Concurrent Quantum Strategies

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Abstract. A game-semantics foundation for quantum computation is presented. It draws on two lines of work: for its temporal dynamics, on concurrent games and strategies, based on event structures; for its quantum interactions, on the mathematical foundations of positive operators and completely positive maps. The two lines are married in the definition of quantum concurrent strategy, obtained via an operator generalisation of the conditions on a probabilistic concurrent strategy. The result is a compact-closed (bi)category of quantum games, whose finite configurations carry finite dimensional Hilbert spaces, and quantum strategies, whose finite configurations carry operators.

1 Introduction

We describe how concurrent strategies, based on event structures, can be extended with quantum effects. The motivation is threefold:

(1) Concurrent strategies have been advanced as a possible foundation for a generalised domain theory, in which concurrent games and strategies take over the roles of domains and continuous functions [1, 2]. A major reason has been to broaden the applicability of denotational semantics. It became important to see how concurrent strategies could be adapted to quantitative semantics, to probabilistic and quantum settings. Although a previous extension of concurrent strategies [3] did generalise quantum game theory as then developed [4], it did not provide a framework rich enough to represent quantum computation; it was insufficient to express the mix of classical and quantum behaviour of quantum lambda-calculi [5]. The extension to truly quantum strategies, has proved elusive. The pioneering attempt [6] placed severe restrictions on entanglement and the recent dynamic account of the execution of a quantum programming language via the geometry of interaction [7] is not compositional.

(2) As quantum information and computation become more sophisticated there is a need to reconcile quantum theory with causality [8], and put any attempt through the strictures of computer science, with its emphasis on compositionality, adequacy and full abstraction. Concurrent quantum strategies expose the causal nature of a quantum process as an event structure, and provide a means to compose quantum processes, in the manner of strategies.

(3) We aim to broaden the semantic basis for quantum programming. The breakthroughs in the denotational semantics of quantum programming of the last decade or so, *e.g.* [9, 5], have been based on insightful generalisations of those

categories used in quantum information, specifically by extending completely positive maps with extra structure to more fully address mixes of classical and quantum effects. But we are now seeing their limitations. Because the generalisations do not capture the dynamics of quantum programs directly it is hard to see whether the models are fully abstract or how they might be refined to fully abstract models. Concurrent quantum strategies form a marriage of concurrent strategies with completely positive maps. They extend to nonlinear features, through symmetry in games, and support the fine-tuning needed to obtain full-abstraction results, along the lines of [10, 11].

An adequate denotational semantics to the full quantum lambda calculus [12] in terms of concurrent quantum strategies is given in [13]. This paper is intended to complement that work by focussing on the fundamental, *linear* concurrent quantum strategies and how they generalise concurrent probabilistic strategies.

2 Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency* relation Con consisting of finite subsets of E. The relation $e' \leq e$ expresses that event e causally depends on the previous occurrence of event e'. That a finite subset of events is consistent conveys that its events can occur together. The relations satisfy several axioms:

$$\begin{aligned}
&[e] =_{\mathrm{def}} \{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \text{ implies } Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \text{ implies } X \cup \{e\} \in \mathrm{Con}.
\end{aligned}$$

There is an accompanying notion of state, or history, those events that may occur up to some stage in the behaviour of the process described. A *configuration* is a, possibly infinite, set of events $x \subseteq E$ which is: *consistent,* $X \subseteq x$ and X is finite implies $X \in \mathrm{Con}$; and *down-closed,* $e' \leq e \in x$ implies $e' \in x$.

Two events e, e' are considered to be causally independent, and called *concurrent* if the set $\{e, e'\}$ is in Con and neither event is causally dependent on the other; then we write $e \;\mathit{co}\; e'$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, plays a very important role. We write $[X]$ for the down-closure of a subset of events X. Write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations. (Sometimes we shall need to distinguish the precise event structure to which a relation is associated and write, for instance, $\leq_E$, $\rightarrowtriangle_E$ or co_E.)

A *map* of event structures $f : E \rightharpoonup E'$ is a partial function f from E to E' such that the image of a configuration x is a configuration fx and any event of fx arises as the image of a unique event of x. When f is total, then written $f : E \to E'$, it induces a bijection $x \cong fx$. Maps compose as functions.

A map $f : E \rightharpoonup E'$ reflects causal dependency locally, in the sense that if e, e' are events in a configuration x of E for which $f(e') \leq f(e)$ in E', then $e' \leq e$ also in E; the event structure E inherits causal dependencies from the event structure E' via the map f. Consequently, a map $f : E \rightharpoonup E'$ preserves concurrency. In general a map of event structures need not preserve causal dependency; a total map which does is called *rigid*.

Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$. Projection hides all events outside V. It is associated with a *partial-total factorization system*. Consider a partial map of event structures $f : E \rightharpoonup E'$. Let

$$V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\} .$$

Then f clearly factors into the composition

$$E \xrightharpoonup{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V. Note that any $x \in \mathcal{C}^\infty(E{\downarrow}V)$ is the image under f_0 of a *minimum* configuration, *viz.* $[x]_E \in \mathcal{C}^\infty(E)$. We call f_1 the *defined part* of the partial map f.

It is sometimes useful to build an event structure out of computation paths. A computation path is described by a partial order $(p, \leq_p)$ for which the set $\{e' \in p \mid e' \leq_p e\}$ is finite for all $e \in p$. We can identify such a path with an event structure in which the consistency relation consists of all finite subsets of events. Between two paths $p = (p, \leq_p)$ and $q = (q, \leq_q)$, we write $p \hookrightarrow q$ when $p \subseteq q$ and the inclusion is a rigid map of event structures.

Proposition 1. *A* rigid family $\mathcal{R}$ *comprises a non-empty subset of finite partial orders which is down-closed w.r.t. rigid inclusion, i.e.* $p \hookrightarrow q \in \mathcal{R}$ *implies* $p \in \mathcal{R}$. *A rigid family determines an event structure* $\mathrm{Pr}(\mathcal{R})$ *whose order of finite configurations is isomorphic to* $(\mathcal{R}, \hookrightarrow)$. *The event structure* $\mathrm{Pr}(\mathcal{R})$ *has events those elements of* $\mathcal{R}$ *with a top event; its causal dependency is given by rigid inclusion; and its consistency by compatibility w.r.t. rigid inclusion. The order isomorphism* $\mathcal{R} \cong \mathcal{C}(\mathrm{Pr}(\mathcal{R}))$ *takes* $q \in \mathcal{R}$ *to* $\{p \in \mathrm{Pr}(\mathcal{R}) \mid p \hookrightarrow q\}$.

