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Further analysis of ternary and 3-point univariate subdivision schemes

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Abstract

The precision set, approximation order and Hölder exponent are derived for each of the univariate subdivision schemes described in [3] and [4].

1 Introduction

In this report we present a method to calculate the Hölder exponent of a ternary subdivision scheme based on Rioul's method for binary schemes [7]. Then we go on to apply his method or our method to each of the univariate subdivision schemes described in [3]. We also derive the approximation order/precision set for each of the schemes. This report should be read as an annex to [3] and [4].

2 Hölder exponent

The Hölder spaces $\dot{C}^r$, $r \in \mathbb{R}_+$, generalize the spaces C^N of N-times continuously differentiable functions.

Definition A function $\varphi(x)$ is said to be *Lipschitz of order* α $(0 < \alpha \leq 1)$, $\varphi(x) \in \dot{C}^\alpha$, if we have for all x and $h \in \mathbb{R}$,

$$|\varphi(x+h) - \varphi(x)| \leq c|h|^\alpha, \tag{1}$$

where c is a constant. Since the spaces $\dot{C}^\alpha$, for $0 < \alpha \leq 1$, interpolate between C^0 and C^1, a $\dot{C}^\alpha$-function will be said to be *regular of order* α. Note that C^1 and $\dot{C}^1$ do not coincide; for example, a linear spline function is $\dot{C}^1$ but not differentiable at its knots.

Definition A function $\varphi(x)$ is regular of order $r = N + \alpha$ $(0 < \alpha \leq 1)$, $\varphi(x) \in \dot{C}^r$, if it is C^N and its Nth derivative $\varphi^{(N)}(x)$ is Lipschitz of order α, $\varphi^{(N)}(x) \in \dot{C}^\alpha$.

As already mentioned in the case $N = 1$, $\dot{C}^N$ contains functions that are not C^N. In fact "$\varphi(x)$ is $\dot{C}^N$" can be thought of as "$\varphi(x)$ is almost C^N," or "$\varphi(x)$ is $C^{N-\epsilon}$," since if $\varphi(x)$ is $\dot{C}^{N+\epsilon}$, for some $\epsilon > 0$, then $\varphi(x)$ is truly C^N. In this sense the limit function produced by the binary four point scheme is $C^{2-\epsilon}$[2].

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2.1 Method

In our framework Rioul's method is as follows: If we have a binary subdivision scheme S, with a mask α satisfying

$$\sum_{j\in\mathbb{Z}} \alpha_{2j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{2j+1} = 1, \tag{2}$$

we can prove $S^\infty P^0 \in \dot{C}^{m+\nu}$, $0 < \nu \leq 1$, by first deriving the mask of $\frac{1}{2}S_{m+1}$ and then computing $\|(\frac{1}{2}S_{m+1})^k\|_\infty$ for $k = 1, 2, 3, \ldots L$, where L is the first integer for which $\|(\frac{1}{2}S_{m+1})^L\|_\infty < 1$. If such an L exists and the mask of S_l satisfies (2) for all $l \leq m$ then $S^\infty P^0 \in \dot{C}^{m+\nu^k}$ for all $k \geq L$, where ν^k is given by

$$2^{-k\nu^k} = \left\| \left(\frac{1}{2} S_{m+1} \right)^k \right\|_\infty . \tag{3}$$

The derivation and proofs for the ternary method are given in [6]. In summary, if we have a ternary subdivision scheme S, with a mask α satisfying

$$\sum_{j\in\mathbb{Z}} \alpha_{3j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+1} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+2} = 1, \tag{4}$$

we can prove $S^\infty P^0 \in \dot{C}^{m+\nu}$, $0 < \nu \leq 1$, by first deriving the mask of $\frac{1}{3}S_{m+1}$ and then computing $\|(\frac{1}{3}S_{m+1})^k\|_\infty$ for $k = 1, 2, 3, \ldots L$, where L is the first integer for which $\|(\frac{1}{3}S_{m+1})^L\|_\infty < 1$. If such an L exists and the mask of S_l satisfies (4) for all $l \leq m$ then $S^\infty P^0 \in \dot{C}^{m+\nu^k}$ for all $k \geq L$, where ν^k is given by

$$3^{-k\nu^k} = \left\| \left(\frac{1}{3} S_{m+1} \right)^k \right\|_\infty . \tag{5}$$

In each of the methods above the integer m can be determined using Dyn's method described in [1] for the binary case and the extension to the ternary case described in [5].

