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# Power-domains, Modalities and the Vietoris Monad

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## *Introduction*

It is perhaps possible to divide, rather crudely, the syntax-directed approaches to programming language semantics into two classes, the "denotational" and the "proof-theoretic". A denotational semantics corresponds (more or less) to model theory; at least *in the* sense that it proceeds by studying objects that represent the interpretations of (denote) syntactic constructs in the programming language. It follows that there are approaches which do not result in a partial–order semantics, but which nevertheless still carry much of the denotational flavour. Work I believe can be placed in this category includes that on semantics done by Martin-Löf (*cf.* Martin-Löf [1982]), and the method of giving a functional programming language a semantics in the (typed) $\lambda$-calculus.

In the second, proof-theoretical, approach the idea is rather to make assertions about (fragments of) the program in a suitable assertion language, and then to use a set of proof rules to combine these into some global assertion about program behaviour. I would include in this category such techniques as the use of Floyd-Hoare logics, and the various modal languages which are chiefly used for describing non-determinism and concurrency. Indeed, it is with one of these modal languages that this paper is chiefly concerned.

There are, of course approaches that do not lie snugly in either of these glib categories. Operational semantics à la Plotkin [1981a] uses proof rules to build up a denotation of a process as a transition system, and so seems to combine elements of both philosophies, while Martin-Löf type theory, with its strong proof-theoretic emphasis, does not really lie securely in the first category.

This paper is based on a different viewpoint which also has the effect of linking the two approaches. The results discussed here stem from the work done in recent years by Beeson, Fourman, Grayson, Hyland, Johnstone, Scott, Wraith, and others (*cf.* Fourman & Grayson [1982], Fourman & Scott [1979], Johnstone [1982]) on locales as formal spaces. We show that this approach provides a way in which we can hope to use a proof-theoretical semantics to give us a denotational one. In the first part of this paper we review those aspects of the general theory which are needed for the sequel. This material can all be found in Peter Johnstone's book on "Stone Spaces" (Johnstone [1982]), which I strongly recommend to the reader who is interested in finding out more about locales than I have space to include here.

The second part of the paper introduces an essentially modal construction on locales, which can be used to obtain a description of the various power-domains of algebraic cpo's. In fact I shall content myself with only giving explicitly the construction of the Plotkin power-domain, the constructions of the Smyth and Hoare power-domains being left as (relatively easy) exercises for the active reader. The point of the construction given here is not that it is in essence new – in fact it is easy to view it either as a reformulation of the power-space construction of Smyth [1983], or as a variant of the modal construction found in Winskel [1985] – but that by giving a presentation of the powerdomain by means of an algebraic description of its lattice of open sets we are able to transform the recursive domain equations and descriptions for semantic maps of a conventional denotational semantics using power-domains directly into proof rules. In this paper this is not, however discussed in any detail, and the reader is referred to Robinson [1986] for a more complete discussion.

In the third section we discuss in greater depth and with a greater degree of technicality the view of power-domains as free non-deterministic algebras. We use results of Johnstone [1985] to show that any algebra for the vietoris monad carries a semi-lattice structure, and results of Plotkin and Hennessy to show that over algebraic domains the continuous semi-lattices and the algebras for the vietoris monad coincide, an extension of Johnstone's results, and obtained by simpler means. The result does not hold for general locales. We then use this fact to indicate how to deduce a complete set of proof rules, which hold when modelling non-determinism in algebraic domains, but which certainly fail for a semantics which allows general localic semi-lattices.

Finally, we include a brief discussion of the relation between our present work and that of Winskel as section four.

I would like to take this opportunity to thank Martin Hyland, Peter Johnstone and Glynn Winskel for many stimulating discussions, and for the interest they have all shown during the preparation of this paper.

## 1 *An introduction to locales*

Locales, the primary objects of our study in this section, can be thought of as the Lindenbaum algebras for constructive propositional theories, or more precisely for intuitionistic propositional theories as formalised by Heyting, but with disjunction over arbitrary sets of formulae and without $\rightarrow$ as a basic propositional operator. Formally, a locale is defined as follows:

DEFINITION A locale, or complete Heyting algebra, is a partially-ordered set $(A,\leq)$ with finite meets and arbitrary suprema, in which finite conjunction distributes over arbitrary disjunction:

$$a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}.$$

I am following the usual convention in which finite includes empty. An immediate consequence of this (which otherwise it would be necessary to state explicitly) is that every locale has a top element ($\top = \bigwedge \phi$) as well as a bottom ($\perp = \bigvee \phi$). Note that a locale is a complete lattice (though not necessarily a completely distributive one), and that the distributivity of $\wedge$ over colimits means (by the adjoint functor theorem) that $a \wedge (\,)$ has a right adjoint, which we can interpret as implication (since the adjunction $a \wedge (\,) \dashv a \rightarrow (\,)$ holds iff for all $b$ and $c \rightarrow (a \wedge b \leq c \text{ iff } b \leq a \rightarrow c)$.

The operations $\wedge$ and $\bigvee$ can be regarded as (infinitary) algebraic (though $\bigvee$ splits into an infinite family of operators, one for each arity), and the distributivity is just an algebraic axiom (or rather, an axiom schema). It follows that we can construct the free complete Heyting algebra on a set of basic propositions, subject, if we wish, to inequalities between terms. For example, if we have a program logic given by proof rules, we can regard the rules as giving inequalities in the complete Heyting algebra of propositions of the logic.

