Social and Technological Network Analysis

Lecture 3: Structure of the Web and Power Laws

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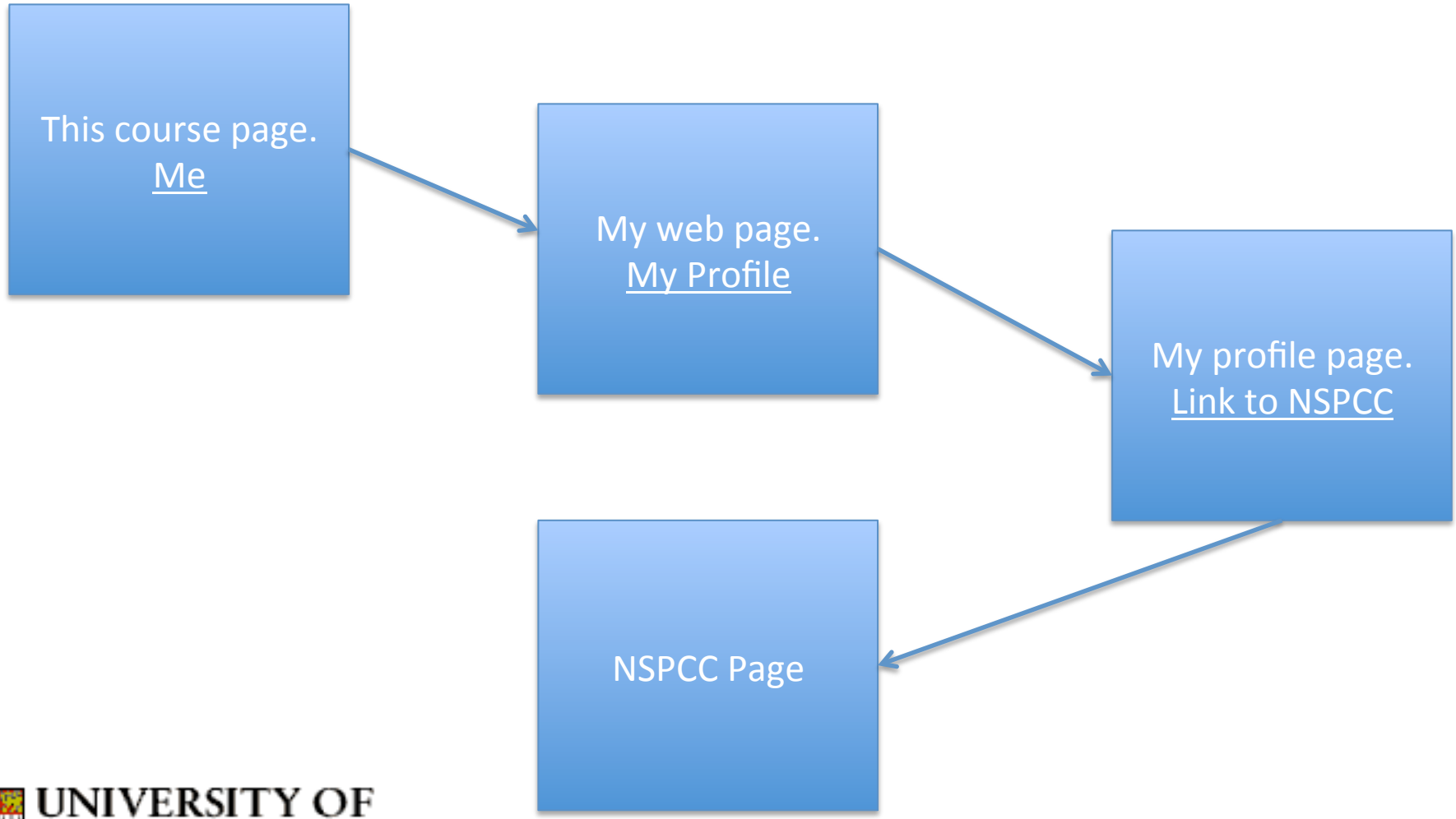
In This Lecture

- We describe power law networks and their properties and show examples of networks which are power law in nature, including the web.
- We present the preferential attachment model which allows the generation of power law networks.

The Web is a Graph…



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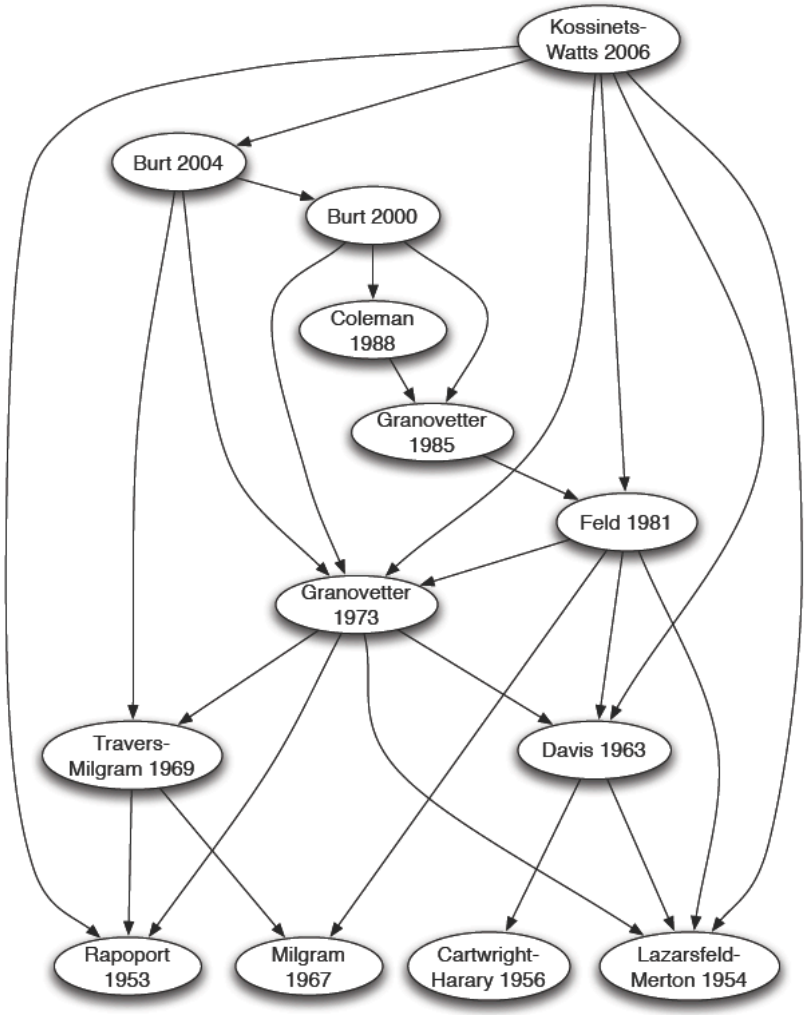
Precursor of hypertexts

