# Lecture 11

# Theory of Combinators

- **Combinators are an alternative theory of functions to the $\lambda$-calculus**

- **Originally introduced by logicians as a way of studying the process of substitution**

- **More recently, Turner has argued that combinators provide a good 'machine code' into which functional programs can be compiled**

- **Several experimental computers have been built based on Turner's ideas**

- **Combinators also provide a good intermediate code for conventional machines**
  - **several of the best compilers for functional languages are based on them**

## Formulations of theory of combinators

- **Two equivalent ways of formulating the theory of combinators:**

  **(i) within the $\lambda$-calculus, or**

  **(ii) as a completely separate theory.**

  - **approach (i) taken here**
  - **approach (ii) was the original one**

- **It will be shown that *any* $\lambda$-expression is equal to an expression built from variables and two particular expressions, `K` and `S`, using only function application**

- **This is done by mimicking $\lambda$-abstractions using combinations of `K` and `S`**

- **$\beta$-reductions can be simulated by simpler operations involving `K` and `S`**

  - **it is these simpler operations that combinator machines implement directly in hardware**

# $\mathtt{S}$ and $\mathtt{K}$

- **The definitions of $\mathtt{K}$ and $\mathtt{S}$ are:**

$$\mathtt{LET\ K} = \lambda x\ y.\ x$$

$$\mathtt{LET\ S} = \lambda f\ g\ x.\ (f\ x)\ (g\ x)$$

- **By $\beta$-reduction, for all $E_1$, $E_2$ and $E_3$:**

$$\mathtt{K}\ E_1\ E_2 = E_1$$

$$\mathtt{S}\ E_1\ E_2\ E_3 = (E_1\ E_3)\ (E_2\ E_3)$$

# Combinators

- **Any expression built by application (i.e. combination) from `K` and `S` is called a *combinator***
  - **`K` and `S` are the *primitive combinators***

- **Combinators have the following syntax:**

  *<combinator>* ::= `K` | `S` | (*<combinator>* *<combinator>*)

- **A *combinatory expression* is an expression built from `K`, `S` and zero or more variables**
  - **a combinator is a combinatory expression not containing variables**

- **Syntax of combinatory expressions:**

  *<combinatory expression>*
  ::= `K` | `S`
  | *<variable>*
  | (*<combinatory expression>* *<combinatory expression>*)

# The identity combinator $\texttt{I}$

- **The identity function $\texttt{I}$ is often taken as a primitive combinator, but this is not necessary as it can be defined from $\texttt{K}$ and $\texttt{S}$**

- **Define $\texttt{I}$ by:**

$$\texttt{LET I} = \lambda x.\ x$$

- **Then $\texttt{I} = \texttt{S K K}$**
  - **exercise!**

# Combinator reduction

- **If $E$ and $E'$ are combinatory expressions then $E \underset{c}{\longrightarrow} E'$ means:**
  - $E \equiv E'$
  - **or $E'$ can be got from $E$ by a sequence of rewritings of the form:**

    **(i)** $\mathtt{K}\ E_1\ E_2 \underset{c}{\longrightarrow} E_1$

    **(ii)** $\mathtt{S}\ E_1\ E_2\ E_3 \underset{c}{\longrightarrow} (E_1\ E_3)\ (E_2\ E_3)$

    **(iii)** $\mathtt{I}\ E \underset{c}{\longrightarrow} E$

- **Example: for any $E$**

$$\begin{array}{rll} \mathtt{S}\ \mathtt{K}\ \mathtt{K}\ E & \underset{c}{\longrightarrow} \mathtt{K}\ E\ (\mathtt{K}\ E) & \textbf{by (ii)} \\ & \underset{c}{\longrightarrow} E & \textbf{by (i)} \end{array}$$

  - **thus (iii) is derivable from (i) and (ii)**

- **Any sequence of combinatory reductions can be expanded into a sequence of $\beta$-conversions**
  - $\mathtt{K}\ E_1\ E_2 \longrightarrow E_1$
  - $\mathtt{S}\ E_1\ E_2\ E_3 \longrightarrow (E_1\ E_3)\ (E_2\ E_3)$

# Functional completeness

- **Every $\lambda$-expression is equal to some combinatory expression**
  - called the functional completeness of combinators
  - basis for compilers for functional languages to the machine code of combinator machines