A map of locales is given by a $\wedge$ and $\bigvee$ (hence $\top$ and $\perp$) preserving map $p^*: \mathrm{B} \to \mathrm{A}$. We follow Johnstone in making the convention that maps in the category of locales go in the geometric direction, and so $p^*$ gives us a map $\mathrm{A} \to \mathrm{B}$ in the category **Loc**. It is easy to check using this definition that **Loc** has a terminal object, the two-point locale $\mathbf{2} = \{\top = 1,\ \perp = 0\}$.

Models of the theory represented by a locale A are given by the $\wedge\bigvee$-preserving maps $\mathrm{A} \to \mathbf{2}$; or in more categorical terms, models are given by the representable functor $\mathrm{pt} = \mathbf{Loc}(\mathbf{2},\ )$. It is easy to check that if for $a \in \mathrm{A}$ we define

$$U_a = \{\, p \in \mathrm{pt}(\mathrm{A}) \mid p^*(a) = 1 \,\},$$

then $U_{\_}$ is a locale map from the powerset of pt(A) to A. Translated into more conventional terms, this tells us that $\{\, U_a \mid a \in \mathrm{A} \,\}$ forms a topology on pt(A), and so, after the easy verification that locale maps give rise to continuous functions we can treat pt(_) as a functor from **Loc** to **Top**, the category of topological spaces.

Conversely, given any topological space, X, its lattice of open sets forms a locale $\Omega$X, and continuous maps of spaces give rise to locale maps between the corresponding topologies. This does not give an equivalence; however the two functors pt and $\Omega$ are adjoint.

1.1 LEMMA $\qquad \Omega \dashv \mathrm{pt} : \mathbf{Loc} \longrightarrow \mathbf{Top}$.

*Proof.* Given a continuous map $f: \mathrm{X} \to \mathrm{pt}\mathrm{Y}$ we must define an $\wedge\bigvee$-preserving map $f^*: \mathrm{Y} \to \Omega\,\mathrm{X}$, in order to give a locale map $\Omega\mathrm{X} \to \mathrm{Y}$. So, given $y \in \mathrm{Y}$, we take $f^*(y)$ to be $f^{-1}\,U_y$, where $U_y$ is defined as above.
Coversely, given $f^*: \mathrm{Y} \to \Omega\mathrm{X}$ we define a continuous map $f$ on $x \in \mathrm{X}$ by

$$f(x)\,(y) = \top \qquad \text{iff} \qquad x \in f^*(y).$$

We leave verification of the necessary equivalences to the reader (alternatively, see Johnstone [1982] II.1). ☐

DEFINITION If the unit of this adjunction $\eta_{\mathrm{X}}: \mathrm{X} \to \mathrm{pt}(\Omega\mathrm{X})$ is an isomorphism, then we say that the space X is sober.

1.2 LEMMA A topological space X is sober if and only if each irreducible closed subset of X is the closure of a unique point. In particular, any sober space is $\mathrm{T}_0$.

*Proof.* The irreducible closed subsets of X are the closed subets of X which cannot be expressed as the union of any finite number of proper closed subsets. Their complements therefore generate the prime principal ideals in $\Omega$(X); but the prime principal ideals correspond naturally to the elements of pt($\Omega$X). ☐

Furthermore

1.3 LEMMA  If A is a locale, then pt(A) is sober.

*Proof.* It is an immediate consequence of the triangle identities for adjunction that $\eta_{\mathrm{pt}}$ is monic. Hence it suffices to show that it is also surjective.

Let $p$ be a point of $\Omega(\mathrm{ptA})$. Then $p^* : \Omega(\mathrm{ptA}) \longrightarrow \mathbf{2}$, and so corresponds to a prime filter $\mathcal{F}$ in the locale $\Omega(\mathrm{ptA})$ that is inaccessible to directed joins. The natural (counit) map $\varphi^* : \mathrm{A} \longrightarrow \Omega(\mathrm{ptA})$ is surjective, so we look at $(\varphi^*)^{-1}\mathcal{F}$.

$(\varphi^*)^{-1}\mathcal{F}$ is a prime filter on A, and it is also inaccessible to directed joins; it thus gives rise to a point $q$ of A. We claim that $\eta q = p$.

By definition of $\eta$, $(\eta q)^*(U) = \top$ $(U \in \Omega(\mathrm{ptA}))$ if and only if $q \in U$. But $q \in U$ if and only if there is an $a \in \mathrm{A}$ with $\varphi^*(a) = U$ and $q(a) - \top$, and, since q was given by $(\varphi^*)^{-1}(\mathcal{F})$, this is equivalent to $U$ being in $\mathcal{F}$. Hence $(\eta q)^* = p^*$ as required. □

We relate these spaces to the usual categories of cpo's by means of the specialisation ordering and Scott topologies.

We can define the specialisation ordering on the points of any space by $x \leq y$ if and only if $x \in \mathrm{cl}\{y\}$. If the space is $\mathrm{T}_0$, in particular if it is sober, then this pre–order is a partial order. In the case of a sober space $\mathrm{X} = \mathrm{ptA}$ we have $x \leq y$ iff $x^* \leq y^*$ in the usual functional ordering (since $x \in \mathrm{cl}\{y\}$ if and only if every open containing $x$ also contains the point $y$).

1.4 LEMMA  $\mathrm{X} = \mathrm{pt\ A}$ has arbitrary directed sups in the specialisation ordering. The ordering has a bottom element iff X is irreducible.

*Proof.* Let S be a subset of X directed in the specialisation ordering. Then $\bigvee \mathrm{S}$ is defined pointwise via $(\bigvee \mathrm{S})^*(a) = \bigvee\{\, p^*a \mid p \in \mathrm{S}\}$. This clearly preserves $\bigvee$'s, and it also preserves $\wedge$, since $\wedge$ commutes with directed sups in $\mathbf{2}$.