- Citation networks of books and articles.
- Difference: links point only backwards in time



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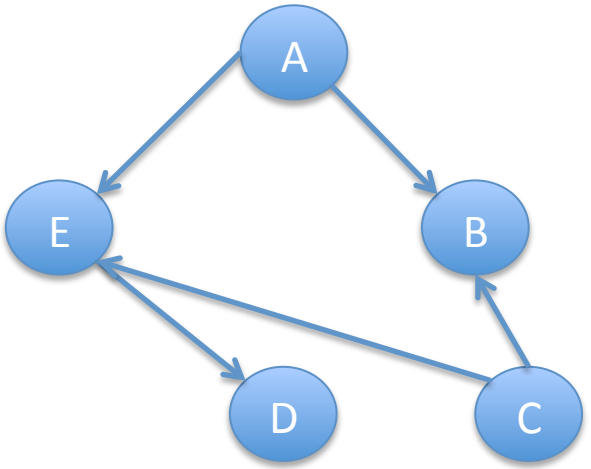
Web is a **Directed** Graph

- **Path**: A path from A to B exists if there is a sequence of nodes beginning with A and ending with B such that each consecutive pair of nodes is connected by an edge pointing in the forward direction.



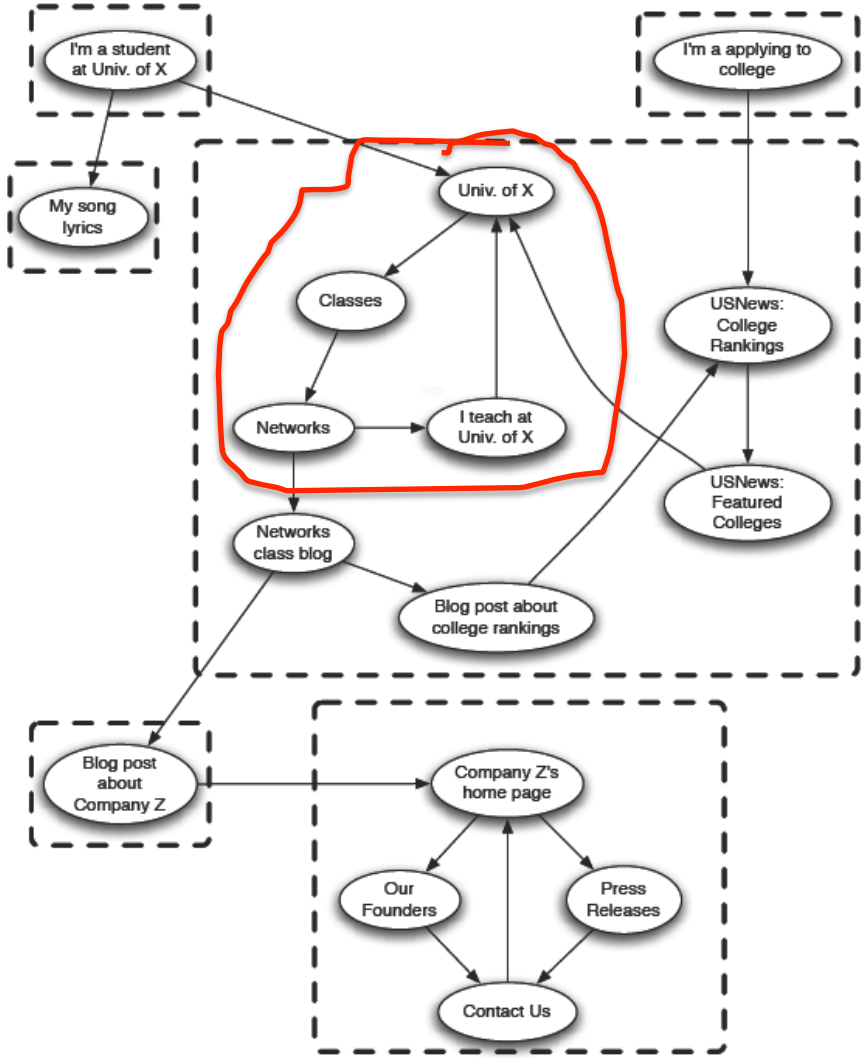
Strongly Connected Component

- A strongly connected component (SCC) in a directed graph is a subset of nodes such that:

  i) Every pair in the subset has a path to each other
  ii) The subset is not part of some larger subset with property i)

- Weakly connected component (WCC) is the connected component in the undirected graph derived from the directed graph.
  - Two nodes can be in the same WCC even if there no directed path between them.

SCC example



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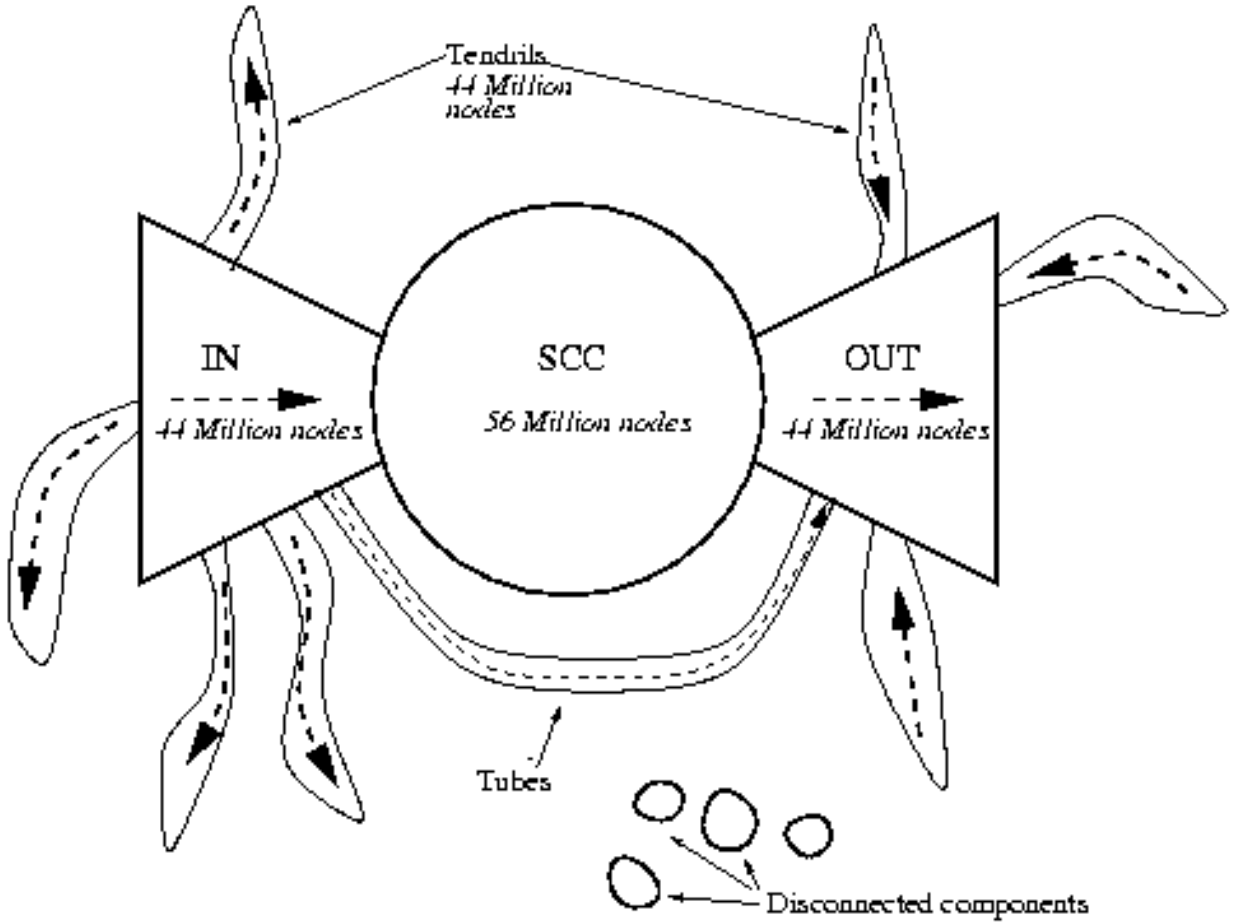
The Web