- **Key idea:**
  - for variable $V$ and combinatory expression $E$ a combinatory expression $\lambda^* V.\ E$ will be defined
  - $\lambda^* V.\ E$ uses `K` and `S` to simulate adding '$\lambda V$' to an expression
  - $\lambda^* V.\ E\ =\ \lambda V.\ E$

# Bracket abstraction $\lambda^* V.\ E$

- **If $V$ a variable and $E$ a combinatory expression, then $\lambda^* V.\ E$ is defined inductively on the structure of $E$ as follows:**

  **(i)** $\lambda^* V.\ V = \mathtt{I}$

  **(ii)** $\lambda^* V.\ V' = \mathtt{K}\ V'$ **(if $V \neq V'$)**

  **(iii)** $\lambda^* V.\ C = \mathtt{K}\ C$ **(if $C$ is a combinator)**

  **(iv)** $\lambda^* V.\ (E_1\ E_2) = \mathtt{S}\ (\lambda^* V.\ E_1)\ (\lambda^* V.\ E_2)$

- **Note that $\lambda^* V.\ E$ is a combinatory expression not containing $V$**

- **Example: if $f$ and $x$ are variables and $f \neq x$, then:**

$$\begin{aligned} \lambda^* x.\ \ f\ x &= \mathtt{S}\ (\lambda^* x.\ f)\ (\lambda^* x.\ x) \\ &= \mathtt{S}\ (\mathtt{K}\ f)\ \mathtt{I} \end{aligned}$$

# Proof of functional completeness

- **THEOREM:**
  - $(\lambda^* V.\ E) = \lambda V.\ E$
- **PROOF:**
  - **show** $(\lambda^* V.\ E)\ V = E$
  - **follows immediately that** $\lambda V.\ (\lambda^* V.\ E)\ V = \lambda V.E$
  - **and hence by** $\eta$**-reduction that** $\lambda^* V.\ E = \lambda V.\ E$

## Proof that $(\lambda^* V.\ E)\ V\ =\ E$

- **Mathematical induction on the 'size' of $E$:**

  **(i) if $E = V$ then:**

  $$(\lambda^* V.\ E)\ V\ =\ \mathtt{I}\ V\ =\ (\lambda x.\ x)\ V\ =\ V\ =\ E$$

  **(ii) if $E = V'$ where $V' \neq V$ then:**

  $$(\lambda^* V.\ E)\ V\ =\ \mathtt{K}\ V'\ V\ =\ (\lambda x\ y.\ x)\ V'\ V\ =\ V'\ =\ E$$

  **(iii) if $E = C$ where $C$ is a combinator, then:**

  $$(\lambda^* V.\ E)\ V\ =\ \mathtt{K}\ C\ =\ (\lambda x\ y.\ x)\ C\ V\ =\ C\ =\ E$$

  **(iv) if $E = (E_1\ E_2)$ then we can assume by induction that:**

  $$(\lambda^* V.\ E_1)\ V = E_1$$
  $$(\lambda^* V.\ E_2)\ V = E_2$$

  **and hence**

  $$\begin{aligned}(\lambda^* V.\ E)\ V &= (\lambda^* V.\ (E_1\ E_2))\ V \\ &= (\mathtt{S}\ (\lambda^* V.\ E_1)\ (\lambda^* V.\ E_2))\ V \\ &= (\lambda f\ g\ x.\ f\ x\ (g\ x))\ (\lambda^* V.\ E_1)\ (\lambda^* V.\ E_2)\ V \\ &= (\lambda^* V.\ E_1)\ V\ ((\lambda^* V.\ E_2)\ V) \\ &= E_1\ E_2 \quad \textbf{(by induction assumption)} \\ &= E\end{aligned}$$

# Translation to combinators

- **The notation**

$$\lambda^* V_1 \ V_2 \ \cdots \ V_n .\ E$$

  **is used to mean**

$$\lambda^* V_1 .\ \lambda^* V_2 . \quad \cdots \ \lambda^* V_n .\ E$$

- **Define the translation of $\lambda$-expression $E$ to a combinatory expression $(E)_{\mathtt{C}}$:**

  **(i)** $(V)_{\mathtt{C}} = V$

  **(ii)** $(E_1 \ E_2)_{\mathtt{C}} = (E_1)_{\mathtt{C}} \ (E_2)_{\mathtt{C}}$

  **(iii)** $(\lambda V.\ E)_{\mathtt{C}} = \lambda^* V.\ (E)_{\mathtt{C}}$

$$E = (E)_{\texttt{C}}$$

- **THEOREM:**
  - **for every $\lambda$-expression $E$ we have: $E = (E)_{\texttt{C}}$**