The second statement is obvious. □

DEFINITION  If X is a partially–ordered set, then the Scott topology $\Sigma(\mathrm{X})$ on X has as its open sets those subsets of X which are upwards closed in the partial order and inaccessible by directed joins.

When the order relation on a set X is given by the specialisation ordering for some topology, the Scott topology is not, of course, completely unrelated to the original.

1.5 LEMMA  If $\mathrm{X} = \mathrm{ptA}$ is sober, then the subsets of X which are open in its given topology also form open sets in the Scott topology on X.

*Proof.* It is easy to see that the open subsets of X are upwards closed in the specialisation preorder on X. Furthermore, since X is sober, any open of X is of the form $U_a$ for some $a \in A$, so if $\bigvee \mathrm{S} \in U_a$ for some directed set S, then $\bigvee\{p^*a \mid p \in \mathrm{S}\}$ and the result follows from the compactness of $\mathbf{2}$. □

1.6 COROLLARY  If X is sober, then $\Sigma\mathrm{X}$ is the finest topology on X compatible with the specialisation pre-order.

It is, however, important to note that this does not imply that the Scott topology on X is necessarily sober.

Thus, we can obtain a predomain (a domain without $\perp$) from a locale. In order to go the other way, we look at the locale which is the Scott topology on the domain. Here we have:

1.7 LEMMA   If D is an algebraic cpo, then it is sober in its Scott topology (or equivalently, it is the space of points of its Scott topology).

*Proof.* The Scott topology on D has as open sets the unions of sets of the form $x\uparrow$, where $x$ is a finite (isolated) element of D (we shall call the collection of these isolated elements $\mathcal{B}[D]$). The Scott topology on D is therefore given by the same locale as the upwards closed topology on $\mathcal{B}[D]$. Now a point of a locale is given by a completely prime filter (the set of elements sent to $\top$). Suppose $\mathcal{F}$ is such a filter on $\Omega\,\mathcal{B}[D]$, then $\{\, x \mid \exists O \in \mathcal{F}.\, x\uparrow \subseteq O \,\}$ is upwards closed and downwards directed in $\mathcal{B}[D]$, and furthermore completely determines $\mathcal{F}$. It follows that the elements of $\mathrm{pt}(\Sigma D)$ correspond precisely to ideals in $\mathcal{B}[D]$, in other words to the elements of D. $\square$

More generally, it is easy to show that continuous cpo's are sober in their Scott topologies. It is also known that there are other cpo's which possess this property. However, the question of finding an order-theoretic characterisation of the complete family remains open.

To sum up, locales are the natural models for (infinitary) intuitionistic propositional logic. Given a set of proof rules and some basic propositions we can construct the locale of provable equivalence classes of propositions. The points of a locale always form a predomain, and if the predomain is actually an algebraic cpo, then there is a very close correspondence between the domain and its Scott topology.

There remains one further point, and a warning. Implicit in the above is a definition of cpo which allows for arbitrary directed sups, and not just colimits of $\omega$-chains. I believe that this can be justified constructively by working in a setting (such as the effective topos, or Martin-Löf type theory, Hyland [1982], Martin-Löf [1982]) which ensures that you can only ever do indexing over those sets whose suprema you might legitimately wish to take. By contrast, restricting oneself to countable but otherwise arbitrary directed sups seems rather hard to justify if it is intended as a measure to introduce some degree of constructibility.

However, it must be admitted that if one does wish to work in the category of cpo's which only have joins of $\omega$-chains, then the approach given above does not function well. The "obvious" fix, of allowing only $\omega$-indexed (or, more generally, $\alpha$-indexed) sups in the definition of an $\omega$-locale ($\alpha$-locale) does not work. One can go on to define $\omega$-spaces and $\omega$-sobriety as before. However, the points of these spaces still have arbitrary directed sups in the specialisation ordering. What this comes down to saying is that you cannot restrict the colimit properties of your category of models simply by limiting your syntax in this way.

## 2 *Vietoris locales and power-domains*

Peter Johnstone has shown in Johnstone [1985] how to define an endofunctor on **Loc** which generalises the Vietoris power-space construction. Given a locale A we construct V(A), the Vietoris locale of A as the free locale on a basis of tokens $\Box a$, $\Diamond a$ (where a runs over the elements of A) subject to the following axioms:

1. if $a \leq b$ then $\Box a \leq \Box b$, and $\Diamond a \leq \Diamond b$
2. if S is directed in A then $\Box \bigvee S = \bigvee \{\Box s \mid s \in A\}$
3. (a) $\Box(a \wedge b) = \Box a \wedge \Box b$

   (b) $\Box \top = \top$
4. If S is any subset of A (including $\phi$), then
   $\Diamond \bigvee S = \bigvee \{\Diamond s \mid s \in A\}$
   (and hence $\Diamond \perp = \perp$.)
5. $\Box(a \vee b) \leq \Box a \vee \Diamond b$
6. $\Diamond(a \wedge b) \geq \Diamond a \wedge \Box b$.

I would like to stress that, despite the fact that I have used the suggestive modal notation, for the purposes of this construction, $\Box a$ should not be regarded as the result of applying the modal operator $\Box$ to the (abstract) proposition $a$, but as a single indivisible token. $\Box(a \wedge b)$ is thus a single token formed from two components whch are $\Box$ and the element $(a \wedge b)$ of the locale.

We refer the reader to Johnstone [1985] for an account of the way in which V extends to a functor on **Loc**, and of the monad it generates. We shall not need this information in this section.

