Provable Intractability

Our aim now is to show that there are languages (*or, equivalently, decision problems*) that we can prove are not in P.

This is done by showing that, for every *reasonable* function f, there is a language that is not in $\mathsf{TIME}(f(n))$.

The proof is based on the diagonal method, as in the proof of the undecidability of the halting problem.

Constructible Functions

A complexity class such as $\mathsf{TIME}(f(n))$ can be very unnatural, if $f(n)$ is.

From now on, we restrict our bounding functions $f(n)$ to be proper functions:

Definition

A function $f : \mathbb{N} \to \mathbb{N}$ is *constructible* if:

- f is non-decreasing, i.e. $f(n+1) \geq f(n)$ for all n; and
- there is a deterministic machine M which, on any input of length n, replaces the input with the string $0^{f(n)}$, and M runs in time $O(n + f(n))$ and uses $O(f(n))$ *work space*.

Examples

All of the following functions are constructible:

- $\lceil \log n \rceil$;
- n^2;
- n;
- 2^n.

If f and g are constructible functions, then so are $f + g$, $f \cdot g$, 2^f and $f(g)$ (this last, provided that $f(n) > n$).

Using Constructible Functions

Recall $\mathsf{NTIME}(f(n))$ is defined as the class of those languages L accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length at most $O(f(n))$.

If f is a constructible function then any language in $\mathsf{NTIME}(f(n))$ is accepted by a machine for which all computations are of length at most $O(f(n))$.

Also, given a Turing machine M and a constructible function f, we can define a machine that simulates M for $f(n)$ steps.

Time Hierarchy Theorem

For any constructible function f, with $f(n) \geq n$, define the f-bounded *halting language* to be:

$$H_f = \{[M], x \mid M \text{ accepts } x \text{ in } f(|x|) \text{ steps}\}$$

where $[M]$ is a description of M in some fixed encoding scheme.

Then, we can show

$H_f \in \mathsf{TIME}(f(n)^3)$ and $H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

Time Hierarchy Theorem
For any constructible function $f(n) \geq n$, $\mathsf{TIME}(f(n))$ is properly contained in $\mathsf{TIME}(f(2n+1)^3)$.

Strong Hierarchy Theorems

For any constructible function $f(n) \geq n$, $\mathsf{TIME}(f(n))$ is properly contained in $\mathsf{TIME}(f(n)(\log f(n)))$.

Space Hierarchy Theorem
For any pair of constructible functions f and g, with $f = O(g)$ and $g \neq O(f)$, there is a language in $\mathsf{SPACE}(g(n))$ that is not in $\mathsf{SPACE}(f(n))$.

Similar results can be established for nondeterministic time and space classes.

Consequences

- For each k, $\mathsf{TIME}(n^k) \neq \mathsf{TIME}(n^{k+1})$.
- $\mathsf{P} \neq \mathsf{EXP}$.
- $\mathsf{L} \neq \mathsf{PSPACE}$.
- Any language that is EXP-complete is not in P.
- There are no problems in P that are complete under linear time reductions.

P-complete Problems

It makes little sense to talk of complete problems for the class P with respect to polynomial time reducibility $\leq_P$.

There are problems that are complete for P with respect to *logarithmic space* reductions $\leq_L$.

One example is CVP—the circuit value problem.

- If $\mathsf{CVP} \in \mathsf{L}$ then $\mathsf{L} = \mathsf{P}$.
- If $\mathsf{CVP} \in \mathsf{NL}$ then $\mathsf{NL} = \mathsf{P}$.