The *pullback* of total maps of event structures is essential in composing strategies. We can define it via a rigid family of *secured bijections*. Let $\sigma : S \to B$ and $\tau : T \to B$ be total maps of event structures. There is a composite bijection

$$\theta : x \cong \sigma x = \tau y \cong y ,$$

between $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$ such that $\sigma x = \tau y$; because σ and τ are total they induce bijections between configurations and their image. The bijection is *secured* when the transitive relation generated on θ by $(s, t) \leq (s', t')$ if $s \leq_S s'$ or $t \leq_T t'$ is a partial order.

Theorem 1. *Let $\sigma : S \to B$ and $\tau : T \to B$ be total maps of event structures. The family $\mathcal{R}$ of secured bijections between $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$ such that $\sigma x = \tau y$ is a rigid family. The functions $\pi_1 : \mathrm{Pr}(\mathcal{R}) \to S$ and $\pi_2 : \mathrm{Pr}(\mathcal{R}) \to T$, taking a secured bijection with top to, respectively, the left and right components of its top, are maps of event structures.* $\mathrm{Pr}(\mathcal{R})$ *with π_1 and π_2 is the pullback of σ and τ in the category of event structures.*

Notation 2 From Proposition 1, finite configurations of the pullback of $\sigma : S \to B$ and $\tau : T \to B$ are order-isomorphic to the rigid family of secured bijections. Define $x \wedge y$ to be the configuration of the pullback which corresponds via this isomorphism to a secured bijection between $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$, necessarily with $\sigma x = \tau y$; any finite configuration of the pullback takes the form $x \wedge y$ for unique x and y.

3 Games and strategies

Both a game and a strategy will be represented by an *event structure with polarity*, which comprises (A, pol_A) where A is an event structure and a polarity function $pol_A : A \to \{+, -, 0\}$ ascribing a polarity $+$ (Player), $-$ (Opponent) or 0 (neutral) to its events. The events correspond to (occurrences of) moves. It will be technically useful to allow events of neutral polarity; they arise, for example, in a play between a strategy and a counterstrategy. Maps are those of event structures which preserve polarity. A *game* is represented by an event structure with polarities restricted to $+$ or $-$, with no neutral events.

Definition 1. In an event structure with polarity, with configurations x and y, write $x \subseteq^- y$ to mean inclusion in which all the intervening events are Opponent moves. Write $x \subseteq^+ y$ for inclusion in which the intervening events are neutral or Player moves. The *Scott order*, between $x, y \in \mathcal{C}^\infty(A)$, where A is a game, is defined by: $y \sqsubseteq_A x \iff y \supseteq^- x \cap y \subseteq^+ x$. (The order $\supseteq^-$ is converse to $\subseteq^-$.)

There are two fundamentally important operations on two-party games. One is that of forming the *dual game*. On a game A this amounts to reversing the polarities of events to produce the dual $A^\perp$. The other operation, a *simple parallel composition* $A \| B$, is achieved on games A and B by simply juxtaposing them, ensuring a finite subset of events is consistent if its overlaps with the two games are individually consistent; any configuration x of $A \| B$ decomposes into $x_A \| x_B$ where x_A and x_B are configurations of A and B respectively.

A *strategy in* a game A is a total map $\sigma : S \to A$ of event structures with polarity such that
(i) if $\sigma x \subseteq^- y$, where $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(A)$, there is a unique $x' \in \mathcal{C}(S)$ with $x \subseteq x'$ and $\sigma x' = y$;
(ii) if $s \rightarrowtriangle s'$ in S & $pol(s) = +$ or $pol(s') = -$, then $\sigma(s) \rightarrowtriangle \sigma(s')$ in A.
The first condition is one of receptivity, ensuring that the strategy is open to all moves of Opponent permitted by the game. The second condition ensures that

the only additional immediate causal dependencies a strategy can enforce are those in which a Player move awaits a move of Opponent.

A strategy *from* a game A *to* a game B is a strategy in the game $A^\perp \| B$. A map $f : \sigma \Rightarrow \sigma'$ of strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ is a map $f : S \to S'$ s.t. $\sigma = \sigma' f$; this determines isomorphism of strategies.

The conditions defining a strategy are precisely those needed to ensure that the copycat strategy behaves as identity w.r.t. composition.

3.1 Copycat

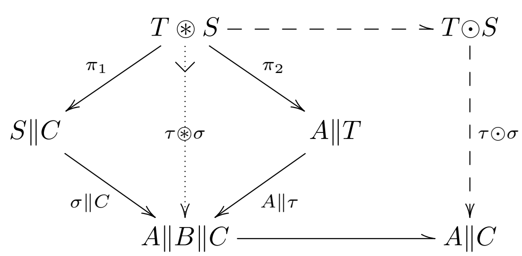
Let A be a game. The copycat strategy $\text{æ}_A : \mathbb{CC}_A \to A^\perp \| A$ is an instance of a strategy from A to A. The event structure $\mathbb{CC}_A$ is based on the idea that Player moves in one component of the game $A^\perp \| A$ always copy previous corresponding moves of Opponent in the other component. For $c \in A^\perp \| A$ we use $\bar{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. The event structure $\mathbb{CC}_A$ comprises $A^\perp \| A$ with extra causal dependencies $\bar{c} \leq c$ for all events c with $pol_{A^\perp \| A}(c) = +$; together with the additional causal dependency they generate a partial order; take a finite subset to be consistent in $\mathbb{CC}_A$ iff its down-closure w.r.t. the relation $\leq$ is consistent in $A^\perp \| A$.

Lemma 1. *Let A be a game. Let $x \in \mathcal{C}(A^\perp)$ and $y \in \mathcal{C}(A)$,*

$$x \| y \in \mathcal{C}(\mathbb{CC}_A) \quad \textit{iff} \quad y \sqsubseteq_A x \,.$$

3.2 Composition

Two strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ compose via pullback and hiding summarised below.