We shall however be interested in $V^+(A)$, the strict Vietoris locale of A, obtained from V(A) by forcing $\Box \perp = \perp$ (or equivalently $\Diamond \top = \top$). Indeed, the principal result of this section is

2.1 THEOREM  If D is an algebraic cpo, then $V^+(\Sigma D)$ is the Scott topology on the Plotkin powerdomain of D.

We recall that if D is an algebraic cpo, with isolated elements $\mathcal{B}[D]$, then the Egli-Milner ordering $\preceq$ on M[D], the set of finite subsets of $\mathcal{B}[D]$ is defined as follows:

$$\begin{array}{lll} A \preceq_0 B & \text{iff} & \forall b \in B\ \exists a \in A\ \ a \leq b \\ A \preceq_1 B & \text{iff} & \forall a \in A\ \exists b \in B\ \ a \leq b \\ A \preceq B & \text{iff} & A \preceq_0 B \text{ and } A \preceq_1 B. \end{array}$$

Note that $\phi$ is an isolated point of M[D] under the Egli-Milner ordering. We shall write $M^*[D]$ for $M[D] \setminus \{\phi\}$.

The Plotkin power-domain of D ( Pl(D) ) is the completion by ideals of $M^*[D]$ with respect to the Egli-Milner ordering. It is thus an algebraic cpo whose isolated points are isomorphic to $(M^*[D], \preceq)$, and its Scott topology is the upwards closed topology on

$M^*[D]$. The Smyth and Hoare power-domains are obtained by similar constructions, also on $M^*[D]$, but using the orderings $\preceq_0$ and $\preceq_1$ respectively.

We shall deduce the theorem above from

2.2 THEOREM  If D is an algebraic cpo, then $V(\Sigma D) = \Sigma(Pl^+(D))$ where $Pl^+(D)$ is the completion by ideals of M[D].

The proof goes via the obvious seqaence of lemmas:

2.3 LEMMA  There is a locale map $\psi: V(\Sigma D) \to \Sigma Pl^+(D)$, given by

$$\psi^* \{a_1, \ldots, a_n\}\uparrow_{EM} = \Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \wedge \Diamond(a_1\uparrow) \wedge \ldots \wedge \Diamond(a_n\uparrow).$$

(As usual we identify the Scott topology on an algebraic domain with the upwards closed topology on its set of isolated elements.)

*Proof.* In this case there are no meets and joins between basic elements whose preservation we have to ensure. We just have to check that

$$\{b_1, \ldots, b_k\} \preceq \{a_1, \ldots, a_n\}$$

implies that

$$\begin{aligned}&\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \wedge \Diamond(a_1\uparrow) \wedge \ldots \wedge \Diamond(a_n\uparrow)\\ &\qquad \leq \Box(b_1\uparrow \vee \ldots \vee b_k\uparrow) \wedge \Diamond(b_1\uparrow) \wedge \ldots \wedge \Diamond(b_k\uparrow).\end{aligned}$$

But we have

$$\{b_1, \ldots, b_k\} \preceq_0 \{a_1, \ldots, a_n\}$$

if and only if

$$a_1\uparrow \vee \ldots \vee a_n\uparrow \leq b_1\uparrow \vee \ldots \vee b_k\uparrow,$$

and this in turn implies that

$$\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \leq \Box(b_1\uparrow \vee \ldots \vee b_k\uparrow)$$

by the monotonicity of $\Box$.

Similarly, the implication

$$\{b_1, \ldots, b_k\} \preceq_1 \{a_1, \ldots, a_n\} \quad \Rightarrow \quad \Diamond(a_1\uparrow) \wedge \ldots \wedge \Diamond(a_n\uparrow) \leq \Diamond(b_1\uparrow) \wedge \ldots \wedge \Diamond(b_k\uparrow)$$

follows from the monotonicity of $\Diamond$. $\Box$

I claim that $\psi$ has an inverse. In order to define it we have first to know that the elements

$$\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \wedge \Diamond(b_1\uparrow) \wedge \ldots \wedge \Diamond(b_k\uparrow)$$

where the $a_i$ and the $b_j$ are isolated elements of D, form a basis of $\mathrm{V}(\Sigma \mathrm{A})$ (in the strong sense that any element is a directed join of elements of this form).

To see this note that for any Scott open $O$:

$$\begin{aligned}\Box O &= \Box \bigvee\{a\uparrow \mid a \in O, a \text{ isolated}\} \\ &= \Box \bigvee\{(a_1\uparrow \vee \ldots \vee a_n\uparrow) \mid a_i \in O, a_i \text{ isolated}\} \\ &= \bigvee \Box\{(a_1\uparrow \vee \ldots \vee a_n\uparrow \mid a_i \in O, a_i \text{ isolated}\}\end{aligned}$$

since this is now a directed join, and also that

$$\begin{aligned}\Diamond O &= \Diamond \bigvee\{b\uparrow \mid b \in O,\ b \text{ isolated}\} \\ &= \bigvee \Diamond\{b\uparrow \mid b \in O,\ b \text{ isolated}\}.\end{aligned}$$

2.4 LEMMA There is a locale map $\varphi: \Sigma(\mathrm{Pl}^+(\mathrm{D})) \to \mathrm{V}(\mathrm{D})$ defined by:
$\varphi^* \Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) = \bigcup\{\mathrm{X}\uparrow_{EM} \mid \mathrm{X} \subseteq \{a_1, \ldots, a_n\}\}$
and $\varphi^* \Diamond b\uparrow = \{\perp, b\}\uparrow_{EM}$.