- **PROOF: induction on the size of $E$**

  **(i) If $E = V$ then $(E)_{\texttt{C}} = (V)_{\texttt{C}} = V$**

  **(ii) If $E = (E_1\ E_2)$ we can assume by induction that**

$$E_1 = (E_1)_{\texttt{C}}$$
$$E_2 = (E_2)_{\texttt{C}}$$

  **hence**

$$(E)_{\texttt{C}} = (E_1\ E_2)_{\texttt{C}} = (E_1)_{\texttt{C}}\ (E_2)_{\texttt{C}} = E_1\ E_2 = E$$

  **(iii) If $E = \lambda V.\ E'$ then we can assume by induction that**

$$(E')_{\texttt{C}} = E'$$

  **hence**

$$\begin{aligned}
(E)_{\texttt{C}} &= (\lambda V.\ E')_{\texttt{C}} \\
&= \lambda^* V.\ (E')_{\texttt{C}} && \textbf{(by translation rules)} \\
&= \lambda^* V.\ E' && \textbf{(by induction assumption)} \\
&= \lambda V.\ E' && \textbf{(by previous theorem)} \\
&= E
\end{aligned}$$

# Consequences of last theorem

- **Every $\lambda$-expression is equal to a $\lambda$-expression built up from $\mathtt{K}$ and $\mathtt{S}$ and variables by application**
  - **the class of $\lambda$-expressions $E$ defined by:**

$$E \;::=\; V \;\mid\; \mathtt{K} \;\mid\; \mathtt{S} \;\mid\; E_1\ E_2$$

  **is equivalent to the full $\lambda$-calculus**

- **A collection of $n$ combinators $C_1, \ldots, C_n$ is called an $n$-element *basis***
  - **if every $\lambda$-expression $E$ is equal to an expression built from $C_i$s and variables by function applications**
  - **theorem above shows $\mathtt{K}$ and $\mathtt{S}$ form a 2-element basis**

- **There exists a 1-element basis!**

  *Exercise*
  **Find a combinator, $\mathtt{X}$ say, such that any $\lambda$-expression is equal to an expression built from $\mathtt{X}$ and variables by application.**
  *Hint*: **Let $\langle E_1, E_2, E_3\rangle = \lambda p.\ p\ E_1\ E_2\ E_3$ and consider $\langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle\ \langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle\ \langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle$ and $\langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle\ \langle\langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle\ \langle \mathtt{K},\mathtt{S},\mathtt{K}\rangle\rangle$**

# Examples

- **Part of Y:**

$$
\begin{aligned}
&\lambda^* f.\ \lambda^* x.\ f\ (x\ x) \\
&\quad = \lambda^* f.\ (\lambda^* x.\ f\ (x\ x)) \\
&\quad = \lambda^* f.\ (\mathtt{S}\ (\lambda^* x.\ f)\ (\lambda^* x.\ x\ x)) \\
&\quad = \lambda^* f.\ (\mathtt{S}\ (\mathtt{K} f)\ (\mathtt{S}(\lambda^* x.\ x)\ (\lambda^* x.\ x))) \\
&\quad = \lambda^* f.\ (\mathtt{S}\ (\mathtt{K} f)\ (\mathtt{S\ I\ I})) \\
&\quad = \mathtt{S}\ (\lambda^* f.\ \mathtt{S}\ (\mathtt{K} f))\ (\lambda^* f.\ \mathtt{S\ I\ I}) \\
&\quad = \mathtt{S}\ (\mathtt{S}\ (\lambda^* f.\ \mathtt{S})\ (\lambda^* f.\ \mathtt{K}\ f))\ (\mathtt{K}\ (\mathtt{S\ I\ I})) \\
&\quad = \mathtt{S}\ (\mathtt{S}\ (\mathtt{K\ S})\ (\mathtt{S}\ (\lambda^* f.\ \mathtt{K})\ (\ \lambda^* f.\ f)))\ (\mathtt{K}\ (\mathtt{S\ I\ I})) \\
&\quad = \mathtt{S}\ (\mathtt{S}\ (\mathtt{K\ S})\ (\mathtt{S}\ (\mathtt{K\ K})\ \mathtt{I}))\ (\mathtt{K}\ (\mathtt{S\ I\ I}))
\end{aligned}
$$