- Broder'00
- Data from Altavista (200 million pages)
- 186M nodes in the WCC (90% of links)



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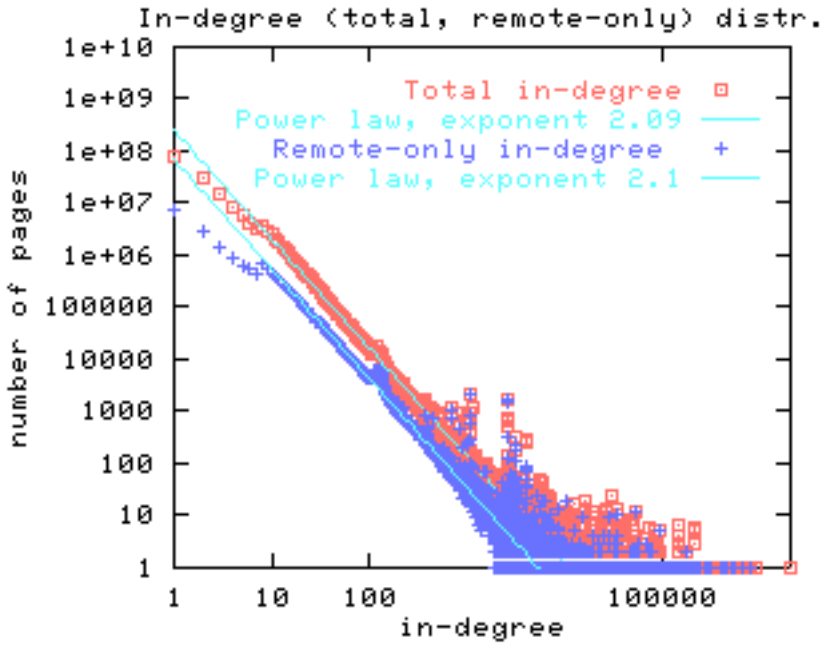
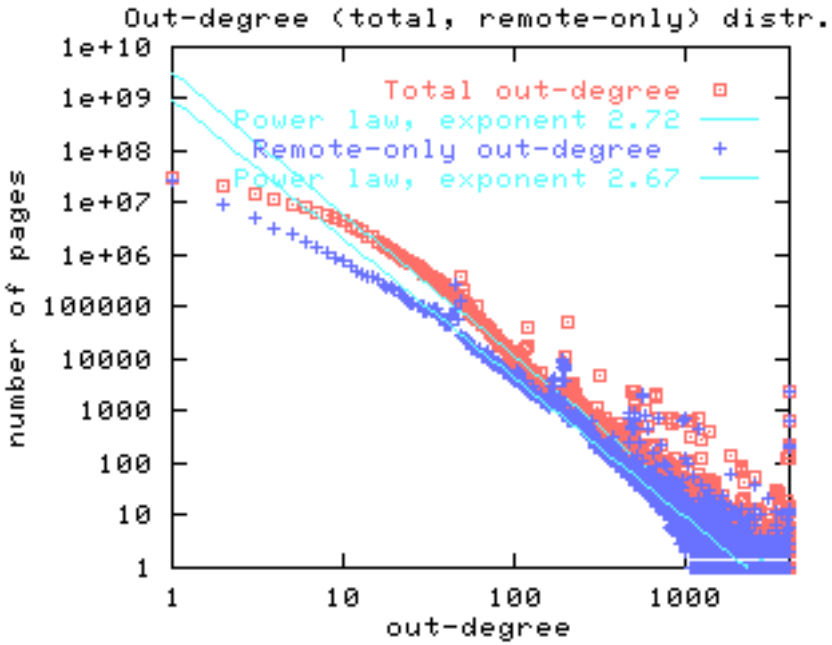
Popularity of Web Pages

- How do we expect the popularity of web pages to be distributed?
  - What fraction of web pages have k in-links?
  - If each page decides independently at random whether to link to any given other page then the n of in-links of a page is the sum of independent random quantities -> normal distribution
  - In this case n pages with k in-links decreases exponentially in k
  - Is this true for the Web?

Degree distribution for the Web

- Finding: degree distr. proportional to ~$1/k^2$
- $1/k^2$ decreases much more slowly than a normal distribution