Note that in defining $\varphi^*$ on elements of the form $\Diamond b\uparrow$, we make use of the fact that D has $\perp$. I believe this to be essential to the proof.

*Proof.* We must check that $\varphi^*$ preserves the defining relations for the Vietoris functor. First we show that if $\alpha \leq \beta$, then $\varphi^*\Box\alpha \leq \varphi^*\Box\beta$.
Without loss of generality we can assume that $\alpha = a_1\uparrow \vee \ldots \vee a_n\uparrow$ and that $\beta = b_1\uparrow \vee \ldots \vee b_k\uparrow$.
From this we know that $\{b_1, \ldots, b_k\} \preceq_0 \{a_1, \ldots, a_n\}$. Now, given $\mathrm{X} \subseteq \{a_1, \ldots, a_n\}$, we have $\{a_1, \ldots, a_n\} \preceq_0 \mathrm{X}$ and hence $\{b_1, \ldots, b_k\} \preceq_0 \mathrm{X}$.
Refining $\{b_1, \ldots, b_k\}$ we obtain $\mathrm{Y} \subseteq \{b_1, \ldots, b_k\}$ such that $\mathrm{Y} \preceq \mathrm{X}$, and so have indeed that

$$\begin{aligned}&\varphi^* \Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \\ &= \bigcup\{\mathrm{X}\uparrow_{EM} \mid \mathrm{X} \subseteq \{a_1, \ldots, a_n\}\} \\ &\leq \bigcup\{\mathrm{Y}\uparrow_{EM} \mid \mathrm{Y} \subseteq \{b_1, \ldots, b_k\}\} \\ &= \varphi^* \Box(b_1\uparrow \vee \ldots \vee b_k\uparrow)\end{aligned}$$

When we come to $\Diamond$ we can assume that $\alpha$ is of the even simpler form $a\uparrow$ and that $\beta$ is similarly $b\uparrow$. Now $a\uparrow \leq b\uparrow$ if and only if $\beta \leq \alpha$, and $X \in \varphi^*(a\uparrow) = \{\perp, a\}\uparrow_{EM}$ if and only if there is an $x$ in $X$ such that $a \leq x$. Hence $\{\perp, a\}\uparrow_{EM} \leq \{\perp, b\}\uparrow_{EM}$, as required.

$\varphi^*$ preserves $\Box$ of directed joins by definition of the extension of $\varphi^*$ to $\mathrm{V}(\Sigma\mathrm{D})$, and from the fact that our basic elements are $\Box$ of quasi-compact opens in D.

We must show that $\varphi^*\Box(\alpha \wedge \beta) \geq \varphi^*\Box\alpha \wedge \varphi^*\Box\beta$
So suppose that $\gamma = \{c_1, \ldots, c_m\}\uparrow_{EM} = \mathrm{C}\uparrow_{EM} \subseteq \mathrm{X}\uparrow_{EM}$ for some $\mathrm{X} \subseteq \{a_1, \ldots, a_n\}$ and that $\mathrm{C}\uparrow_{EM} \subseteq \mathrm{Y}\uparrow_{EM}$ for some $\mathrm{Y} \subseteq \{b_1, \ldots, b_k\}$; then $\mathrm{X} \preceq \mathrm{C}$, and $\mathrm{Y} \preceq \mathrm{C}$, and hence $\mathrm{A} \preceq_0 \mathrm{X} \preceq_0 \mathrm{C}$, $\mathrm{B} \preceq_0 \mathrm{Y} \preceq_0 \mathrm{C}$.
Therefore, by monotonicity, $\Box\gamma \geq \Box(\alpha \wedge \beta)$, and hence $\mathrm{C}\uparrow_{EM}$ is a finite subset of the image of $\Box(\alpha \wedge \beta)$.

To show that $\varphi^*$ preserves the inequality $\Box(\alpha \vee \beta) \leq \Box\alpha \vee \Diamond\beta$, we note first that $\mathrm{C}\uparrow_{EM}$ is in the image of $\Box(\alpha \vee \beta)$ iff for some $\mathrm{X} \cup \mathrm{Y}(\mathrm{X} \subseteq \{a_1, \ldots, a_n\}, \mathrm{Y} \subseteq \{b_1, \ldots, b_k\})$

we have $X \cup Y \preceq C$. Now, if $Y = \phi$ then $C\uparrow_{EM}$ is in the image of $\Box\alpha$. If, on the other hand $Y \neq \phi$, then for some $y \in Y$ we must have $\{\bot, y\} \preceq C$, which implies that $C\uparrow_{EM}$ is contained in the image of $\Diamond\beta$.

Finally, we must show that $\varphi^*$ preserves the inequality $\Box\alpha \wedge \Diamond\beta \leq \Diamond(\alpha \wedge \beta)$. If $C\uparrow_{EM}$ is contained in the image of $\Box\alpha \wedge \Diamond\beta$ then there is both an $X \preceq C$ and a $b_i \in \{b_1, \ldots, b_k\}$ such that $\{\bot, b_i\} \preceq C$. This implies that there is a $c$ in C such that $b_i \leq c$. It follows that $\alpha \wedge \beta \geq c\uparrow$ in A. However, for any $c \in C$ we have $\{\bot, c\} \preceq C$, and putting these together we see that $C\uparrow_{EM} \leq \varphi^*\Diamond(\alpha \wedge \beta)$. $\Box$

To conclude the proof of theorem 2.2 we have to show that the maps $\psi$ and $\varphi$ of lemmas 2.3 and 2.4 are mutual inverses.