Ignoring polarities, by forming the pullback of $\sigma \| C$ and $A \| \tau$ we obtain the synchronisation of complementary moves of S and T over the common game B; subject to the causal constraints of S and T, the effect is to instantiate the Opponent moves of T in $B^\perp$ by the corresponding Player moves of S in B, and *vice versa*. Reinstating polarities we obtain the *interaction* of σ and τ

$$\tau \circledast \sigma : T \circledast S \to A^\perp \| B^0 \| C \,,$$

where we assign neutral polarities to all moves in or over B. Moves over the common game B remain unhidden. The map $A \| B \| C \rightharpoonup A \| C$ is undefined on B

and otherwise mimics the identity. Pre-composing this map with $\tau \circledast \sigma$ we obtain a partial map $T \circledast S \rightharpoonup A^\perp \| C$; it is undefined on precisely the neutral events of $T \circledast S$. The defined parts of its partial-total factorization yields

$$\tau \odot \sigma : T \odot S \to A^\perp \| C$$

on reinstating polarities; this is the *composition* of σ and τ.

Notation 3 For $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$, let $\sigma x = x_A \| x_B$ and $\tau y = y_B \| y_C$ where $x_A \in \mathcal{C}(A)$, $x_B, y_B \in \mathcal{C}(B)$, $y_C \in \mathcal{C}(C)$. Define $y \circledast x = (x \| y_C) \wedge (x_A \| y)$. This is a partial operation. Any finite configuration of $T \circledast S$ has the form $y \circledast x =_{\text{def}} (x \| y_C) \wedge (x_A \| y)$ for unique $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$.

3.3 A bicategory of strategies

We obtain a bicategory for which the objects are games, the arrows $\sigma : A \to\!\!\!\!+\!\!\!> B$ are strategies $\sigma : S \to A^\perp \| B$; with 2-cells $f : \sigma \Rightarrow \sigma'$ maps of strategies. The vertical composition of 2-cells is the usual composition of maps. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback and partial-total factorisation).

As $A^\perp \| B \cong (B^\perp)^\perp \| A^\perp$, a strategy $\sigma : A \to\!\!\!\!+\!\!\!> B$ corresponds to a strategy $\sigma^\perp : B^\perp \to\!\!\!\!+\!\!\!> A^\perp$. The bicategory of strategies is compact-closed; the *unit* $\emptyset \to\!\!\!\!+\!\!\!> A^\perp \| A$ and *counit* $A \| A^\perp \to\!\!\!\!+\!\!\!> \emptyset$ being the obvious modifications of the copycat strategy.

We can restrict the 2-cells to be rigid maps and still obtain a bicategory. This is important later, when the 2-cells for probabilistic and quantum strategies will be rigid.

A strategy $\sigma : S \to A$ is *deterministic* if S is deterministic, *viz.*

$$\forall X \subseteq_{\text{fin}} S.\ [X]^- \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $[X]^- =_{\text{def}} \{s' \in S \mid \exists s \in X.\ pol_S(s') = - \ \&\ s' \leq s\}$. In other words, a strategy is deterministic if consistent behaviour of Opponent is answered by consistent behaviour of Player. Copycat æ_A is deterministic iff the game A is *race-free*, *i.e.* if $x \subseteq^- y$ and $x \subseteq^+ z$ in $\mathcal{C}(A)$ then $y \cup z \in \mathcal{C}(A)$.

4 Quantum foundations

The category **FdHilb** of finite dimensional Hilbert spaces has as objects finite dimensional vector spaces, over the complex numbers $\mathbb{C}$, with an inner product $\langle \phi | \psi \rangle$, which is conjugate-linear in the first argument and linear in the second. Its arrows are linear maps between the underlying vector spaces. Any map $f : H \to K$ has an adjoint $f^\dagger : K \to H$ specified by $\langle f^\dagger(\phi) | \psi \rangle_H = \langle \phi | f(\psi) \rangle_K$.

The category **FdHilb** is symmetric monoidal w.r.t. the well-known operation of tensor product of Hilbert spaces, where the tensor unit I is the one-dimensional vector space, comprising the complex numbers $\mathbb{C}$ with inner product $\langle c | d \rangle = c^* \cdot d$ where c^* is the complex conjugate of c.

As observed in [9], the category **FdHilb** is compact-closed w.r.t. the operation of dual space. A finite dimensional Hilbert space H with inner product $\langle\phi|\psi\rangle_H$ has a *dual space* H^* given concretely as the vector space of linear maps from H to I; as any linear map from H to I can be represented by $\phi^* = \langle\phi| = \langle\phi|\_\rangle_H$, for some $\phi \in H$, its inner product is specified by taking $\langle\phi^*|\psi^*\rangle_{H^*} =_{\text{def}} \langle\psi|\phi\rangle_H$. The *unit* of the compact-closure $\eta_H : \mathrm{I} \to H^* \otimes H$ takes $1 \in \mathrm{I}$ to the identity matrix $\sum_i \langle i| \otimes |i\rangle$ w.r.t. an orthonormal basis $|1\rangle, \cdots, |n\rangle$, of size the dimension $\dim(H)$. The *counit*, $\epsilon_H : H \otimes H^* \to \mathrm{I}$ is given by the inner product and takes $\phi \otimes \psi$ to $\langle\psi|\phi\rangle$.

As is well-known, via this compact-closed structure, **FdHilb** admits a *partial trace* to form a traced monoidal category [14]. Given a map $f : H \otimes L \to K \otimes L$ in **FdHilb** its partial trace is a map $\mathrm{Tr}_L(f) : H \to K$. When H and K are the unit space, $\mathrm{tr}(f) = \mathrm{Tr}_L(\mathrm{I} \otimes f) : \mathrm{I} \to \mathrm{I}$, so is a scalar factor, which coincides with the usual trace of the matrix of the operator f.

