Machine Learning and Autoformalisation

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Machine Learning

Machine learning (ML) is a methodology for leveraging data to improve machine performance on various types of tasks. One such type of tasks is "autoformalisation" (AF), namely turning informal mathematical proofs into formal ones with minimal input from humans. There are various approaches to ML, but each of them requires some data and some "learning" algorithms. Regarding algorithms for AF, one can repurpose algorithms for translating natural language to code. In this talk, I will focus on the data aspect. One approach to ML is *supervised* learning. This approach requires a *training set* that pairs some inputs (tasks) with their desired outputs (solutions).

Machine Learning for Autoformalisation

Since we don't have a large training set for AF, supervised learning alone is not sufficient. To do ML for AF (MLAF), one needs to adopt the *unsupervised* learning approach. In this approach, an algorithm is given an "outputless" dataset, namely a set of inputs without their desired outputs.

A Promising Path

How to perform AF using the unsupervised approach? Some people have proposed to build a training set by bootstrapping from an outputless dataset to initiate a positive feedback loop. How? See Szegedy's article *A Promising Path Towards Autoformalisation and General Artificial Intelligence*. In a nutshell, one needs:

- a source language for the informal material, *e.g.* LaTeX
- a target language for the formal one, *e.g.* Isabelle/HOL
- a translation component to translate informal statements of the source language into formal ones of the target language
- a reasoning engine to prove the resulting formal statements.

If the search engine fails to prove a translated statement, then this statement is discarded. Otherwise, the formal proof is paired with its informal counterpart and added to a database which in return will be used to train the reasoning engine. Does it work?

First Steps

Recently, Isabelle/HOL has been used as a target language to perform the autoformalisation of competition mathematics problems. See Yuhuai Wu *et al*, *Autoformalization for Neural Theorem Proving*, AITP, 2022. Out of 3908 statements from a dataset, 3363 statements were automatically translated into Isabelle/HOL. As a result of the automatic translation, a statement and its translation may be misaligned. Their reasoning engine, trained on Isabelle's standard library and its Archive of Formal Proofs, was able to prove 23.3% of the translated statements.

See also Wang *et al*, *Exploration of neural machine translation in autoformalization of mathematics in Mizar* (2020), for experiments in Mizar.

Current Limitations

Current methods have performed only the autoformalisation of fairly small pieces of mathematics and may not be able to scale up.

If autoformalisation of high school mathematics is feasible, more advanced mathematics seems out of reach.

Creating Synergy

How can users of interactive theorem provers contribute to MLAF?
We propose to use the above boostrapping and the positive feedback loop in conjunction with a proper database.
This database should be made of aligned pairs of informal artefacts and formal ones, where an artefact is either a definition, a statement or the proof of a given statement.
By "informal" artefacts we mean artefacts as written by mathematicians in LaTeX and by "formal" artefacts we mean artefacts translated by expert users of the chosen formal system.
This would result in semi-supervised machine learning for autoformalisation and it could improve current results.

Summary

I will describe our early achievements and future plans for developing such a database. Many competitive formal systems (Agda, Coq, Lean . . .) could be used as a target language for this project. I will first discuss Isabelle's fitness as a target language.

Outline

1. Isabelle's Fitness as a Target Language for MLAF
2. A Database Infrastructure
3. Current State of the Database and Future Extension

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Isabelle, a Contender?

To perform AF on many parts of mathematics at various levels of abstraction, one needs a target language sufficiently expressive for all kinds of mathematical objects.

Isabelle/HOL is based on Church's simple type theory, but dependently-typed systems (Agda, Coq, Lean . . .) are more expressive than simply-typed systems.

Is it possible to formalise more abstract mathematics into Isabelle/HOL, so that we could be able to collect enough data points for improving MLAF?

Can Schemes be formalised in Isabelle/HOL?

Schemes are abstract spaces introduced in 1960 by Alexander Grothendieck and studied in algebraic geometry.

Schemes have been recently formalised in Lean.

Following this formalisation, the mathematician Kevin Buzzard challenged all the other interactive theorem provers to formalise schemes.

Following Hartshorne's textbook *Algebraic Geometry*, Isabelle/HOL met the challenge in 2021.