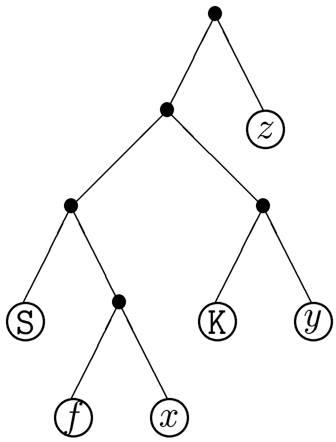
- $\mathtt{Y}$:

$$
\begin{aligned}
(\mathtt{Y})_{\mathtt{C}} &= (\lambda f.\ (\lambda x.\ f(x\ x))\ (\lambda x.\ f(x\ x)))_{\mathtt{C}} \\
&= \lambda^* f.\ ((\lambda x.\ f(x\ x))\ (\lambda x.\ f(x\ x)))_{\mathtt{C}} \\
&= \lambda^* f.\ ((\lambda x.\ f(x\ x))_{\mathtt{C}}\ (\lambda x.\ f(x\ x))_{\mathtt{C}}) \\
&= \lambda^* f.\ (\lambda^* x.\ (f(x\ x))_{\mathtt{C}})\ (\lambda^* x.\ (f(x\ x))_{\mathtt{C}}) \\
&= \lambda^* f.\ (\lambda^* x.\ f(x\ x))\ (\lambda^* x.\ f(x\ x)) \\
&= \mathtt{S}\ (\lambda^* f.\ \lambda^* x.\ f(x\ x))\ (\lambda^* f.\ \lambda^* x.\ f(x\ x)) \\
&= \mathtt{S(S(S(KS)(S(KK)I))(K(SII)))(S(S(KS)(S(KK)I))(K(SII)))}
\end{aligned}
$$

# Reduction machines

- **Represent combinatory expressions by trees**

- **Example:** $\mathtt{S}\ (f\ x)\ (\mathtt{K}\ y)\ z$ **represented by:**

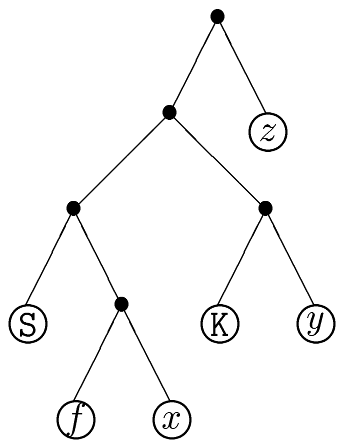



- **Such trees are represented as pointer structures in memory**

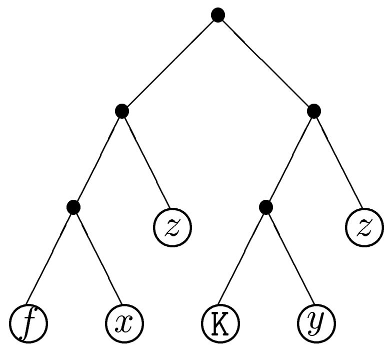
  - **special hardware or firmware can then be implemented to transform such trees according to the rules of combinator reduction defining** $\xrightarrow[c]{}$