We reserve the term *operator* for a linear map with the same domain and codomain. An operator preserving the inner product is called *unitary*; unitaries are associated with the undisturbed evolution of a quantum system. An operator $f : H \to H$ in **FdHilb** is *positive* if $\langle\phi|f(\phi)\rangle$ is a non-negative real for all $\phi \in H$. Write $\mathbf{Op}(H)$, and $\mathbf{Pos}(H)$, for the set of operators, respectively positive operators, on a finite dimensional Hilbert space H. Given operators f and g on a finite dimensional Hilbert space H we can define the *Löwner order* on $\mathbf{Op}(H)$ by taking $f \leq_L g$ iff $g - f$ is positive. Those $\rho \in \mathbf{Pos}(H)$ for which $\mathrm{tr}(\rho) \leq 1$ are called subdensity operators. They play the role of "mixed" quantum states to be thought of as subprobabilistic combinations of pure quantum states.[3]

In order to represent operations on quantum systems, such as those taking quantum states to quantum states, one derives a category **CPM** based on a rich class of completely positive maps.The objects of **CPM** are again finite dimensional Hilbert spaces but now a *completely positive map* $f : H \to K$ in **CPM** is a linear map $f : H^* \otimes H \to K^* \otimes K$ in **FdHilb** such that its correspondent $\bar{f} : H^* \otimes K \to H^* \otimes K$ in **FdHilb**, got via compact-closure, is a positive operator. We write $\mathrm{CJ} : f \mapsto \bar{f}$ for the 1-1 correspondence between completely positive maps $f \in \mathbf{CPM}(H, K)$ and positive operators $\bar{f} \in \mathbf{Pos}(H^* \otimes K)$; it is the well-known *Choi-Jamiolkowski isomorphism*.

We represent the Hilbert space $H^* \otimes H$ as that of matrices of the isomorphic space of operators $\mathbf{Op}(H)$; w.r.t. an orthonormal basis of H, an operator on H can be described as a vector $\sum_{i,j} c_{ij} |i\rangle\langle j|$ or as a matrix with entries c_{ij}. It is helpful conceptually and technically to regard a map $f : H \to K$ in **CPM** as taking operators on H to operators on K, so as a map $f : \mathbf{Op}(H) \to \mathbf{Op}(K)$ in **FdHilb**. A linear map $f : \mathbf{Op}(H) \to \mathbf{Op}(K)$ is *positive* if it takes positive operators to positive operators. Those $f : \mathbf{Op}(H) \to \mathbf{Op}(K)$ arising from completely positive maps are those for which $f \otimes \mathrm{id}_L$ is positive for any $\mathrm{id}_L : \mathbf{Op}(L) \to \mathbf{Op}(L)$. If a completely positive map f further satisfies $\mathrm{tr}(f(A)) \leq \mathrm{tr}(A)$ it is called a *superoperator*. Superoperators represent the physically realisable operations on

[3] The use of subdensity rather than density operators, where $\mathrm{tr}(\rho) = 1$, is natural in quantum systems which may stick with a non-trivial probability.

quantum states. In strategies, due to the presence of Opponent moves, we shall have call for completely positive maps which are not superoperators, for maps of **CPM** which act on positive operators which are not identifiable with the usual states of quantum mechanics.

We can describe a map in **CPM**, regarded as a map between operators, as function from matrices of the argument to matrices of the result. A qubit is represented by a vector so a column matrix in $\mathbb{C}^2$, w.r.t. the standard basis, and an operator on qubits by a 2-by-2 matrix. The measurement of a value 0 or 1 of a qubit in $\mathbb{C}^2$ is described, respectively, by the two superoperators $\mathsf{meas}_0, \mathsf{meas}_1 \in \mathbf{CPM}(\mathbb{C}^2, \mathrm{I})$ where

$$\mathsf{meas}_0 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \quad \text{and} \quad \mathsf{meas}_1 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d\,.$$

The two superoperators representing the creation of qubit initially set to 0 or 1, respectively, are given by $\mathsf{new}_0, \mathsf{new}_1 \in \mathbf{CPM}(\mathrm{I}, \mathbb{C}^2)$ where

$$\mathsf{new}_0 : a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{new}_1 : d \mapsto \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\,.$$

For U a unitary on H, the superoperator $\hat{U} \in \mathbf{CPM}(H, H)$ takes an operator $M \in \mathbf{Op}(H)$ to $UMU^\dagger$, which restricts to the usual application of a unitary operation to a quantum state.

Two maps in **CPM** play an early role. They derive from the unit and counit associated with the compact closure of **FdHilb**. Let H be a finite dimensional Hilbert. The unit η_H^{Hilb} of **FdHilb** viewed as completely positive map gives $1_\mathrm{H} \in \mathbf{CPM}(\mathrm{I}, \mathrm{H})$ which on argument 1 returns the identity operator id_H; it is not a superoperator. The counit $\epsilon_{H^*}^{Hilb}$, makes a completely positive map $\mathrm{tr}_H \in \mathbf{CPM}(H, \mathrm{I})$ which on an operator on H returns its trace.

The category **CPM** inherits its symmetric monoidal structure from **FdHilb**. Its compact-closed structure, $\eta_H^{cpm} \in \mathbf{CPM}(\mathrm{I}, H^* \otimes H)$ and $\epsilon_H^{cpm} \in \mathbf{CPM}(H \otimes H^*, \mathrm{I})$, is also induced by the compact-closed structure of **FdHilb** once we identify an object H in **CPM** with its space of operators $\mathbf{Op}(H)$:

$$\eta_H^{cpm} = \eta_{\mathbf{Op}(H)}^{Hilb} : \mathrm{I} \to \mathbf{Op}(H)^* \otimes \mathbf{Op}(H)\,;$$
$$\epsilon_H^{cpm} = \epsilon_{\mathbf{Op}(H)}^{Hilb} : \mathbf{Op}(H) \otimes \mathbf{Op}(H)^* \to \mathrm{I}\,.$$

More explicitly, w.r.t. an orthonormal basis $|1\rangle, \cdots |n\rangle$ of H, we have an orthornormal basis $E_{ij} =_{\mathrm{def}} |i\rangle\langle j|$ of $\mathbf{Op}(H)$. The unit η_H^{cpm} takes 1 to the identity $\sum_{i,j} E_{ij}^* \otimes E_{ij}$. The counit ϵ_H^{cpm} takes $v \otimes f$ to $f(v)$; explicitly, on the basis, it takes $E_{ij} \otimes E_{kl}^*$ to $\delta_{ik}\delta_{jl}$, described using the Kronecker delta.