The details are in B., Paulson and Li, *Simple Type Theory is not too Simple: Grothendieck's Schemes Without Dependent Types*, Experimental Mathematics, 2022.

Another Case Study: Strict Omega Categories

In recent years, higher-dimensional structures, such as omega-groupoids and omega-categories, have been formalised in some dependently-typed systems like Agda and Coq (both extended by a strict equality).

These structures seem to make use of dependent types in a decisive way. In particular, the aforementioned achievements are based on type-theoretic reformulations of these structures.

Can strict omega-categories be formalised without dependent types?

Yes, we have formalised strict omega-categories in Isabelle/HOL following Leinster's book *Higher Operads, Higher Categories*.

We report on this formalisation in our preprint: B. and Doña Mateo, *Encoding Dependently-Typed Constructions into Simple Type Theory*, 2022.

Strict Omega-Categories

Definition (Strict Omega-Categories)

A strict ω-category is a globular set X equipped with

- a function $\circ_n \colon X_m \times_{X_n} X_m \to X_m$ for all natural numbers m and n with $n < m$; we write $\circ_n(y, x)$ as $y \circ_n x$ and call it a composite of x and y
- a function $i \colon X_n \to X_{n+1}$ for each natural number n; we write $i(x)$ as 1_x and call it the identity of x,

satisfying the following axioms

1. (sources and targets of composites) if $n < m$ and $(y, x) \in X_m \times_{X_n} X_m$ then

$$\left.\begin{aligned} s(y \circ_n x) &= s(x) \\ t(y \circ_n x) &= t(y) \end{aligned}\right\} \text{if } n = m - 1$$

   and

Strict Omega-Categories (continued)

Definition (continued)

$$\left.\begin{aligned} s(y \circ_n x) &= s(y) \circ_n s(x) \\ t(y \circ_n x) &= t(y) \circ_n t(x) \end{aligned}\right\} \text{ if } n \leq m - 2$$

2. (sources and targets of identities) if $x \in X_n$ then

$$s(1_x) = x = t(1_x)$$

   for all natural number n

3. (associativity) if $n < m$ and $x, y, z \in X_m$ with $(z, y), (y, x) \in X_m \times_{X_n} X_m$ then

$$(z \circ_n y) \circ_n x = z \circ_n (y \circ_n x)$$

Strict Omega-Categories (continued)

Definition (continued)

4. (identities) if $n < m$ and $x \in X_m$ then

$$i^{m-n}(t^{m-n}(x)) \circ_n x = x = x \circ_n i^{m-n}(s^{m-n}(x)),$$

where $i^{m-n} \colon X_n \to X_m$ denotes the $(m-n)$-th iterate of i

5. (nullary interchange) if $q < p$ and $(y, x) \in X_p \times_{X_q} X_p$ then $1_y \circ_q 1_x = 1_{y \circ_q x}$

Strict Omega-Categories (end)

Definition (end)

6. (binary interchange) if $q < p < m$ and $x, x', y, y' \in X_m$ with

$$(y', y), (x', x) \in X_m \times_{X_p} X_m$$

and

$$(y', x'), (y, x) \in X_m \times_{X_q} X_m$$

then

$$(y' \circ_p y) \circ_q (x' \circ_p x) = (y' \circ_q x') \circ_p (y \circ_q x)$$