# Examples of tree reduction

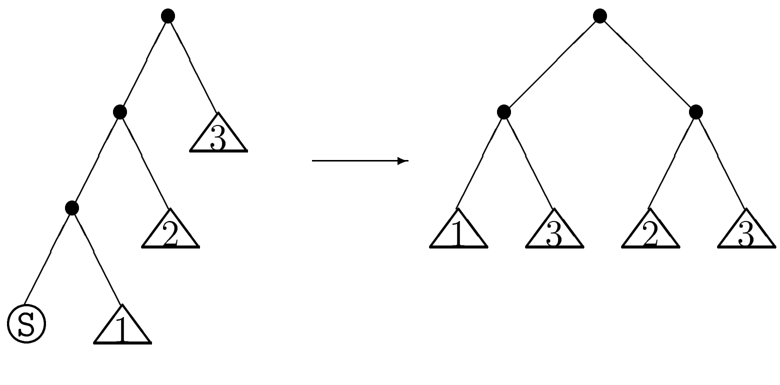
- **The tree:**




**Could be transformed to:**



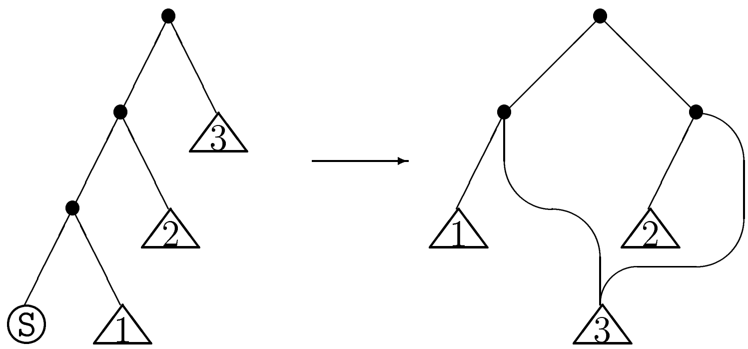

**Using the transformation:**




- **Implements:** $\mathtt{S}\ E_1\ E_2\ E_3 \xrightarrow[c]{} (E_1\ E_3)\ (E_2\ E_3)$

# Graph reduction

- **Tree transformation for `S` just given duplicates a subtree**
  - **wastes space**
  - **a better transformation would be to generate one subtree with two pointers to it:**




- **Generates a *graph* rather than a tree**

# Using combinators for evaluation

- **Valid way of reducing $\lambda$-expressions is:**

  **(i) translating to combinators**

  - **i.e.** $E \mapsto (E)_{\mathtt{C}}$

  **(ii) applying the rewrites**

$$\begin{array}{l} \mathtt{K}\ E_1\ E_2 \underset{\mathrm{c}}{\longrightarrow} E_1 \\ \mathtt{S}\ E_1\ E_2\ E_3 \underset{\mathrm{c}}{\longrightarrow} (E_1\ E_3)\ (E_2\ E_3) \end{array}$$

  **until no more rewriting is possible**

- **If** $E_1 \longrightarrow E_2$ **in the $\lambda$-calculus**

  - **then** *not* **necessarily** $(E_1)_{\mathtt{C}} \underset{\mathrm{c}}{\longrightarrow} (E_2)_{\mathtt{C}}$

  - **for example, take**

$$E_1 = \lambda y.\ (\lambda z.\ y)\ (x\ y)$$

$$E_2 = \lambda y.\ y$$

# Combinatory normal form

- **A combinatory expression is in** *combinatory normal form* **if it contains no subexpressions of the form** $\mathtt{K}\ E_1\ E_2$ **or** $\mathtt{S}\ E_1\ E_2\ E_3$

- **Normalization theorem holds for combinatory expressions**

  - **i.e. always reducing the leftmost combinatory redex will find a combinatory normal form if it exists**

- **If** $E$ **is in combinatory normal form, then it does not necessarily follow that it is a** $\lambda$**-expression in normal form**

  - $\mathtt{S}\ \mathtt{K}$ **is in combinatory normal form, but it contains a** $\beta$**-redex, namely:**

$$(\lambda f.\ (\lambda g\ x.\ (f\ x\ (g\ x)))\ (\lambda x\ y.\ x)$$

# Improving translation to combinators

- **Simple $\lambda$-expressions can translate to complex combinatory expressions**

- **To make the 'code' executed by reduction machines more compact, various optimizations have been devised**

- **Let $E$ be a combinatory expression and $x$ a variable not occurring in $E$**

  - **then:**

$$\mathtt{S}\ (\mathtt{K}\ E)\ \mathtt{I}\ x \xrightarrow[\mathrm{c}]{} (\mathtt{K}\ E\ x)\ (\mathtt{I}\ x) \xrightarrow[\mathrm{c}]{} E\ x$$

  - **hence** $\mathtt{S}\ (\mathtt{K}E)\ \mathtt{I}\ x = E\ x$ **(because** $E_1 \xrightarrow[\mathrm{c}]{} E_2$ **implies** $E_1 \longrightarrow E_2$**)**

  - **so by extensionality:**

$$\mathtt{S}\ (\mathtt{K}\ E)\ \mathtt{I} = E$$

- **Whenever** $\mathtt{S}\ (\mathtt{K}\ E)\ \mathtt{I}$ **is generated**

  - **it can be 'peephole optimized' to just** $E$

# Another optimisation

- **Let $E_1$, $E_2$ be combinatory expressions and $x$ a variable not occurring in either of them**
  - **then:**

$$\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2)\ x \underset{\mathrm{c}}{\longrightarrow} \mathtt{K}\ E_1\ x\ (\mathtt{K}\ E_2)\ x \underset{\mathrm{c}}{\longrightarrow} E_1\ E_2$$