CPM provides a conveniently rich category, supporting all quantum operations, and the diagrammatic reasoning which derives from compact-closure. In fact, **CPM** inherits a *dagger (a.k.a. strong) compact-closed* structure from **FdHilb** [9, 15]. The mathematics that follows could be explained more axiomatically w.r.t. dagger compact-closed categories enriched over cancellative commutative monoids; the enrichment is needed to support subtraction in the "monotone" condition on quantum strategies.

In what follows, often Hilbert spaces will come presented as explicit tensor products $A = \bigotimes_{a\in x} H(a)$ or $B = \bigotimes_{b\in y} H(b)$; in such cases we adopt the convention that $A \otimes B = \bigotimes_{c\in x\cup y} H(c)$ when $x \cap y = \emptyset$; the associated structural maps, symmetry and the left and right unit maps, will become identities.

5 From probabilistic to quantum strategies

Taking guidance from probabilistic strategies we are led to a definition of quantum strategy in a quantum game. Probabilistic strategies are recovered as a special case, when the quantum game is classical.

5.1 Probabilistic strategies

A *probabilistic strategy* in a game A is a strategy $\sigma : S \to A$ together with a probability valuation which endows S with probability, while taking account of the fact that in the strategy Player can't be aware of the probabilities assigned by Opponent. We should restrict to race-free games, precisely those for which copycat is deterministic, so that we have probabilistic identity strategies; it follows that S is race-free. Precisely, a *probability valuation* is a function $v : \mathcal{C}(S) \to [0, \infty)$ which is

(normalised) $v(\emptyset) = 1$;

(oblivious) if $x \subseteq^- y$ then $v(x) = v(y)$, for $x, y \in \mathcal{C}(S)$; and

(monotone) if $y \subseteq^+ x_1, \cdots, x_n$ then $d_v[y; x_1, \cdots, x_n] \geq 0$,

where the *drop function*,

$$d_v[y; x_1, \cdots, x_n] =_{\mathrm{def}} v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(x_I)\,,$$

$y, x_1, \cdots, x_n \in \mathcal{C}(S)$ and we take $x_I = \bigcup_{i\in I} x_i$ and $v(x_I) = v(\bigcup_{i\in I} x_i)$ when the union x_I is a configuration and 0 otherwise. Together the three conditions ensure that the range of a probability valuation stays within the interval $[0, 1]$.

When there are no Opponent moves in S, a probability valuation v makes S into a probabilistic event structure [16]. Then v extends to a continuous valuation w on the Scott-open[4] sets of $\mathcal{C}^\infty(S)$, one in which $w(\{y \in \mathcal{C}^\infty(S) \mid x \subseteq y\}) = v(x)$; this yields a 1-1 correspondence between valuations on configurations and continuous valuations on open sets [16]. Hence, by [17], the valuation v determines a probability distribution on the Borel sets. In this case $v(x)$ reads as $\mathrm{Prob}(x)$, the probability that the result includes the events of the finite configuration x. When S has Opponent moves, the reading of v involves conditional

[4] A *Scott-open* subset of configurations is upwards-closed w.r.t. inclusion and such that if it contains the union of a directed subset S of configurations then it contains an element of S. A *continuous valuation* is a function w from the Scott-open subsets of $\mathcal{C}^\infty(E)$ to $[0, 1]$ which is *((normalised)* $w(\mathcal{C}^\infty(E)) = 1$; *(strict)* $w(\emptyset) = 0$; *(monotone)* $U \subseteq V \implies w(U) \leq w(V)$; *(modular)* $w(U \cup V) + w(U \cap V) = w(U) + w(V)$; and *(continuous)* $w(\bigcup_{i\in I} U_i) = \sup_{i\in I} w(U_i)$, for *directed* unions.

probabilities. When $x \subseteq^+ y$ in $\mathcal{C}(S)$, provided $v(x) \neq 0$, the conditional probability of Player making moves y given x, is expressed by $\mathrm{Prob}(y \mid x) = v(y)/v(x)$. Player is *oblivious* to Opponent in the sense that if two events, $\oplus$, $\ominus$, of opposite polarities can occur at a configuration x, then not only are they causally independent there (because S is race-free), they are also probabilistically independent: $\mathrm{Prob}(\oplus|x) = \mathrm{Prob}(\oplus|x, \ominus)$. The *monotone* condition expresses that we assign non-negative probabilities to generalised intervals $[y; x_1, \cdots, x_n]$, consisting of those configurations which include the finite configuration y but do not include any of the finite configurations $x_1, \cdots, x_n$.

The composition of strategies extends to probabilistic strategies, $\sigma : S \to A^\perp \| B$ with valuation v_σ and $\tau : T \to B^\perp \| C$ with v_τ. A configuration of their interaction, of the form $y \circledast x \in \mathcal{C}(T \circledast S)$ for $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$, is assigned valuation $v_{\tau \circledast \sigma}(y \circledast x) = v_\tau(y) \cdot v_\sigma(x)$. Their composition $\tau \odot \sigma$ has probability valuation $v_{\tau \odot \sigma}(z) = v_{\tau \circledast \sigma}([z]_{T \circledast S})$ for z a finite configuration of $T \odot S$. The proof that we so obtain probability valuations relies heavily on properties of drop functions.

We obtain a bicategory of probabilistic strategies on race-free games. Because copycat is deterministic it can be assigned the constantly 1 valuation and remains an identity w.r.t. composition. The 2-cells are rigid maps of strategies which relate probability valuations across 2-cells via a *push-forward* result:

Lemma 2. *Let $f : \sigma \Rightarrow \sigma'$ be a rigid 2-cell between strategies $\sigma : S \to A$ and $\sigma' : S' \to A$. Let v be a probability valuation for σ. Taking, for $y \in \mathcal{C}(S')$, $(fv)(y) =_{\mathrm{def}} \sum_{x : fx = y} v(x)$ defines a probability valuation fv for σ', the* push-forward *of v.*

A 2-cell between probabilistic strategies σ, v to σ', v' is a rigid 2-cell $f : \sigma \Rightarrow \sigma'$ of strategies for which $(fv)(x') \leq v'(x')$, for all configurations $x' \in \mathcal{C}(S')$. Vertical and horizontal composition are inherited from strategies.