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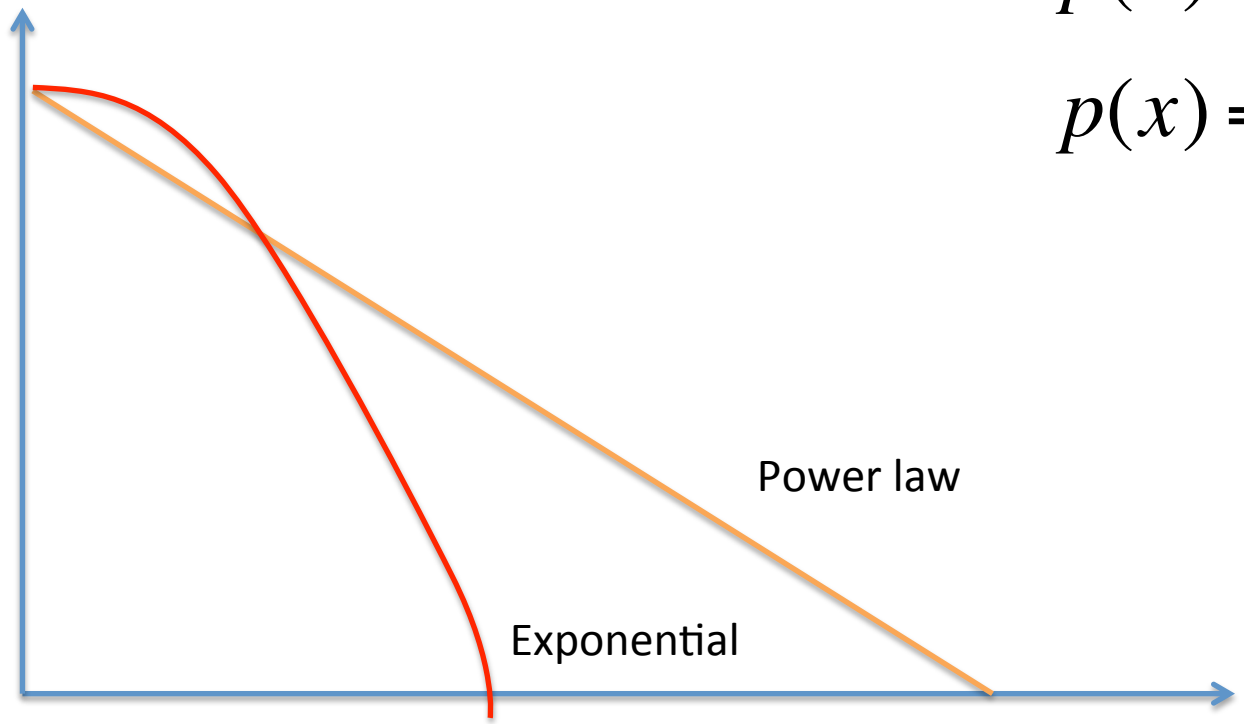
Power Law vs Exponential

$$p(x) = x^{-\alpha}$$

$$p(x) = e^{-\lambda x}$$



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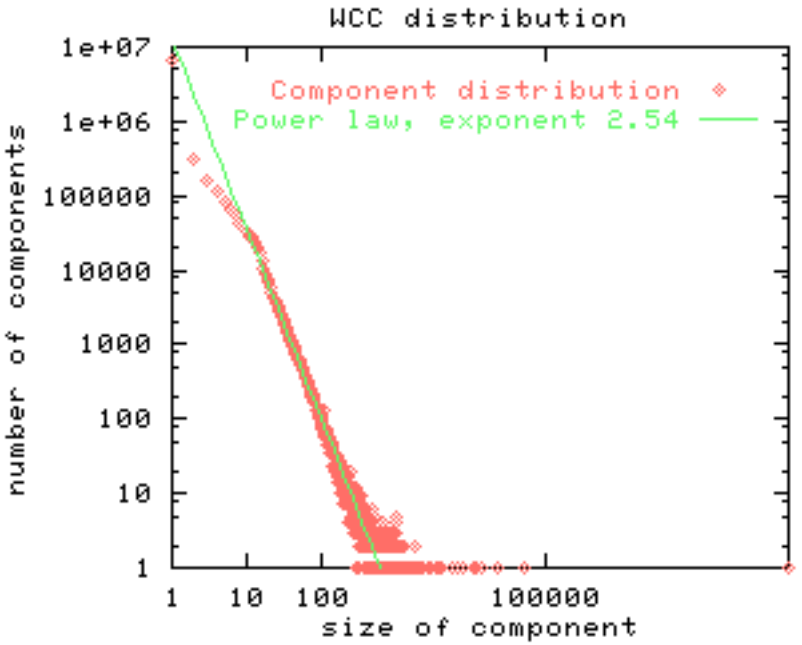
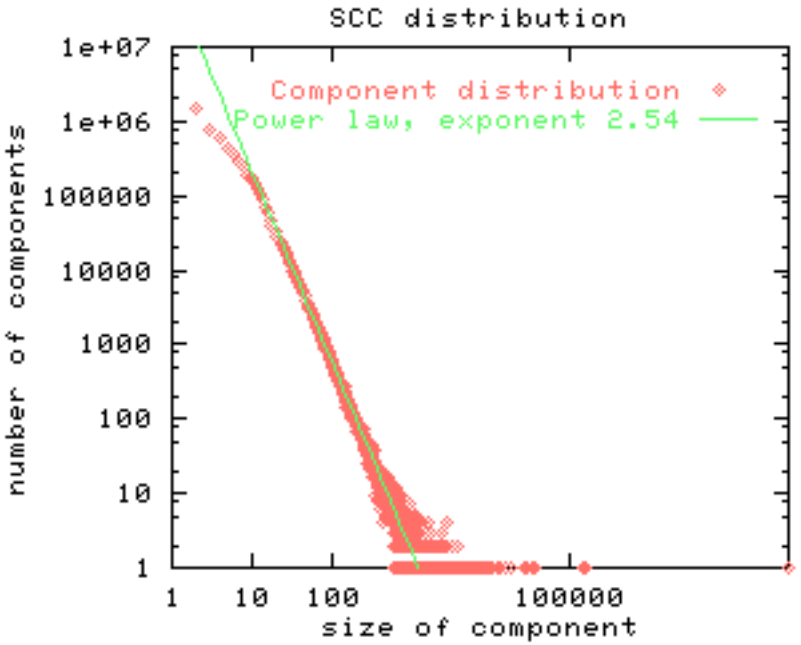
Distribution of WCC and SCC