3 Binary 3-point approximating scheme

The derivation of this scheme is given in [3]. Here we will simply give the mask: $\alpha = \frac{1}{16}[1, 5, 10, 10, 5, 1]$.

3.1 Precision set

Suppose $\{p_i\}, i \in \mathbb{N}_0$ is a sequence of points lying at equally spaced parameter values on a quartic. Without loss of generality we can write

$$\begin{aligned} P(t) = &\ \tfrac{1}{24} \ (1-t)(2-t)(3-t)(4-t)p_0 \\ + &\ \tfrac{1}{6} \ t(2-t)(3-t)(4-t)p_1 \\ - &\ \tfrac{1}{4} \ t(1-t)(3-t)(4-t)p_2 \\ + &\ \tfrac{1}{6} \ t(1-t)(2-t)(4-t)p_3 \\ - &\ \tfrac{1}{24} \ t(1-t)(2-t)(3-t)p_4 \end{aligned} \tag{6}$$

so that $P(i) = p_i$.

Now let $\{q_i\}$ be the sequence of points after one subdivision step:

$$q_{2i} = 5p_i + 10p_{i+1} + p_{i+2} \tag{7}$$
$$q_{2i+1} = p_i + 10p_{i+1} + 5p_{i+2} \tag{8}$$

and define

$$\begin{aligned} Q(t) = & \tfrac{1}{24} (1-t)(2-t)(3-t)(4-t)q_0 \\ + & \tfrac{1}{6} t(2-t)(3-t)(4-t)q_1 \\ - & \tfrac{1}{4} t(1-t)(3-t)(4-t)q_2 \\ + & \tfrac{1}{6} t(1-t)(2-t)(4-t)q_3 \\ - & \tfrac{1}{24} t(1-t)(2-t)(3-t)q_4. \end{aligned} \tag{9}$$

From (6),(7) we have

$$\begin{aligned} q_{2i} = & \quad 5P(i) + 10P(i+1) + P(i+2) \\ = & \quad \dots \qquad (10) \\ = & \tfrac{1}{24}(1-2i)(2-2i)(3-2i)(4-2i)(5p_0 + 10p_1 + p_2) \\ & +\tfrac{1}{6}2i(2-2i)(3-2i)(4-2i)(p_0 + 10p_1 + 5p_2) \\ & -\tfrac{1}{4}2i(1-2i)(3-2i)(4-2i)(5p_1 + 10p_2 + p_3) \\ & +\tfrac{1}{6}2i(1-2i)(2-2i)(4-2i)(p_1 + 10p_2 + 5p_3) \\ & -\tfrac{1}{24}2i(1-2i)(2-2i)(3-2i)(5p_2 + 10p_3 + p_4) \\ = & \quad Q(2i) \end{aligned} \tag{11}$$

Similarly from (6),(8) we can show

$$q_{2i+1} = Q(2i+1) \tag{12}$$

Hence $q_i = Q(i)$ and so the subdivided points also lie on a quartic. This cannot be shown for a quintic. Therefore the precision set of this scheme is the quartics and the approximation order is $O(h^5)$.

3.2 Hölder exponent

Using Rioul's method [7] we find that the Hölder exponent for this scheme is $\dot{C}^4$.

4 Ternary 3-point interpolating scheme

Hassan's PhD thesis [6] gives a much fuller analysis of this scheme. Here we will simply give the mask:

$$\alpha = [\dots, 0, 0, a, 0, b, 1-a-b, 1, 1-a-b, b, 0, a, 0, 0, \dots] \tag{13}$$

where $-\frac{1}{9} < a < 0$, $b = a + \frac{1}{3}$, and $(1-a-b) = \frac{2}{3} - 2a$.

4.1 Precision set

Suppose $\{p_i\}, i \in \mathbb{N}_0$ is a sequence of points lying at equally spaced parameter values on a straight line. Without loss of generality we can write

$$P(t) = (1-t)p_0 + tp_1 \tag{14}$$

so that $P(i) = p_i$.