Strict Omega-Categories Formalised

```
locale strict_omega_category = globular_set +
  fixes comp :: "nat ⇒ nat ⇒ 'a ⇒ 'a ⇒ 'a"
    and i' :: "nat ⇒ nat ⇒ 'a ⇒ 'a"
  assumes i'_fun: "n ≤ m ⟹ i' m n ∈ X n → X m"
    and i'_n_n: "i' n n = id"
    and comp_fun: "is_composable_pair m n x' x ⟹ comp m n x' x ∈ X m"
    and s_comp_Suc: "is_composable_pair (Suc m) m x' x ⟹ s m (comp (Suc m) m x' x) = s m x"
    and t_comp_Suc: "is_composable_pair (Suc m) m x' x ⟹ t m (comp (Suc m) m x' x) = t m x'"
    and s_comp: "⟦is_composable_pair (Suc m) n x' x; n < m⟧ ⟹
          s m (comp (Suc m) n x' x) = comp m n (s m x') (s m x)"
    and t_comp: "⟦is_composable_pair (Suc m) n x' x; n < m⟧ ⟹
          t m (comp (Suc m) n x' x) = comp m n (t m x') (t m x)"
    and s_i: "x ∈ X n ⟹ s n (i' (Suc n) n x) = x"
    and t_i: "x ∈ X n ⟹ t n (i' (Suc n) n x) = x"
    and comp_assoc: "⟦is_composable_pair m n x' x; is_composable_pair m n x'' x'⟧ ⟹
            comp m n (comp m n x'' x') x = comp m n x'' (comp m n x' x)"
    and i_comp: "⟦n < m; x ∈ X m⟧ ⟹ comp m n (i' m n (t' m n x)) x = x"
    and comp_i: "⟦n < m; x ∈ X m⟧ ⟹ comp m n x (i' m n (s' m n x)) = x"
    and bin_interchange: "⟦q < p; p < m;
          is_composable_pair m p y' y; is_composable_pair m p x' x;
          is_composable_pair m q y' x'; is_composable_pair m q y x⟧ ⟹
          comp m q (comp m p y' y) (comp m p x' x) = comp m p (comp m q y' x') (comp m q y x)"
    and null_interchange: "⟦q < p; is_composable_pair p q x' x⟧ ⟹
          comp (Suc p) q (i' (Suc p) p x') (i' (Suc p) p x) = i' (Suc p) p (comp p q x' x)"
```

Did we face some limitations of Isabelle's type system?

Breaking Out of Limitations

Let's backpedal and discuss globular sets, a higher-dimensional analogue of a directed graph and a prerequisite for strict omega-categories.

Definition (globular sets)

A globular set X is a diagram

$$\cdots \overset{s}{\underset{t}{\rightrightarrows}} X_n \overset{s}{\underset{t}{\rightrightarrows}} X_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} \cdots \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

of sets and maps satisfying the so-called globular identities

$$s(s(x)) = s(t(x))$$
$$t(t(x)) = t(s(x))$$

for all element $x \in X_m$ and all natural number $m \geq 2$.

Breaking Out (continued)

```
locale globular_set =
  fixes X :: "nat ⇒ 'a set" and s :: "nat ⇒ 'a ⇒ 'a" and t :: "nat ⇒ 'a ⇒ 'a"
  assumes s_fun: "s n ∈ X (Suc n) → X n"
    and   t_fun: "t n ∈ X (Suc n) → X n"
    and  s_comp: "x ∈ X (Suc (Suc n)) ⟹ s n (t (Suc n) x) = s n (s (Suc n) x)"
    and  t_comp: "x ∈ X (Suc (Suc n)) ⟹ t n (s (Suc n) x) = t n (t (Suc n) x)"
```

We can't have in Isabelle/HOL an indexed family of type variables (here, a family indexed by the natural numbers), hence we used a single type variable 'a to type all the carriers of the sets X_n.

Is it a problem?

Breaking Out (continued)

As a sanity check, we had to provide a standard example of strict omega-category, so we decided to prove that pasting diagrams form a strict omega-category. For the underlying globular set pd of pasting diagrams, there are two equivalent definitions and only one of them can be easily formalised given the above constraint.
First definition: the underlying globular set pd of pasting diagrams is defined inductively as follows,

- $\mathrm{pd}_0 = 1$
- $\mathrm{pd}_{n+1} = (\mathrm{pd}_n)^*$

where $(\_)^*$ is the so-called free monoid functor on Set. This means that pd_{n+1} is the set of finite lists of elements in pd_n.

Assuming that the unique element of the singleton 1 has type 'a, pd_0 should have type 'a set, while pd_1 (*resp.* pd_2 ...) should have type ('a list) set (*resp.* (('a list) list) set ...).

Breaking Out (end)

Second definition:

define pd_n as the set of trees of height less than n.

Take 'a := tree, then all the types of the carriers for the pd_n can be embedded in the single type tree set.

Conclusion

As a formal system Isabelle/HOL is able to express a large class of mathematical structures.

A large part of the undergraduate curriculum has already been covered by Isabelle's standard library and its Archive of Formal Proofs.

It seems reasonable to think that the graduate curriculum could be formalised as well.

The Road Less Travelled

Even though Isabelle/HOL is based on simple type theory, locales can be used as a substitute for dependent types.