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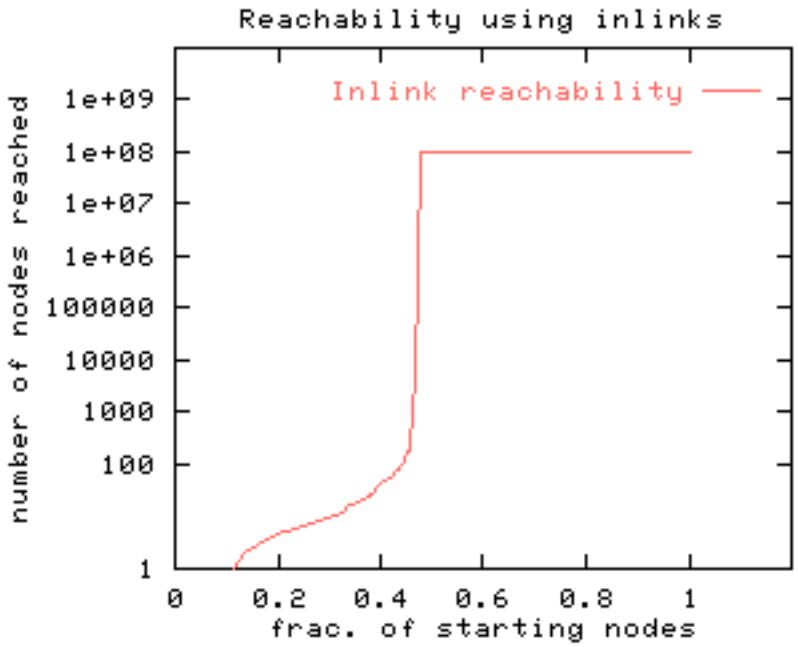
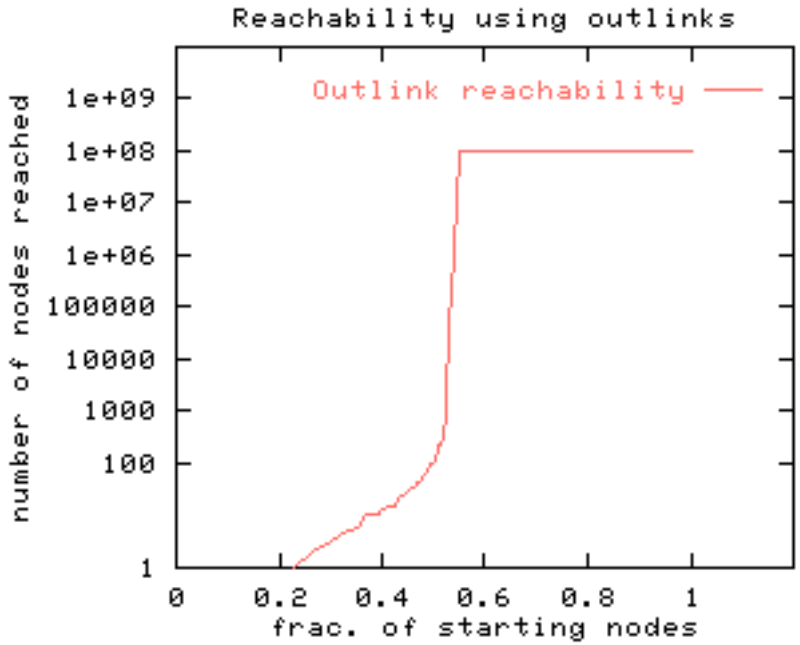
Reachability

- Followed links backwards and forward





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Diameter of the Web

- 75% of the time there is no directed path between two random nodes
- Average distance of existing paths: 16
- Average distance of undirected paths: 6.83

- Diameter in the SCC is at least 28

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Power Laws aka Scale Free Networks

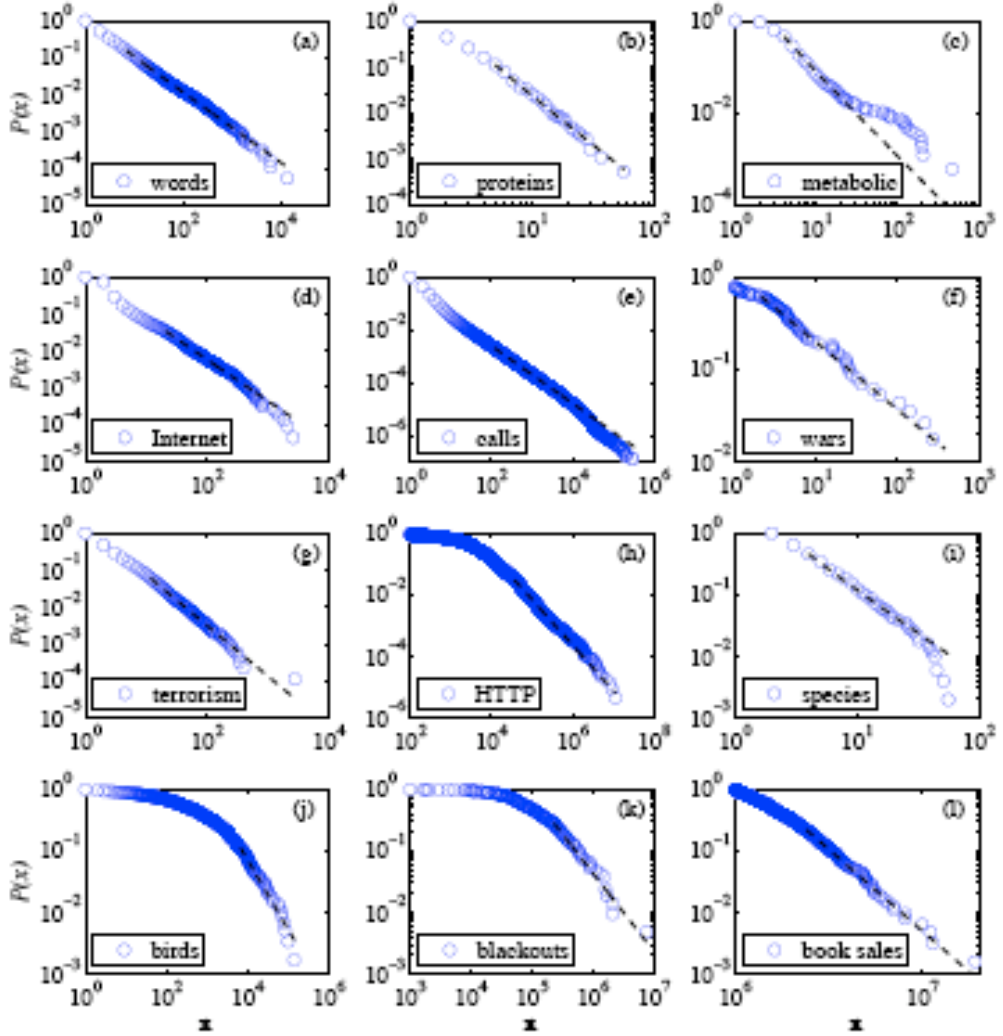
- We have seen that the degree distribution followed a straight line in log-log

$$\ln p_k = -\alpha \ln k + c$$

$$p_k = Ck^{-\alpha}$$

- α defines the slope of the curve
- α is typically between 2 and 3.