In one direction we have

2.5 LEMMA $\psi\varphi$ is the identity on $\Sigma(\mathrm{Pl}^+ \mathrm{D})$.

*Proof.* We look at its effect on basic opens:

$$\begin{aligned}
&\varphi^*\psi^*(\{a_1, \ldots, a_n\}\uparrow_{EM}) \\
&= \varphi^*[\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow)] \wedge \Diamond(a_1\uparrow) \wedge \ldots \wedge \Diamond(a_n\uparrow) \\
&= \bigcup\{X\uparrow_{EM} \mid X \subseteq \{a_1, \ldots, a_n\}\} \;\cap\; \bigwedge\!\!\!\bigwedge\{\bot, a_i\}\uparrow_{EM}.
\end{aligned}$$

Now, suppose $\{a_1, \ldots, a_n\} \preceq Y$, then we have $\{\bot, a_i\} \preceq Y$ for all i.

Conversely, if $X \preceq Y$, for some $X \subseteq \{a_1, \ldots, a_n\}$, then for all $y$ in Y there is an $i$ such that $a_i \leq y$ and so $\{a_1, \ldots, a_n\} \preceq_0 Y$.
On the other hand, $\{\bot, a_i\} \preceq_1 Y$ for all i implies that $\{a_1, \ldots, a_n\} \preceq_1 Y$.
Hence $\{a_1, \ldots, a_n\} \preceq Y$. $\Box$

The converse is only slightly more difficult.

2.6 LEMMA $\varphi\psi$ is the identity on $V(\Sigma D)$.

*Proof.* We have

$$\begin{aligned}
\psi^*\varphi^*(\Diamond a\uparrow) &= \psi^*\{\bot, a\}\uparrow_{EM} \\
&= \Box(\bot\uparrow \vee a\uparrow) \wedge \Diamond(\bot\uparrow) \wedge \Diamond(a\uparrow) \\
&= \Diamond(a\uparrow)
\end{aligned}$$

since $\bot\uparrow = \top$ and by the monotonicity of $\Diamond$.

We also have $\psi^*\varphi^*(\Box a\uparrow) = \psi^*[\phi\uparrow_{EM} \cup \{a\}\uparrow_{EM}] = \Box\phi \vee (\Box(a\uparrow) \wedge \Diamond(a\uparrow))$.
But $\Box(a\uparrow) \geq \Box\phi$ and $\Box(a\uparrow) \wedge \Diamond(a\uparrow) = \Box(a\uparrow) \wedge \Diamond(\bot\uparrow)$, and so this comes out to $\Box(a\uparrow)$.

Now, whereas $\Diamond$ respects finite unions, $\Box$ does not. We must prove the general case for $\Box$ by induction.

$$\begin{aligned}
&\psi^*\varphi^*(\ \Box[a_1\uparrow \vee \ldots \vee a_n\uparrow]\ ) \\
&= \psi^*[\ \bigcup\{X\uparrow \mid X \subseteq \{a_1, \ldots, a_n\}\}\ ] \\
&= \bigvee\!\!\!\bigvee[\Box(a_{i_1}\uparrow \vee \ldots \vee a_{i_r}\uparrow) \wedge \Diamond(a_{i_1}\uparrow) \wedge \ldots \wedge \Diamond(a_{i_r}\uparrow)] \\
&= [\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \wedge \Diamond(a_1\uparrow) \wedge \ldots \wedge \Diamond(a_n\uparrow)] \\
&\qquad \vee \bigvee\!\!\!\bigvee_{\{i_1,\ldots,i_r\} \subset \{1,\ldots,n\}} \Box(a_{i_1}\uparrow \vee \ldots \vee a_{i_r}\uparrow) \\
&= [\ \Box(a_1\uparrow \vee \ldots \vee a_n\uparrow) \vee \bigvee\!\!\!\bigvee \Box(a_{i_1}\uparrow \vee \ldots \vee a_{i_r}\uparrow)\ ] \\
&\qquad \wedge [\ \Diamond(a_1\uparrow) \vee \bigvee\!\!\!\bigvee \Box(a_{i_1}\uparrow \vee \ldots \vee a_{i_r}\uparrow)\ ] \wedge \ldots
\end{aligned}$$

Now for each $\Diamond(a_i\uparrow)$, one of the components of this join is

$$\Box(a_1\uparrow \vee \ldots \vee \widehat{a_i\uparrow} \vee \ldots \vee a_n\uparrow),$$

and we have

$$\Diamond(a_i\uparrow) \vee \Box(a_1\uparrow \vee \ldots \vee \widehat{a_i\uparrow} \vee \ldots \vee a_n\uparrow) \quad \geq \quad \Box(a_1\uparrow \vee \ldots \vee a_n\uparrow).$$

Hence the whole expression is at least $\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow)$.

On the other hand, since it is clearly contained in $\Box(a_1\uparrow \vee \ldots \vee a_n\uparrow)$, we must have equality. $\Box$

This concludes the proof of theorem 2.2.

We obtain theorem 2.1 by corresponding forcing operations in the two locales. The Plotkin powerdomain is obtained from $\mathrm{Pl}^+(\mathrm{D})$ by forcing $\phi$ to be identified with the bottom element. Now, $\varphi^*(\phi\uparrow_{EM}) = \Box\bot$, and so a locale isomorphic to the Scott topology on the Plotkin powerdomain is obtained from the Vietoris locale by forcing $\Box\bot$ to be $\bot$, or in other words, by taking the strict Vietoris locale.