5.2 Quantum strategies

The probabilistic case provides loose guidelines in extending to quantum strategies. As usual probabilities are replaced by operators but there is now the question of their type, which we take as given by the game.[5]

A *quantum game* (A, H) comprises A, a race-free event structure with polarity, together with H assigning a finite dimensional Hilbert space $H(a)$ to each event $a \in A$. It is convenient to extend the assignment to any finite $y \subseteq A$ and write $H(y) =_{\mathrm{def}} \bigotimes_{a \in y} H(a)$; in particular $H(\emptyset) = \mathrm{I}$, the one-dimensional Hilbert space.

At this point we are guided to a quantum extension of strategies in which finite configurations of a strategy are positive operators of type given by the game. (In order to extend the probabilistic case and model the non-local nature

[5] We eschew the other obvious possibility in which the game also determines the operators because we want strategies to be quantum, not just probabilistic, in line with the quantum lambda-calculus [5] and earlier definition [6].

of quantum theory we do not assign operators just to events.) Once here, it is hard to escape the quantum generalisations of the first two conditions on quantum strategies. There is though the issue of how to generalise the remaining monotone condition and the drop function on which it is based. For reasons explained shortly we adopt a strong condition in which positivity is expressed by the Löwner order between operators.

A *quantum strategy* in a quantum game (A, H) is a strategy $\sigma : S \to A$ together with a *quantum valuation* for σ, an assignment $Q(x)$ of a positive operator on $H(\sigma x)$ to each $x \in \mathcal{C}(S)$, which is

(normalised) $Q(\emptyset) = 1$, the identity on I;

(oblivious) if $x \subseteq^- y$ then $Q(x) \otimes \mathrm{id}_{H(\sigma y \setminus \sigma x)} = Q(y)$; and

(monotone) if $y \subseteq^+ x_1, \cdots, x_n$ then $d_Q[y; x_1, \cdots, x_n] \geq_L 0$,

where $d_Q[y; x_1, \cdots, x_n] =_{\mathrm{def}} Q(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} \mathrm{Tr}_{H(\sigma x_I \setminus \sigma y)} Q(x_I)$.

Analogously to the probabilistic case, we take $x_I = \bigcup_{i \in I} x_i$ and $Q(x_I) = Q(\bigcup_{i \in I} x_i)$ when the union is a configuration and to be 0, the zero operator, otherwise. The role of the partial trace in the "monotone" condition is to hide the effects of operators outside the space $H(\sigma y)$, and reduce an operator on larger spaces $H(\sigma x_I)$ to one on $H(\sigma y)$.

Note a special case, when the quantum game is *classical*, in the sense that each $H(a)$ is the one-dimensional Hilbert space. Then, by the "monotone" condition, every non-zero operator $Q(x)$ is necessarily multiplication by a positive scalar, less than or equal to 1. Identifying operators on one-dimensional Hilbert space with scalars, we recover probabilistic strategies.

Another special case is that in which all the moves in the game A are those of Player. Then, by "monotone", each $Q(x)$ is a subdensity operator; so in this case states of an event structure, *viz.* configurations, are assigned *quantum states*. In moving from probabilistic to quantum strategies what were formerly probabilistic states have become quantum states. Without Opponent events we have uncovered a notion of quantum event structure (in some ways stricter, in others more general, than those defined previously [16].)

When the games contain Opponent events the operators $Q(x)$ need not have trace less than or equal to one; consider, for instance, the identity operator assigned to the singleton configuration of a strategy over a quantum game comprising a single Opponent event with a space of dimension 2. The operators $Q(x)$ will however be *1-bounded*—the output's norm never exceeds that of the input, see Proposition 3.

There is the issue of the choice of "monotone" condition. Why not weaken it to one in which the drop is reduced to a real number using the full trace operation? Because the weaker form is not preserved by composition of strategies.

Quantum strategies and superoperators We characterise those positive operators $Q(x)$ on $H(\sigma x)$ which are assigned to $x \in \mathcal{C}(S)$ in a quantum strategy $\sigma : S \to A$ w.r.t. a quantum game (A, H). This involves splitting a configuration x into its Opponent and Player events, x^- and x^+ respectively.

Lemma 3. *Let $\sigma : S \to A$ with Q be a quantum strategy in a quantum game (A, H). For any $x \in \mathcal{C}(S)$, $Q(x)$ is a positive operator for which*

$$\mathrm{Tr}_{H(\sigma x^+)}(Q(x)) \leq_L \mathrm{id}_{H(\sigma x^-)} \textit{ in the Löwner order.}$$

Given a positive operator Q on $N \otimes P$, for which $\mathrm{Tr}_P(Q) \leq_L \mathrm{id}_N$, it is easy to arrange a quantum strategy in which Q is assigned to a finite configuration.

Example 1. Let A be the quantum game $\ominus \rightarrowtriangle \oplus$ with $\ominus$ assigned Hilbert space N and $\oplus$ the space P. Imagine a quantum strategy $\sigma : S \to A$ where S has the same shape as A, *viz.* $\ominus \rightarrowtriangle \oplus$. It will necessarily assign id_N to the configuration $\{\ominus\}$ and the operators Q that can be assigned to $\{\ominus, \oplus\}$ are precisely those positive operators Q on $N \otimes P$, for which $\mathrm{Tr}_P(Q) \leq_L \mathrm{id}_N$.

Lemma 3 informs us how to rescale a quantum valuation to obtain subdensity operators whose trace is a probability valuation:

Proposition 2. *Let Q be a quantum valuation for a strategy $\sigma : S \to A$. Defining $\rho(x) = Q(x)/\dim(H(\sigma x^-))$ we obtain subdensity operators for all $x \in \mathcal{C}(S)$. Their trace $v(x) = \mathrm{tr}(\rho(x)) = \mathrm{tr}(Q(x)/\dim(H(\sigma x^-))$, for $x \in \mathcal{C}(S)$, yields a probability valuation v for σ.*

Via the Choi-Jamiolkowski isomorphism, the positive operators $Q(x)$ assigned to a strategy correspond to superoperators. In more detail, a positive operator $Q(x) \in \mathbf{Pos}(H(\sigma x))$ is an operator

$$Q(x) \in \mathbf{Pos}((H(\sigma x^-)^*)^* \otimes H(\sigma x^+))$$

which corresponds under Choi-Jamiolkowski to a completely positive map

$${}^-Q^+(x) : \mathbf{CPM}(H(\sigma x^-)^*, H(\sigma x^+))\,.$$

In quantum strategies, the operators $Q(x)$ are precisely those which correspond to superoperators ${}^-Q^+(x)$ —a corollary of the following refinement of the Choi-Jamiolkowski isomorphism (read $H(\sigma x^-)$ for N and $H(\sigma x^+)$ for P):