Power Laws in various domains



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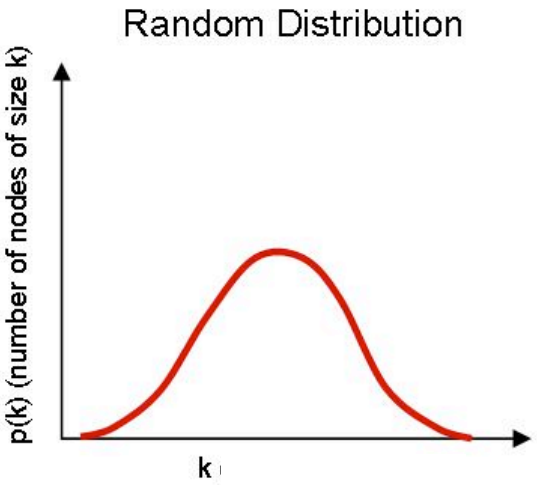
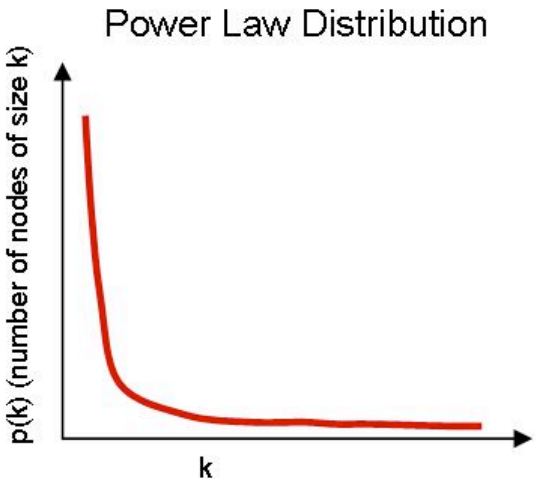
What does it mean?





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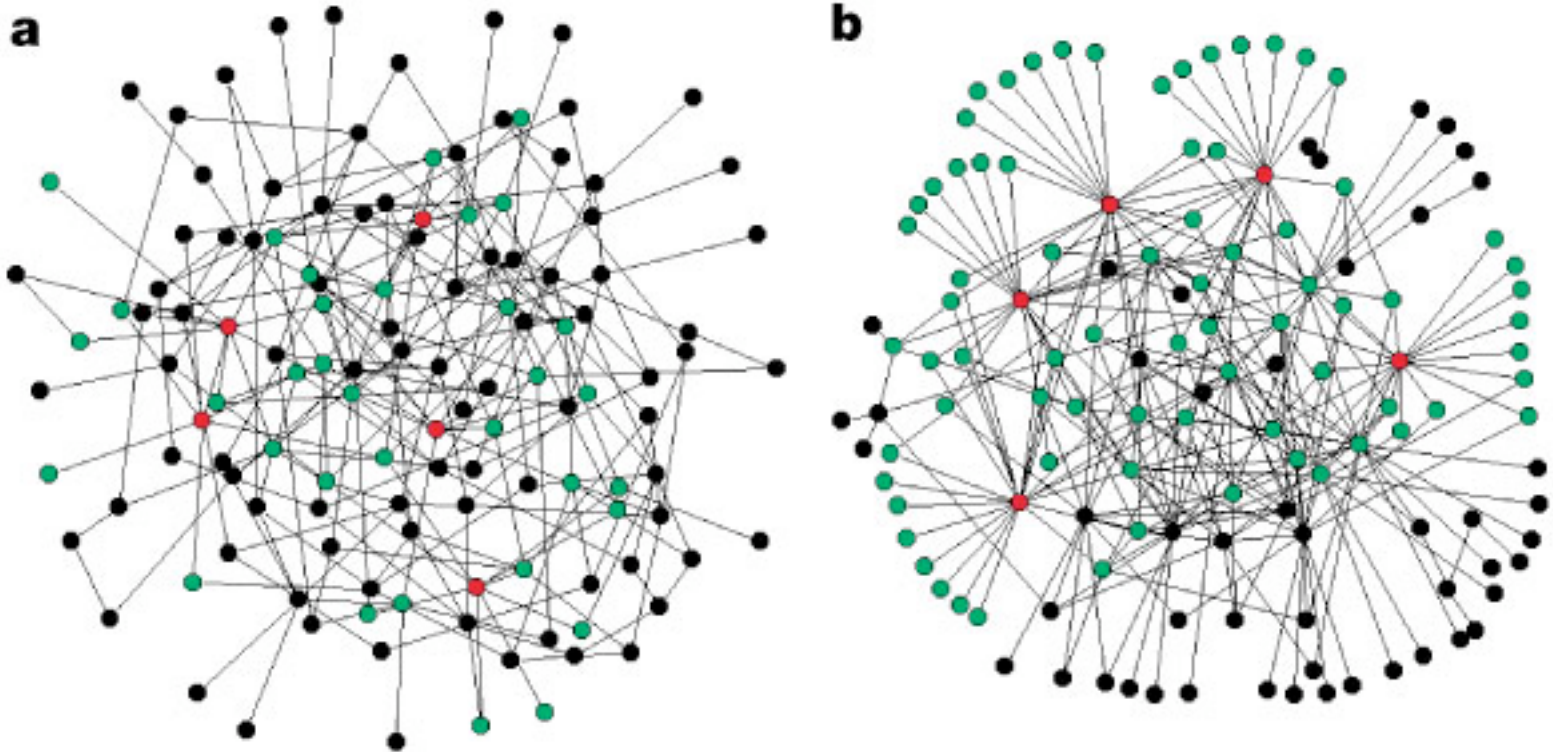
Random vs Power Law Networks



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What's a good model for scale free networks

- Let's use the web network as example:
- Pages are created in order (1,2,3..)
- Page j created and it links to an earlier page in the following way:
  - With prob. p, j chooses page i at random and links it;
  - With prob. 1-p, j chooses page i and links to the page i points to.
  - Repeat.
- The middle step is essentially a copy of the node i behaviour…

Preferential attachment

- Pages are created in order (1,2,3..)
- Page j created and it links to an earlier page in the following way:
  - With prob. p, j chooses page i at random and links it;
  - **With prob. 1-p, j chooses a page z with prob. proportional to z's current number of in-links and links to z (ie proportional to degree).**
  - Repeat.

Rich-get-richer model

If we run this for many pages the fraction of pages with k in-links will be distributed approximately according to a power law $1/k^c$

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Let's get formal…

- Number of in-links Xj(t) at time t.
- Initially (creation of j at time j) Xj(j)=0
- What's the probability that j gains a link at step t+1?
  - Node t+1 links with prob p to an earlier node: so probability of linking to j is 1/t
  - Node t+1 links with prob 1-p to a node proportionally to its links. Total number of links is t of which Xj(t) point to j: Xj(t)/t so:

$$\frac{p}{t} + \frac{(1-p)X_j(t)}{t}$$

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And more formal…

- If we use a continuous function $x_j(t)$

- Initial condition $x_j(j)=0$
- Growth condition:

$$\frac{dx_j}{dt} = \frac{p}{t} + \frac{(1-p)x_j}{t}$$

- Let's use q=1-p
- Divide both sides by $p+qx_j$

$$\frac{dx_j}{dt} = \frac{p}{t} + \frac{qx_j}{t}$$

$$\frac{1}{p+qx_j}\frac{dx_j}{dt} = \frac{1}{t}$$

- Integrating

$$\int \frac{1}{p+qx_j}\frac{dx_j}{dt}dt = \int \frac{1}{t}dt$$

$$\ln(p+qx_j) = q\ln t + c$$

- $A=e^c$

$$p+qx_j = At^q$$

$$x_j(t) = \frac{1}{q}(At^q - p)$$

- Initial xj(j)=0

$$0 = x_j(j) = \frac{1}{q}(Aj^q - p)$$

...