Rather simpler, though essentially similar methods, enable us to give analogous descriptions of the other two powerdomains:

2.7 PROPOSITION

(i) Let $\mathrm{V}_0(A)$ be the locale generated by tokens $\Box a$ subject to the axioms:

(a) if $S$ is directed in $A$ then $\Box \bigvee S = \bigvee \{\Box s \mid s \in S\}$

(b) i. $\Box(a \wedge b) = \Box a \wedge \Box b$

ii. $\Box\top = \top$

(c) $\Box\bot = \bot$.

Then $\mathrm{V}_0(\Sigma\, D) = \Sigma\, \mathrm{P}_0(D)$, the Scott topology on the Smyth powerdomain of $D$.

(ii) Let $\mathrm{V}_1(A)$ be the locale generated by the $\Diamond a$ subject to the axioms:

(a) If $S$ is any subset of $A$ (including $\phi$), then $\Diamond \bigvee S = \bigvee \{\Diamond s \mid s \in S\}$

(b) $\Diamond\top = \top$.

Then $\mathrm{V}_1(\Sigma\, D) = \Sigma\, \mathrm{P}_1(D)$, the Scott topology on the Hoare powerdomain of $D$.

## 3 *Non-deterministic algebras*

In this section we study the recent work of Peter Johnstone on the Vietoris monad and examine it in the context of some results of Plotkin. The results are not in essence new, though the fact that any algebraic semi-lattice carries a Vietoris structure is an extension of results in Johnstone [1985]. The importance of the material is indicated briefly at the end of the section, where the presentation of the join map on the Vietoris locale is used to give proof rules for the extension of a simple imperative language to one including an or-operator.

Since $\Box\bot$ and $\Diamond\top$ are complementary, V(A) decomposes as the disjoint coproduct of $V^+(A)$ and a locale called $V_0(A)$ by Johnstone, and obtained by forcing $\Box\bot = \top$. We apologise for the slight clash of notation.

3.1 LEMMA $V_0(A)$ is $\Omega$, the locale of opens of the one point space (which in our case is singleton $\phi$).

*Proof.* The claim is that $V_0(A)$ is the free locale on no generators. Johnstone observes that $\Box\bot = \top$ forces $\Box a = \top$ and $\Diamond a = \bot$ for all a. But given these, the remaining equations and inequations in the definition of V(A) are satisfied automatically. This means that all the generators are, in fact, redundant. $\Box$

We recall that the product of locales corresponds to the cartesian product of topological spaces, and the sum to disjoint union. We use the notation "$\alpha \times \beta$" to denote the element of the product locale to be thought of as the elementary open rectangle of sides $\alpha$ and $\beta$, and similarly $(\alpha, \beta)$ to represent disjoint union.

Now V(A) carries a semi-lattice structure given by

$$0 : \Omega \longrightarrow V(A)$$

the injection of $V_0$, and

$$\vee : V(A) \times V(A) \xrightarrow{q} V(A+A) \xrightarrow{V(\nabla)} V(A)$$

where q is defined by

$$\begin{aligned} q^*(\Box(a,b)) &= \text{"}\Box a \times \Box b\text{"} \\ q^*(\Diamond(a,b)) &= \text{"}\Diamond a \times 1\text{"} \vee \text{"}1 \times \Diamond b\text{"} \end{aligned}$$

and $\nabla$ is the co-diagonal

$$\nabla^*(a) = (a, a).$$

Since q restricts to a map

$$V^+(A) \times V^+(A) \longrightarrow V^+(A+A)$$

$V^+(A)$ has the structure of a non-deterministic algebra in the sense of Hennessy & Plotkin [1979] (in other words, it is a locale which is equipped with an associative, commutative and idempotent binary operation).

V(A) is the free algebra over A for the Vietoris monad on the category of locales, however, if A is any algebra for this monad (with algebra structure given by $V(A) \xrightarrow{\alpha} A$) then A also carries a natural semi-lattice structure given by

$$p_0 : \Omega \longrightarrow V(A) \xrightarrow{\alpha} A$$

and

$$\sqcup : A \times A \xrightarrow{\eta\times\eta} V(A) \times V(A) \xrightarrow{\vee} V(A) \xrightarrow{\alpha} A$$

where $\eta$ is the unit of the monad, the singleton map. We still have to show that these maps satisfy the algebraic axioms of a semi-lattice. Having done this we can of course use this process to induce another "semi-lattice" structure on V(A), however, when this is done, the new structure is fortunately found to be the same as the first. Thus these operations certainly induce a genuine semi-lattice structure on free algebras. In the case of a general algebra, we use the fact that $V(A) \to A$ is split epi, and the naturality of the definition of the operations given above in order to transfer the semi-lattice axioms from V(A) to A. Exactly the same proof shows that if A is an algebra for the monad induced by $V^+(A)$, then A carries a natural non-deterministic algebra structure.

I shall, rather loosely, call the algebras induced by this monad Lawson.

One of the major results of Johnstone's paper is that if a semi-lattice structure (resp. non-deterministic algebra structure) is induced by a V–algebra structure (resp. $V^+$–algebra structure) on A, then that V–algebra structure (resp. $V^+$–algebra structure) is unique. In other words, the category of V–algebras (resp. $V^+$–algebras) forms a sub–category of the category of localic semi-lattices (resp. non–deterministic algebras).

It is, however, known that this is a strict sub–category (there is a compact Hausdorff topological semi–lattice which is not Lawson).

On the other hand, we have the result of Hennessy and Plotkin (Hennessy & Plotkin [1979], but better presented in Plotkin [1981]) that the power–domain of an algebraic cpo is the initial non–deterministic algebra. The corresponding property for locales follows immediately from this spatial result:
given a map $A \to P$ where A is the Scott topology on an algebraic cpo and P is a non-deterministic algebra, there is a unique factorisation through the spatial co–reflection of P (since A is itself spatial). The Plotkin–Hennessy result now gives a lifting of this to a homomorphism from the power–domain into this spatial co–reflection, and composing with the co–unit of the adjunction gives the required map into P.