Now let $\{q_i\}$ be the sequence of points after one subdivision step:

$$\begin{aligned} q_{3i} &= bp_i + (1-a-b)p_{i+1} + ap_{i+2} && (15) \\ q_{3i+1} &= p_{i+1} && (16) \\ q_{3i+2} &= ap_i + (1-a-b)p_{i+1} + bp_{i+2} && (17) \end{aligned}$$

As this is an interpolating scheme, we can immediately see that $q_{3i+1} = P(i+1)$. We also have

$$\begin{aligned} q_{3i} &= bP(i) + (1-a-b)P(i+1) + aP(i+2) \\ &= [(1-i)b - i(\tfrac{4}{3} - 2b) - (i+1)(b - \tfrac{1}{3})]p_0 + [ib + (i+1)(\tfrac{4}{3} - 2b) + (i+2)(b - \tfrac{1}{3})]p_1 \\ &= (\tfrac{1}{3} - i)p_0 + (i + \tfrac{2}{3})p_1 \\ &= P(i + \tfrac{2}{3}) && (18) \end{aligned}$$

and similarly we can show that

$$q_{3i+2} = P(i + \frac{4}{3}) \tag{19}$$

Hence the subdivided points lie on the original line. This is not true for a quadratic, and so this scheme has linear precision giving an approximation order of $O(h^2)$[1].

4.2 Hölder exponent

In order to calculate the Hölder exponent for this scheme, we used the method described in Section 2.1 to conduct the following experiment. First we selected 20 values for a equally distributed within the range for which the scheme is C^1 $(-\frac{1}{9} < a < 0)$. For each of these values we then calculated ν_k, given by

$$3^{-k\nu_k} = \left\| \left(\frac{1}{3} S_2 \right)^k \right\|_\infty , \tag{20}$$

for $k = 1, \ldots, 20$. Surprisingly, we found that

$$\left\| \left(\frac{1}{3} S_2 \right)^k \right\|_\infty = \left(\left\| \frac{1}{3} S_2 \right\|_\infty \right)^k \tag{21}$$

for all the values of a and k that we considered. This indicates that, for this particular mask, the method converges at the first step, giving

$$3^{-\nu_k} = \left\| \frac{1}{3} S_2 \right\|_\infty . \tag{22}$$

[1]For $b = \frac{2}{9}$ this scheme has quadratic precision (approximation order $O(h^3)$). However for this value of b we cannot show that the limit curve is C^1.

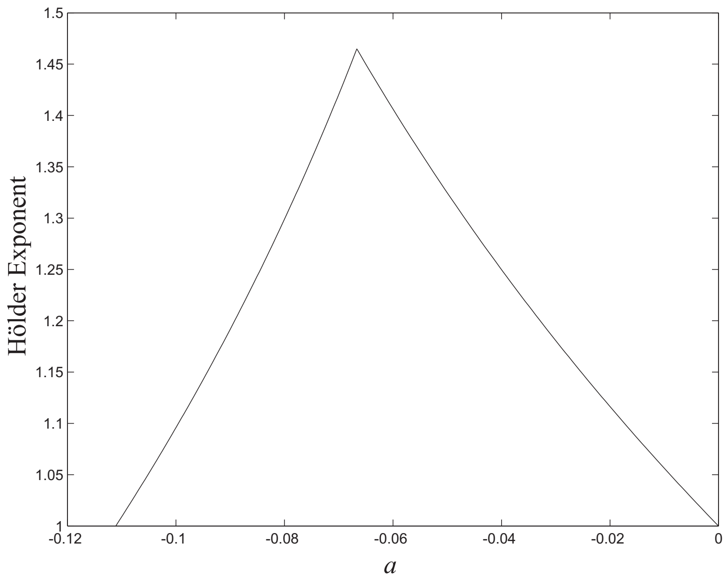

Figure 1: Graph of the Hölder exponent against a for the 3-point interpolating scheme. Note that the Hölder exponent is only defined for $-\frac{1}{9} < a < 0$.