- hence: $A = p / j^q$

$$x_j(t) = \frac{1}{q}\left(\frac{p}{j^q}t^q - p\right) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right]$$

- For a given k, and a given t what fraction of nodes have at least k in-links at time t? x_j approximates j's in-links number so:
- For a given k and a given t what fraction of all functions xj satisfy xj(t)>=k?

...

$$x_j(t) = \frac{1}{q}\left(\frac{p}{j^q}t^q - p\right) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right] \geq k$$

$$j \leq t\left[\frac{q}{p}k + 1\right]^{-1/q}$$

- Out of x1, x2…xt at time t, the fraction of values j that satisfy this is

$$\frac{1}{t}t\left[\frac{q}{p}k + 1\right]^{-1/q} = \left[\frac{q}{p}k + 1\right]^{-1/q}$$

Proportional to k

...

- Fraction of xj which are at least k (F(k)) is proportional to $k^{-1/q}$
- Fraction of nodes with exactly k in-links (fk)
  - Take the derivative (approximation) –dF/dk

$$\frac{1}{q}\frac{q}{p}[\frac{q}{p}k+1]^{-1-1/q}$$

- So the fraction of nodes with k in-links is proportional to $k^{-(1+1/q)}$ ie, a powerlaw with exponent

$$1+\frac{1}{q}=1+\frac{1}{1-p}$$

Preferential Attachment

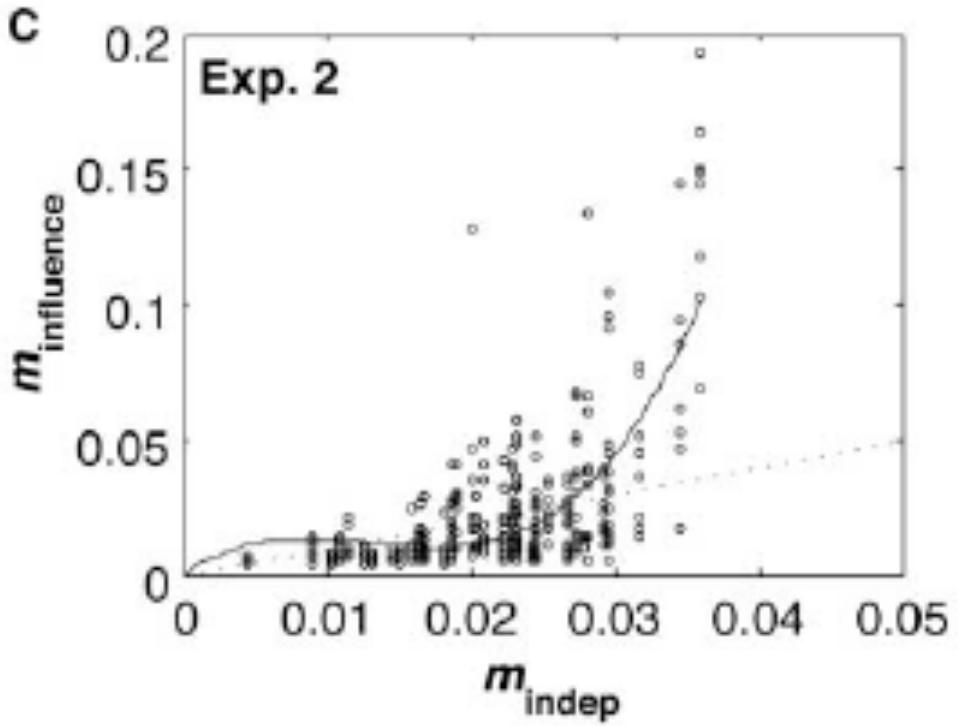
- What have we shown?
- There is a “copying” behaviour happening in these networks where node seem to emulate other nodes.
- This is shown true for selection of books, songs, web pages, movies etc.

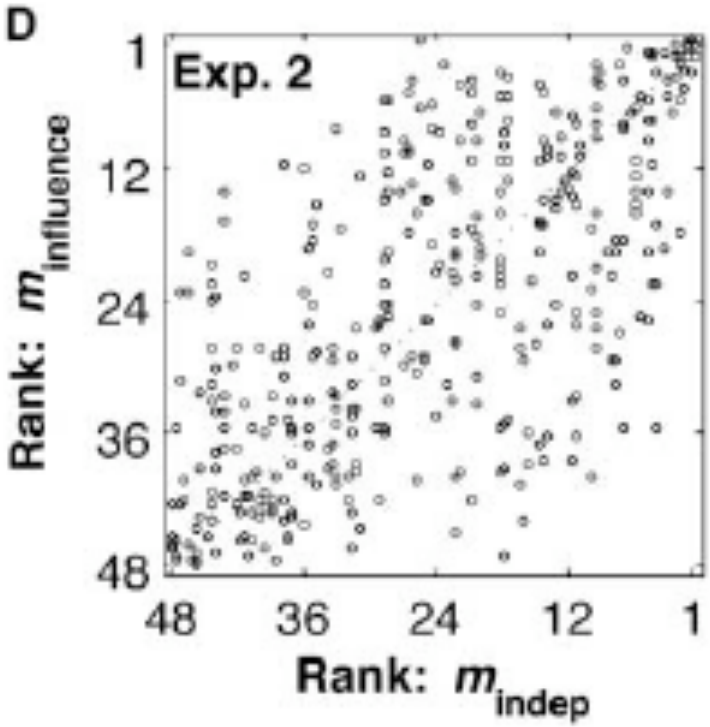
How predictable is the rich-get-richer process?