Lemma 4. *Let N and P be finite dimensional Hilbert spaces. Positive operators $Q \in \mathbf{Pos}(N \otimes P)$, for which $\mathrm{Tr}_P(Q) \leq_L \mathrm{id}_N$ in the Löwner order, correspond via the Choi-Jamiolkowski isomorphism to trace non-increasing completely positive maps $\mathrm{CJ}^{-1}(Q) \in \mathbf{CPM}(N^*, P)$, i.e. superoperators.*

The view espoused by Leifer and Spekkens of this refinement of the CJ-isomorphism is that it establishes a correspondence between *conditional* quantum states of P, conditional on N, and superoperators from N to P, understood as the quantum analogue of stochastic maps [18]. Their view is underscored here in strategies where the explicit contingency on the environment through Opponent moves leads to matching intuitions.

It follows as a corollary of Lemma 4 that any positive operator $Q(x)$, where $x \in \mathcal{C}(S)$, is necessarily 1-bounded:

Proposition 3. *A positive operator Q on $N \otimes P$ for which $\mathrm{Tr}_P(Q) \leq_L \mathrm{id}_N$ in the Löwner order is 1-bounded.*

Let $\sigma : S \to A$ be a strategy in a race-free game A, expanded to a quantum game (A, H). In summary, in moving from a probabilistic valuation v to a quantum valuation Q, w.r.t. $x \in \mathcal{C}(S)$, we replace: the valuation $v(x) \in [0, \infty)$, at $x \in \mathcal{C}(S)$, by a bounded positive operator $Q(x)$; that the value $v(x)$ is in $[0, 1]$, by $Q(x)$ being a 1-bounded positive operator; the order $\leq$ on the reals by the Löwner order $\leq_L$ on operators; that $v(x) = \mathrm{Prob}(x)$, when $x = x^+$, by $Q(x)$ being a sub-density operator, *i.e.* a quantum state; the conditional probability $v(x)$ by a conditional state $Q(x)$; multiplication in the reals by composition in **CPM**. Indeed, in the next section, composition in **CPM** will play a central role in the composition of quantum valuations, replacing the role of multiplication in composing probabilistic valuations.

6 Quantum strategies between games

We extend the operations on games, simple parallel composition and dual, to quantum games (A, H_A) and (B, H_B). Any finite subset z of $A \| B$ splits as $z = x \| y$ for unique finite subsets x of A and y of B; we take $H_{A\|B}(z) = H_A(x) \otimes H_B(y)$. A quantum game (A, H_A) has *dual* $(A^\perp, H_{A^\perp})$ where $H_{A^\perp}(z)$ is the dual Hilbert space $H_A(z)^*$, for any finite subset z of $A^\perp$.

6.1 Quantum valuations as completely positive maps

Before we compose quantum strategies we reformulate quantum valuations as maps in **CPM**. Let $\sigma : S \to A^\perp \| B$ be a quantum strategy with valuation Q_S. For $x \in \mathcal{C}(S)$ its image in the game $A^\perp \| B$ decomposes into $x_A \| x_B = \sigma x$, where $x_A \in \mathcal{C}(A)$ and $x_B \in \mathcal{C}(B)$. Thus $Q_S(x)$ is a positive operator on $H_{A^\perp \| B}(\sigma x) = H_A(x_A)^* \otimes H_B(x_B)$. As such, it corresponds via the Choi-Jamiolkowski isomorphism to a completely positive map

$$\boldsymbol{Q}_S(x) \in \mathbf{CPM}(H_A(x_A), H_B(x_B))\,.$$

(The map need not be a superoperator, but note, in general, it acts between conditional quantum states not merely mixed states.)

Via the compact-closure of **FdHilb**, we can reformulate the conditions required of a quantum strategy now with the corresponding assignments $\boldsymbol{Q}$ of completely positive maps. In the reformulation, when $x \subseteq^- y$, we shall require the *expansion* of a map $Q \in \mathbf{CPM}(H_A(x_A), H_B(x_B))$ to

$$\Uparrow^y (Q) = Q \otimes (1_{\mathrm{H_B(y_B \setminus x_B)}} \circ \mathrm{tr}_{\mathrm{H_A(y_A \setminus x_A)}})$$

in $\mathbf{CPM}(H_A(y_A), H_B(y_B))$. Similarly, in rephrasing the "monotone" condition, when $y \subseteq^+ x$, we need the *reduction* of a map $Q \in \mathbf{CPM}(H_A(x_A), H_B(x_B))$ to

$$\Downarrow_y (Q) = (\mathrm{id}_{H_B(y_B)} \otimes \mathrm{tr}_{H_B(x_B \setminus y_B)}) \circ Q \circ (\mathrm{id}_{H_A(y_A)} \otimes 1_{\mathrm{H_A(x_A \setminus y_A)}})$$

in $\mathbf{CPM}(H_A(y_A), H_B(y_B))$. The expansion and reduction operations on completely positive maps correspond via the CJ-isomorphism to the earlier operations (tensoring with an identity and partial trace) we saw earlier on positive operators. The conditions on a quantum valuation become

(normalised) $\boldsymbol{Q}(\emptyset) = 1 \in \mathbf{CPM}(\mathrm{I}, \mathrm{I})$;

(oblivious) if $x \subseteq^- y$ then $\Uparrow^y(\boldsymbol{Q}(x)) = \boldsymbol{Q}(y)$; and

(monotone) if $y \subseteq^+ x_1, \cdots, x_n$ then $d_{\boldsymbol{Q}}[y; x_1, \dots, x_n]$ is in $\mathbf{CPM}(H(x_A), H(x_B))$, where

$$d_{\boldsymbol{Q}}[y; x_1, \dots, x_n] = \boldsymbol{Q}(y) - \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \Downarrow_y (\boldsymbol{Q}(x_I)) \,,$$

again with the understanding that $\boldsymbol{Q}(x_I) = \boldsymbol{Q}(\bigcup_{i\in I} x_i)$ when the union is a configuration and the zero map otherwise.