Recalling

$$\alpha^{(1)} = 3\left[\ldots, 0, 0, a, -a, a+\frac{1}{3}, \frac{1}{3}-2a, a+\frac{1}{3}, -a, a, 0, 0, \ldots\right] \tag{23}$$

$$\alpha^{(2)} = 9\left[\ldots, 0, 0, a, -2a, 2a+\frac{1}{3}, -2a, a, 0, 0, \ldots\right], \tag{24}$$

we deduce that this scheme is $\dot{C}^r$, where r is given by

$$r = \begin{cases} 1 - \log_3(-9a) & -\frac{1}{9} < a \leq -\frac{1}{15} \\ 1 - \log_3(1+6a) & -\frac{1}{15} \leq a < 0. \end{cases} \tag{25}$$

Figure 1 shows a plot of the Hölder exponent against a. Notice that the highest smoothness is achieved at $a = -\frac{1}{15}$, which also gives the best trade-off for the magnitude of the third eigenvalue. For $a = -\frac{1}{15}$ the scheme is $\dot{C}^{1.46}$ (3sf).

5 Ternary 3-point approximating scheme

The derivation of this scheme is given in [3]. Here we will simply give the mask: $\alpha = \frac{1}{27}[1, 4, 10, 16, 19, 16, 10, 4, 1]$

5.1 Precision set

Suppose $\{p_i\}, i \in \mathbb{N}_0$ is a sequence of points lying at equally spaced parameter values on a cubic. Without loss of generality we can write

$$
\begin{aligned}
P(t) = \ & \tfrac{1}{6} \quad (1-t)(2-t)(3-t))p_0 \\
+ \ & \tfrac{1}{2} \quad t(2-t)(3-t)p_1 \\
- \ & \tfrac{1}{2} \quad t(1-t)(3-t))p_2 \\
+ \ & \tfrac{1}{6} \quad t(1-t)(2-t)p_3
\end{aligned} \tag{26}
$$

so that $P(i) = p_i$.

Now let $\{q_i\}$ be the sequence of points after one subdivision step:

$$q_{3i} = 10p_i + 16p_{i+1} + p_{i+2} \tag{27}$$

$$q_{3i+1} = 4p_i + 19p_{i+1} + 4p_{i+2} \tag{28}$$

$$q_{3i+2} = p_i + 16p_{i+1} + 10p_{i+2} \tag{29}$$

and define

$$
\begin{aligned}
Q(t) = \ & \tfrac{1}{6} \quad (1-t)(2-t)(3-t))q_0 \\
+ \ & \tfrac{1}{2} \quad t(2-t)(3-t)q_1 \\
- \ & \tfrac{1}{2} \quad t(1-t)(3-t))q_2 \\
+ \ & \tfrac{1}{6} \quad t(1-t)(2-t)q_3
\end{aligned} \tag{30}
$$

From (26),(27) we have

$$
\begin{aligned}
q_{2i} = \ & 10P(i) + 16P(i+1) + P(i+2) \\
= \ & \ldots \qquad (31) \\
= \ & \tfrac{1}{6}(1-2i)(2-2i)(3-2i)(10p_0 + 16p_1 + p_2) \\
& +\tfrac{1}{2}2i(2-2i)(3-2i)(4p_0 + 19p_1 + 4p_2) \\
& -\tfrac{1}{2}2i(1-2i)(3-2i)(p_0 + 16p_1 + 10p_2) \\
& +\tfrac{1}{6}2i(1-2i)(2-2i)(10p_1 + 16p_2 + p_3) \\
= \ & Q(2i)
\end{aligned} \tag{32}
$$

Similarly from (26),(28), and (29) we can show

$$q_{2i+1} = Q(2i+1) \tag{33}$$

$$q_{2i+2} = Q(2i+2) \tag{34}$$

Hence $q_i = Q(i)$ and so the subdivided points also lie on a cubic. This cannot be shown for a quartic. Therefore the precision set of this scheme is the cubics and the approximation order is $O(h^4)$.

5.2 Hölder exponent

Using the method described in Section 2.1 we find that the Hölder exponent for this scheme is $\dot{C}^3$.

Scheme	$O(h^{n+1})/\Pi_n$	Support Size	C^n	$\dot{C}^n$
Binary 3-point approximating	4	5	3	4
Ternary 3-point interpolating	1	4	1	1.46
Ternary 3-point approximating	3	4	2	3

Table 1: Comparison of the main properties of the schemes. The highest Hölder Exponent has been given for the interpolating scheme, rounded down to three significant figures. The Hölder Exponent for the approximating schemes are exact.

6 Summary

Table 1 gives a comparison of the main properties of the schemes. The highest Hölder Exponent has been given for the interpolating scheme ($a = \frac{1}{15}$), rounded down to three significant figures. The Hölder Exponent for the approximating schemes are exact.

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