- Is the popularity of items in the power law predictable?
- Would a popular book still be popular if we go back in time and start the process again?
- Experiments show it would not…

Unpredictability [Salganik et al 06]

- 48 songs, 14,000 participants, 8 servers





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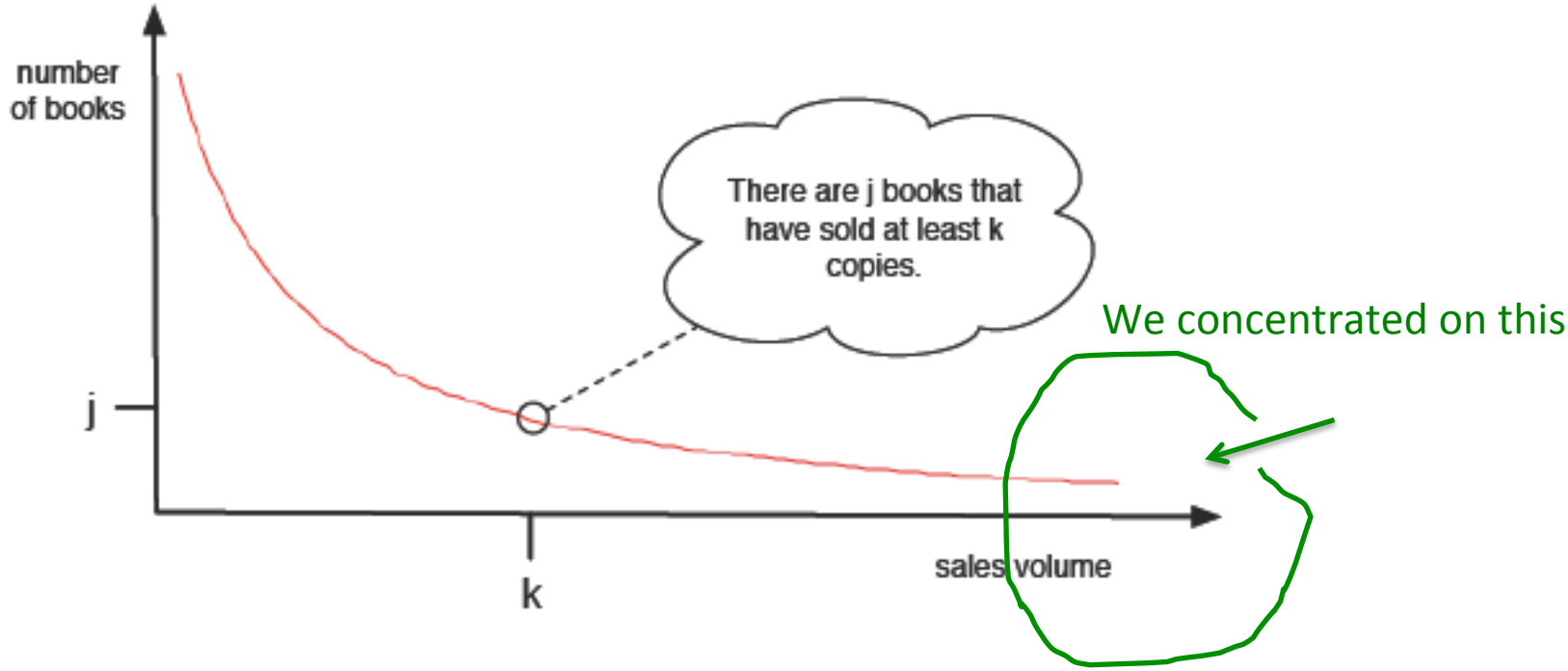
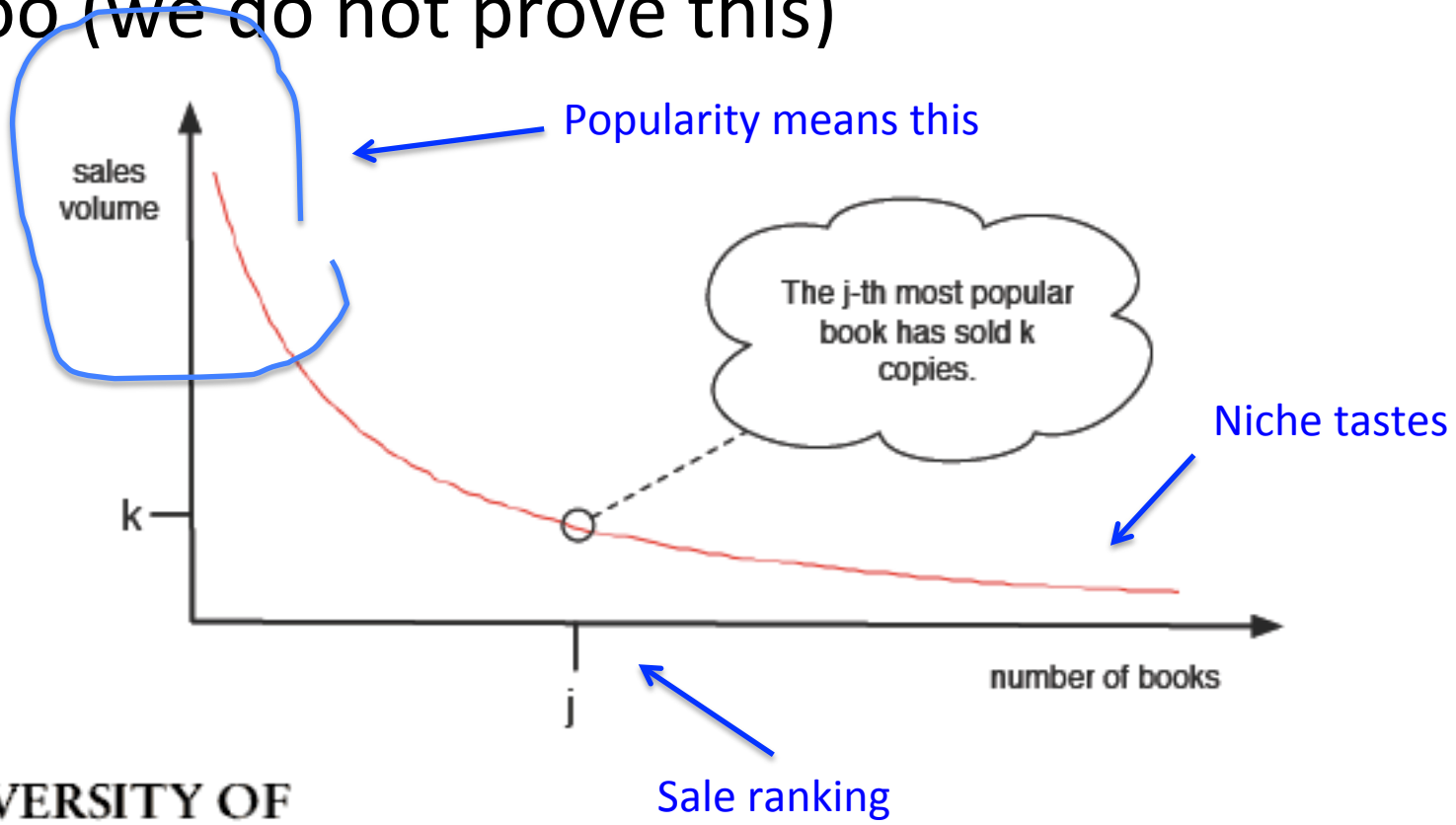
View of the curve

- The way we have seen the curve so far…



Let's transform the function

- If the initial function is a power law, this one is too (we do not prove this)



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- Chapter 13 and 18

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