Isabelle/HOL could be well suited as a target language for MLAF.

Isabelle/HOL has relatively large libraries, a simple yet expressive logic, a vernacular for structured statements/proofs and powerful automation.

Can Isabelle/HOL's simple type theory ease the work of an autoformalisation system that would be able to translate at least partially an informal proof into a formal one and then leverage Isabelle's powerful automation?

Outline

The Mathematician's Dream

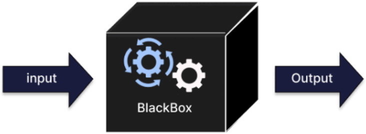


Figure: LaTeX source file as input, residual goals as output

certificate of correctness = no residual goals
Who wants dozens of ITP users burning the candle at both ends inside our black box to produce certificates of correctness?

Ideally, inside the black box we want a model for autoformalisation working hard for us.

A Database for the Mathematician's Dream

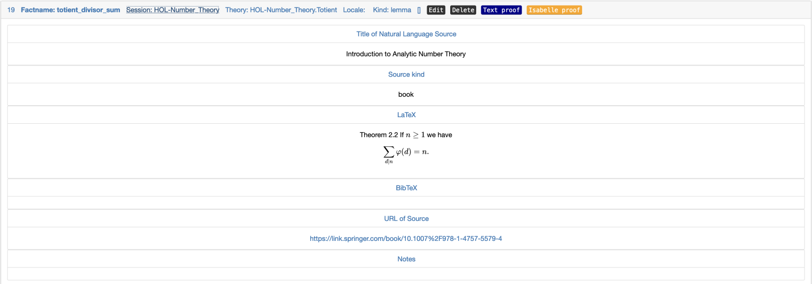
Instead of aligned pairs, think in terms of aligned tuples $(a_0 \in L_0, \ldots, a_{n-1} \in L_{n-1})$, where $n \geq 5$ and $L_0 = \text{\LaTeX}, L_1 = \mathit{Agda}, L_2 = \mathit{Coq}, L_3 = \mathit{Isabelle}, L_4 = \mathit{Lean} \ldots$
We want to build a database whose infrastructure could be used for creating as many parallel corpora as there are competitive target languages. Think of such a database as a large library containing only "dual-language" books or "side-by-side books".

The Isabelle Parallel Corpus

Our current database contains one such corpus: the Isabelle Parallel Corpus (IPC). Through an interface, one can search (with words like in Google!) for a formal artefact in Isabelle/HOL's libraries and its Archive of Formal Proofs. Once the desired artefact is found, one can attach to it its natural language counterpart using LaTeX and possibly a BibTeX reference and an URL for the source. This way, we are building a parallel corpus made of pairs of natural and Isabelle artefacts.

Check out our article: B., Stathopoulos and Paulson, *A Parallel Corpus for Natural Language Machine Translation to Isabelle*, CICM, 2022.

An Entry of the IPC


19 Factname: totient_divisor_sum Session: HOL-Number_Theory Theory: HOL-Number_Theory.Totient Locale: Kind: lemma [] Edit Delete Text proof Isabelle proof

Title of Natural Language Source
Introduction to Analytic Number Theory
Source kind
book
LaTeX
Theorem 2.2 If $n \geq 1$ we have $\sum_{d \mid n} \varphi(d) = n.$
BibTeX
URL of Source
https://link.springer.com/book/10.1007%2F978-1-4757-5579-4
Notes

Figure: Visit the website: `https://behemoth.cl.cam.ac.uk/ipc/`.