  - **thus**

$$\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2)\ x \ = \ E_1\ E_2$$

  - **now**

$$\mathtt{K}\ (E_1\ E_2)\ x \ \underset{\mathrm{c}}{\longrightarrow}\ E_1\ E_2$$

  - **hence** $\mathtt{K}\ (E_1\ E_2)\ x = E_1\ E_2$
  - **thus**

$$\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2)\ x \ = \ E_1\ E_2 \ = \ \mathtt{K}\ (E_1\ E_2)\ x$$

  - **it follows by extensionality that:**

$$\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2) \ = \ \mathtt{K}\ (E_1\ E_2)$$

- **Whenever $\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2)$ is generated**
  - **it can be optimized to** $\mathtt{K}\ (E_1\ E_2)$

# Example optimisation

- **Example: showed earlier that:**

  $\lambda^* f.\ \lambda^* x.\ f(x\ x) = \mathtt{S}\ (\mathtt{S}\ (\mathtt{K}\ \mathtt{S})\ (\mathtt{S}\ (\mathtt{K}\ \mathtt{K})\ \mathtt{I}))\ (\mathtt{K}\ (\mathtt{S}\ \mathtt{I}\ \mathtt{I}))$

- **Using the optimization**

$$\mathtt{S}\ (\mathtt{K}\ E)\ \mathtt{I} = E$$

- **This simplifies to:**

$$\lambda^* f.\ \lambda^* x.\ f(x\ x) = \mathtt{S}\ (\mathtt{S}\ (\mathtt{K}\ \mathtt{S})\ \mathtt{K})\ (\mathtt{K}\ (\mathtt{S}\ \mathtt{I}\ \mathtt{I}))$$

# More combinators

- **Easy to recognize applicability of optimization $\texttt{S}\ (\texttt{K}\ E)\ \texttt{I} = E$ if $\texttt{I}$ has not been expanded to $\texttt{S}\ \texttt{K}\ \texttt{K}$**
  - **i.e. if $\texttt{I}$ is taken as a primitive combinator**

- **Other combinators similarly useful**

- **Define $\texttt{B}$ and $\texttt{C}$ by:**

$$\texttt{LET B} = \lambda f\ g\ x.\ f\ (g\ x)$$

$$\texttt{LET C} = \lambda f\ g\ x.\ f\ x\ g$$

- **These have the following reduction rules:**

$$\begin{array}{l} \texttt{B}\ E_1\ E_2\ E_3 \underset{\text{c}}{\longrightarrow} E_1\ (E_2\ E_3) \\ \texttt{C}\ E_1\ E_2\ E_3 \underset{\text{c}}{\longrightarrow} E_1\ E_3\ E_2 \end{array}$$

- **It follows that:**

$$\begin{array}{l} \texttt{S}\ (\texttt{K}\ E_1)\ E_2 = \texttt{B}\ E_1\ E_2 \\ \texttt{S}\ E_1\ (\texttt{K}\ E_2) = \texttt{C}\ E_1\ E_2 \end{array}$$

**($E_1$, $E_2$ are any two combinatory expressions)**

# Curry's algorithm

- **Combining the various optimizations yields** *Curry's algorithm* **for translating $\lambda$-expressions to combinatory expressions**

- **Use the definition of** $(E)_{\mathtt{C}}$

- **Whenever an expression of the form** $\mathtt{S}\ E_1\ E_2$ **is generated, try to apply the following rewrite rules:**

  **1.** $\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2) \longrightarrow \mathtt{K}\ (E_1\ E_2)$

  **2.** $\mathtt{S}\ (\mathtt{K}\ E)\ \mathtt{I} \longrightarrow E$

  **3.** $\mathtt{S}\ (\mathtt{K}\ E_1)\ E_2 \longrightarrow \mathtt{B}\ E_1\ E_2$

  **4.** $\mathtt{S}\ E_1\ (\mathtt{K}\ E_2) \longrightarrow \mathtt{C}\ E_1\ E_2$

- **Always use earliest applicable rule**

- $\mathtt{S}\ (\mathtt{K}\ E_1)\ (\mathtt{K}\ E_2)$ **is translated to** $\mathtt{K}\ (E_1\ E_2)$

- $\mathtt{Y}$ **is translated to** $\mathtt{S}\ (\mathtt{C}\ \mathtt{B}\ (\mathtt{S}\ \mathtt{I}\ \mathtt{I}))\ (\mathtt{C}\ \mathtt{B}\ (\mathtt{S}\ \mathtt{I}\ \mathtt{I}))$