6.2 Quantum copycat

Let (A, H_A) be a race-free quantum game. We can extend a copycat strategy $\text{æ}_A : \mathbb{C}_A \to A^\perp \| A$ with a quantum valuation. Recall a finite configuration of $\mathbb{C}_A$ comprises $x\|y$ where $x, y \in \mathcal{C}(A)$ are in the Scott order $y \sqsubseteq x$, so $y \supseteq^- x \cap y \subseteq^+ x$. We thus have the inclusion $(x \cap y)\|(x \cap y) \subseteq x\|y$ in $\mathcal{C}(\mathbb{C}_A)$. Define the quantum valuation of copycat as

$$\boldsymbol{Q}_{\text{æ}_A}(x\|y) = \Uparrow^{x\|y}(\mathrm{id}_{H_A(x\cap y)}) \in \mathbf{CPM}(H_A(x), H_A(y)) \,,$$

the expansion of the identity on $H_A(x \cap y)$ in $\mathbf{CPM}$. Its being a quantum valuation depends on A being race-free to validate the "monotone" condition.

6.3 Composition of quantum strategies

Consider quantum strategies $\sigma : S \to A^\perp\|B$, $\boldsymbol{Q}_\sigma$ and $\tau : T \to B^\perp\|C$, $\boldsymbol{Q}_\tau$. We assign a quantum valuation, $\boldsymbol{Q}_{\tau \circledast \sigma}$ to their interaction. Recall, the interaction

$$\tau \circledast \sigma : T \circledast S \to A^\perp \| B^0 \| C \,,$$

in which the events of B are reset to have neutral polarity, and are now additionally assigned the one-dimensional Hilbert space. Recall a configuration of $T \circledast S$ has the form $y \circledast x$, for unique $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$. We have

$$\boldsymbol{Q}_\sigma(x) \in \mathbf{CPM}(H_A(x_A), H_B(x_B)) \,,$$

where $x_A\|x_B = \sigma x$, with $x_A \in \mathcal{C}(A)$ and $x_B \in \mathcal{C}(B)$. Similarly,

$$\boldsymbol{Q}_\tau(y) \in \mathbf{CPM}(H_B(y_B), H_C(y_C)) \,,$$

for a decomposition $y_B\|y_C = \tau y$. Define

$$\boldsymbol{Q}_{\tau\circledast\sigma}(y \circledast x) =_{\mathrm{def}} \boldsymbol{Q}_\tau(y) \circ \boldsymbol{Q}_\sigma(x) \in \mathbf{CPM}(H_A(x_A), H_C(y_C)) \,.$$

The composition is well-defined because for $y \circledast x$ to be defined configurations x and y must share a common image, $x_B = y_B$, in the game B.

The composition $\tau \odot \sigma : T \odot S \to A^\perp \| C$ has quantum valuation $\boldsymbol{Q}_{\tau\odot\sigma}(z) = \boldsymbol{Q}_{\tau\circledast\sigma}([z]_{T\circledast S})$ for z a finite configuration of $T \odot S$.

In particular, the interaction $\tau \circledast \sigma : T \circledast S \to B^0$, of a strategy $\sigma : S \to B$ against a counterstrategy $\tau : T \to B^\perp$, has $\boldsymbol{Q}_{\tau\circledast\sigma}$ assign a non-negative real to each finite configuration of $T \circledast S$ to form a probability valuation, making $T \circledast S$ into a probabilistic event structure. We can push forward the probability valuation of $T \circledast S$ to a probability valuation of B (via the continuous valuation induced on the Scott open sets of $\mathcal{C}^\infty(T \circledast S)$) and consequently to a probability distribution over $\mathcal{C}^\infty(B)$, the possible resulting end positions of the play.

The proof that composition yields a quantum strategy mimics that in the probabilistic case, but generalising from the reals to quantum operations.

Theorem 4. *The composition of two quantum strategies is a quantum strategy and, up to isomorphism, has quantum copycat is its identity.*

6.4 A bicategory of quantum strategies

In analogy with the probabilistic case, 2-cells between quantum strategies are rigid maps of strategies which relate quantum valuations across 2-cells via a *push-forward* result:

Lemma 5. *Let $f : \sigma \Rightarrow \sigma'$ be a rigid 2-cell between strategies $\sigma : S \to A$ and $\sigma' : S' \to A$. Let $\boldsymbol{Q}$ be a quantum valuation for σ. Taking, for $y \in \mathcal{C}(S')$,*

$$(f\boldsymbol{Q})(y) =_{\mathrm{def}} \sum_{x : fx = y} \boldsymbol{Q}(x)$$

defines a quantum valuation $f\boldsymbol{Q}$ for σ', the quantum push-forward*.*

A bicategory of quantum strategies on race-free quantum games ensues. Its maps are quantum strategies. A 2-cell between quantum strategies from $\sigma, \boldsymbol{Q}$ to $\sigma', \boldsymbol{Q}'$ is a rigid 2-cell $f : \sigma \Rightarrow \sigma'$ of strategies for which $\boldsymbol{Q}'(x') - (f\boldsymbol{Q})(x')$ is completely positive for all configurations $x' \in \mathcal{C}(S')$. The bicategory of quantum strategies inherits compact-closure from that of plain strategies and **CPM**.

There are notable special cases.

Proposition 4. *The sub-bicategory of quantum strategies on games (A, H_A), in which H_A is constantly the one-dimensional Hilbert space, is isomorphic to the bicategory of probabilistic strategies.*

Consider the sub-(bi)categories in which the games consist purely of Player moves. When there is no additional quantum structure, the strategies in this case yield a sub-bicategory equivalent to *stable spans*, a model which underlies treatments of nondeterministic dataflow [19]; restricting to deterministic strategies on countable games, the sub-bicategory is equivalent to Berry's dI-domains and stable functions. Broadened to quantum games and quantum strategies, all the quantum assignments $\boldsymbol{Q}(x)$ will be a superoperators and we obtain a framework for quantum dataflow and, in particular, for the semantics of quantum flowcharts [20]. Of more interest though, are interpretations of higher-order languages such as quantum λ-calculi where interactions are more complicated and in which polarities play a more intricate role [13].

Acknowledgments We are grateful to the ERC for Advanced Grant ECSYM. Discussions with Frank Roumen and Benoît Valiron have been very helpful.

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