Misalignment Between Textual Proofs and Formal Proofs

Sentence	Text
①	$\underbrace{\text{If } n \geq 1}_{I}$ we have $$\overbrace{\sum_{d\mid n} \varphi(d) = n.}^{\alpha}$$
②	Let S denote the set $\{1, 2, \ldots, n\}$.
③	We distribute the integers of S into disjoint sets as follows.
④	For each divisor d of n, let $$A(d) = \{k : (k, n) = d, 1 \leq k \leq n\}.$$
⑤	That is, $A(d)$ contains those elements of S which have the gcd d with n.
⑥	The sets $A(d)$ form a disjoint collection whose union is S.
⑦	Therefore if $f(d)$ denotes the number of integers in $A(d)$ we have $$\sum_{d\mid n} f(d) = n. \quad (1)$$
⑧	But $(k, n) = d$ if and only if $(k/d, n/d) = 1$, and $0 < k \leq n$ if and only if $0 < k/d \leq n/d$.
⑨	Therefore, if we let $q = k/d$, there is a one-to-one correspondence between the elements in $A(d)$ and those integers q satisfying $0 < q \leq n/d$, $(q, n/d) = 1$.
⑩	The number of such q is $\varphi(n/d)$.
⑪	Hence $\underbrace{f(d) = \varphi(n/d)}_{\beta}$ and (1) becomes $$\underbrace{\sum_{d\mid n} \varphi(n/d) = n\,.}_{\gamma}$$
⑫	But this is equivalent to the statement $\sum_{d\mid n} \varphi(d) = n$ because when d runs through all divisors of n so does n/d.
⑬	This completes the proof.

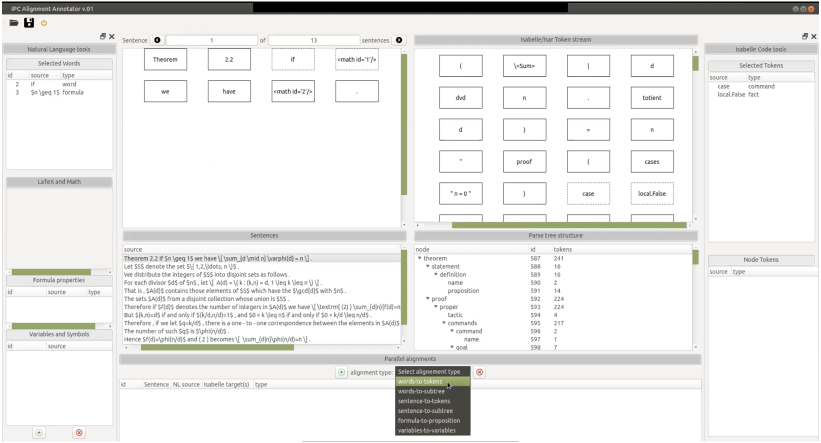
Theory: HOL-Number_Theory.Totient

<table>
<tr><th>Sentence</th><th></th><th>Isabelle command</th></tr>
<tr><td>α</td><td>1</td><td><code>lemma totient_divisor_sum: "(∑d | d dvd n. totient d) = n"</code>
<code>proof (cases "n = 0")</code></td></tr>
<tr><td>I</td><td>2</td><td><code>case False</code></td></tr>
<tr><td></td><td>3</td><td><code>hence "n > 0" by simp</code></td></tr>
<tr><td>④</td><td>4</td><td><code>define A where "A = (λd. {k∈{0<..n}. gcd k n = d})"</code></td></tr>
<tr><td>β</td><td>5</td><td><code>have *: "card (A d) = totient (n div d)" if d: "d dvd n" for d</code>
<code>using ‹n > 0› and d unfolding A_def by (rule card_gcd_eq_totient)</code></td></tr>
<tr><td></td><td>6</td><td><code>have "n = card {1..n}" by simp</code></td></tr>
<tr><td>② ⑥</td><td>7</td><td><code>also have "{1..n} = (⋃d∈{d. d dvd n}. A d)" by safe (auto simp: A_def)</code></td></tr>
<tr><td>(1) in ⑦</td><td>8</td><td><code>also have "card ... = (∑d | d dvd n. card (A d))"</code>
<code>using ‹n > 0› by (intro card_UN_disjoint) (auto simp: A_def)</code></td></tr>
<tr><td>γ</td><td>9</td><td><code>also have "... = (∑d | d dvd n. totient (n div d))" by (intro sum.cong refl *) auto</code></td></tr>
<tr><td>⑫</td><td>10</td><td><code>also have "... = (∑d | d dvd n. totient d)" using ‹n > 0›</code>
<code>by (intro sum.reindex_bij_witness[of _ "(div) n" "(div) n"]) (auto elim: dvdE)</code></td></tr>
<tr><td></td><td>11</td><td><code>finally show ?thesis ..</code></td></tr>
<tr><td>⑬</td><td>12</td><td><code>qed auto</code></td></tr>
</table>

Figure: An example of a many-to-many mapping.

