \( V(h) \overset{\text{def}}{=} (\eta_E \circ h)^\uparrow \); concretely, \( V(h) \) maps \( \nu \in VD \) to \( \lambda U \in \sigma E. \nu(h^{-1}(U)) \). Jones [13] also showed that \( V \) is a strong monad over DCPO: its strength at \( (D,E) \) is given by

\[
\tau^D_{VE} : D \times VE \to V(D \times E) \overset{(x, \nu) \mapsto \lambda U. \int_{y \in E} \chi_U(x,y)\,d\nu},
\]

where \( \chi_U \) is the characteristic function of \( U \in \sigma(D \times E) \). Whether \( V \) is a commutative monad on DCPO has remained an open problem for decades. Proving this to be true requires showing the following Fubini-type equation holds:

\[
\int_{x \in D} \int_{y \in E} \chi_U(x,y)\,d\nu = \int_{y \in E} \int_{x \in D} \chi_U(x,y)\,d\nu\,d\xi,
\]

for dcpo’s \( D \) and \( E \), for \( U \in \sigma(D \times E) \) and for \( \nu \in VD, \xi \in VE \) [12] Section 6]. The difficulty lies in the well-known fact that a Scott open set \( U \in \sigma(D \times E) \) might not be open in the product topology \( \sigma D \times \sigma E \) in general [18 Exercise II-4.26].

We obtain a commutative monad of valuations on DCPO by restricting to a suitable completion of \( SD \) inside \( VD \). There are several possibilities (cf. [21]), and we choose the smallest and simplest – the d-closure of \( SD \) in \( VD \).

**Definition 1.** For each dcpo \( D \), we define \( MD \) to be the intersection of all sub-dcpo’s of \( VD \) that contain \( SD \).

**Theorem 2.** \( M \) is a commutative monad on DCPO. The monad operations are (co)restrictions of those of \( V \).

**Remark 3.** We note that \( MD \) is the first example of a commutative valuations monad on DCPO that contains the simple valuations. And, since every valuation on a domain \( D \) is a directed supremum of simple valuations [13 Theorem 5.2], it follows that \( M = V \) on the category DOM.

In fact, the construction of the monad \( M \) is a special case of a more general construction based on \( \kappa \)-categories.

**Theorem 4.** Any \( \kappa \)-category \( K \) with \( K \subseteq D \) determines a commutative valuations monad \( V_K \) on DCPO. In particular, \( M = VD \). The categories of sober spaces \( SOB \) and that of well-filtered spaces \( WF \) are both \( \kappa \)-categories and subcategories of \( D \) and so they determine two additional commutative monads on DCPO, denoted \( P \) and \( W \), respectively. These monads satisfy the following relationship \( SD \subseteq MD \subseteq WD \subseteq PD \subseteq VD \) for each dcpo \( D \).

**Remark 5.** All subsequent results stated here hold for all three monads \( M, W \) and \( P \).

Kegelspitzen [16] are dcpo’s equipped with a convex structure. A Kegelspitz which is also a domain is called *continuous*. We show every continuous Kegelspitz \( K \) has a linear barycenter map \( \beta : MK \to K \) making \( (K, \beta) \) an \( M \)-algebra and conversely, every \( M \)-algebra \( (K, \beta) \) on DCPO admits a Kegelspitz structure on \( K \) making \( \beta : MK \to K \) a linear map.

**Theorem 6.** The Eilenberg-Moore category \( DOM^M \) of \( M \) over DOM is isomorphic to the category of continuous Kegelspitzen and Scott-continuous linear maps.

Since \( DOM^M = DOM^V \), the above theorem also characterises the algebras of \( V \) on domains. This corrects an error in [13], where it is claimed that *continuous abstract probabilistic domains* and linear maps are isomorphic to \( DOM^V \). A separating example is the extended non-negative reals \( [0, \infty] \), which is a continuous Kegelspitz but not an abstract probabilistic domain.

Our final contribution is to show that our monads can be used to study the semantics of probabilistic programming languages.

**Theorem 7.** The Kleisli category \( DCPO_M \) of \( M \) is a sound and (strongly) adequate denotational model of PFPC.
REFERENCES