Building Tools for Alignment

Tools have been built to fine-tune the alignment, *e.g.* LaTeX to Isabelle/HOL tokens, with parsers and automated extraction of variables and symbols.

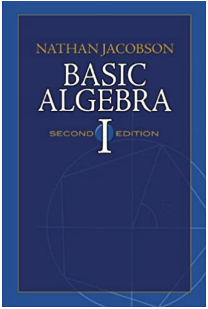


Tracking Dependencies

A suite of graph analysis tools has been developed for modelling the relationships between formal artefacts. Given a formal definition or a formal statement, one will be able to track the definitions used in them. Given a formal proof, one will be able to track the facts called within this proof. Tracking dependencies allows to understand which artefacts could be paired within a parallel corpus to better the alignment between informal and formal artefacts.

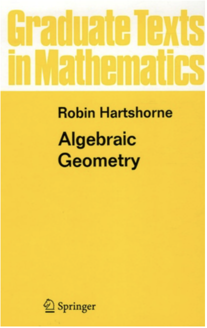
Outline

IPC's Table of Contents



Clemens Ballarin formalised parts of Jacobson's *Basic Algebra* first three chapters covering monoids, groups and rings. This formal development includes 175 theorems. The formal definitions and statements have been paired with their informal counterparts in the IPC.

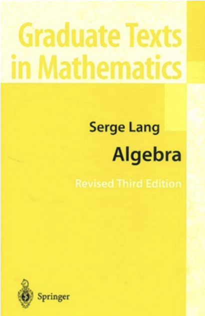
IPC's Table of Contents (continued)



Schemes

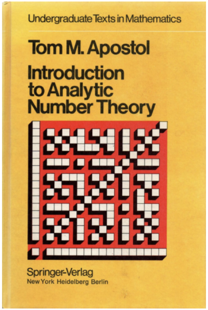
are part of the IPC. For machine learning, the definition alone wouldn't be useful. We had to pair all the formal prerequisites (affine schemes, sheaves and presheaves of rings . . .) with their informal counterparts, covering some material in chapter II of Hartshorne's textbook.

IPC's Table of Contents (continued)



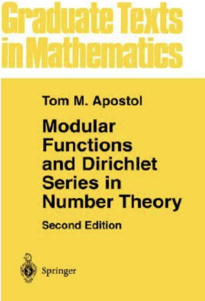
We took
also the opportunity for pairing the prerequisites in algebra not covered by Ballarin's development (prime and maximal ideals . . .), bridging the gap in the IPC with this last development.

IPC's Table of Contents (end)



Apostol's *Introduction to Analytic Number Theory* has been largely formalised in Isabelle/HOL ($\sim$ 80%, mainly by Manuel Eberl). It represents $\sim$ 30,000 lines of code. The formal definitions and statements have been paired with their informal counterparts in the IPC.

Future Extension of the IPC



With Manuel Eberl, Wenda Li and Larry Paulson, we're formalising Apostol's follow-up textbook. The definitions and statements in all eight chapters have been formalised. Proofs in chapters 1,2,3 and 7 have been formalised and also a few bits and bobs in chapters 4 and 6 ($\sim$ 55,000 lines of code as of November 2022). This project features elliptic functions, Eisenstein series, Ramanujan's τ function, modular forms . . .

Since its inception in 2021, the project has been aimed at extending the IPC.

Future Extension of the IPC (continued)

1. Add to the IPC all the pen-and-paper proofs from Apostol's undergraduate textbook. Consider contributing, credit will be given.
2. Can we autoformalise some of the remaining parts of Apostol's graduate textbook by combining current methods for autoformalisation with the IPC?
3. It's probably a far away goal, but more approachable goals for which the IPC could be useful include:
   - developing natural language search for Isabelle's libraries
   - "informalisation", *i.e.* generating natural language descriptions of Isabelle artefacts.

Future Extension of the Database

Our project methodology and a suite of similar tools could be used for various target languages (Agda, Coq, Lean . . .) resulting in various parallel corpora.

Take-Home Message

- We promote an integrative approach to autoformalisation.
- Autoformalisation is not restricted to mathematics and we should cover the entire scope of formal verification (algorithms, protocols . . .).

Thank You