There is also a direct proof of this, which makes use of the rather simple localic structure of A.

Put another way, the (localic) Plotkin–Hennessy theorem says that if we restrict to the category of locales which are Scott topologies of algebraic cpo's, then the free localic semi-lattices are precisely the free Lawson semi–lattices. This has the immediate consequence that the two associated monads are the same, and hence that any algebraic (localic) semi-lattice has a unique algebra structure for the Vietoris monad.

To give an example of the use of this we consider the addition of an <u>or</u> combinator to a simple imperative language. We suppose that we are given an algebra *Comm* of commands, and we freely extend this to an algebra *Comm'* over a signature enlarged by the addition of <u>or</u> as an extra binary operator. We suppose the original language has been given a semantics by imposing some algebra structure on $State \to State$, and giving a homomorphism $\Phi : Comm \to (State \to State)$. We shall consider what are essentially Hoare-style proof rules for the semantics, which we shall write

$$c, p \models q.$$

Here $c$ is a command, $p$ is an assertion about the state (equals Scott open subset of states) at the beginning of the execution of the command, and $q$ is an assertion which is intended to hold on completion of execution. Proof rules can be divided into two classes, those concerned with describing the semantic domain (*State* $\to$ *State*), and those concerned with giving a presentation of the map $\Phi$.

When we adjoin <u>or</u> we extend $\Phi$ to a map

$$\Phi' : Comm' \to \mathrm{Hom}(Pl(States), Pl(States)),$$

where Hom is the semi-lattice homomorphisms. $\mathrm{Hom}(Pl(States), Pl(States))$ is of course equivalent to $(States \to Pl(States))$. Since, given $f : X \to Y$, $PL(f)$ is a semi-lattice homomorphism $Pl(X) \to \mathrm{Pl}(\mathrm{Y})$, it is easy to extend the proof rules dealing with the operators of *Comm* to the new semantics. We only have to cope with <u>or</u>, which is of interpreted via the join operation in the semi-lattice. Using $[\![\mathrm{c}\ \underline{\mathrm{or}}\ \mathrm{c}']\!] = \vee \circ ([\![\mathrm{c}]\!] \times [\![\mathrm{c}']\!]) \circ \Delta$, and the expression given above for $\vee$, we see immediately that the rules we need are:

$$\frac{\mathrm{c}, \alpha \models \Box\beta \quad \mathrm{c}', \alpha \models \Box\beta}{\mathrm{c}\ \underline{\mathrm{or}}\ \mathrm{c}', \alpha \models \Box\beta} \qquad \frac{\mathrm{c}, \alpha \models \Diamond\beta}{\mathrm{c}\ \underline{\mathrm{or}}\ \mathrm{c}', \alpha \models \Diamond\beta}$$

and the similar symmetric rule for $\Diamond$.

## 4 *The modal approach of Winskel*

The approach of Winskel [1985] is founded on the notion of a non-deterministic D-computation, where D is an algebraic domain. These are finitely-branching trees whose nodes are labelled by elements of $\mathcal{B}[\mathrm{D}]$ in such a way that if a node $t'$ is a successor of the node $t$, then $val\, t' \sqsubseteq val\, t$.

Winskel also introduces a small modal language which has as basic propositions the elements $a$ of $\mathcal{B}[\mathrm{D}]$, together with the modal operators $\Box$ and $\Diamond$, and $\vee$ as the sole connective. This is interpreted via Kripke forcing; the relation $t \Vdash p$ is defined inductively as follows:

$$\begin{aligned} t \Vdash a \quad & \text{iff} \quad a \sqsubseteq val\, t \\ t \Vdash p \vee q \quad & \text{iff} \quad t \Vdash p \text{ or } t \Vdash q \\ t \Vdash \Diamond p \quad & \text{iff} \quad t \Vdash p \text{ or for some } t', \text{ successor of } tt' \Vdash \Diamond p \\ t \Vdash \Box p \quad & \text{iff} \quad t \Vdash p \text{ or for all } t', \text{ successor of } tt' \Vdash \Box p \end{aligned}$$

We shall write $T \Vdash p$ for $root_T \Vdash p$.

The essential definition is now

$$V(T) = \{\, \Box s \mid T \Vdash \Box s \,\},$$

and we note that if $T$ is infinite, then $V(T) = \bigcup V(T_n)$, where $T_n$ is the restriction of $T$ to the first n levels.

Write now $p \longleftrightarrow p'$ if for all $T$, $T \Vdash p$ iff $T \Vdash p'$. Winskel produces axioms to show that any $\Box s$ is equivalent to an expression of the form

$$\Box(a_0, \ldots, a_n) \vee \Diamond b_0 \vee \ldots \vee \Diamond b_m.$$

Using the fact that $T \Vdash p \vee p'$ iff $T \Vdash p$ or $T \Vdash p'$, we restrict our attention to the expressions $\Box(a_0, \ldots, a_n)$ and $\Diamond b$.

Now for finite $T$, $T \Vdash \Box(a_0, \ldots, a_n)$ iff $\{a_0, \ldots, a_n\} \preceq_0$ *leaves*$(T)$, and $T \Vdash \Diamond b$ iff $\{b\} \preceq_1$ *leaves*$(T)$, which implies that for finite trees $T_1$ and $T_2$, $V(T_1) \subseteq V(T_2)$ iff *leaves*$(T_1) \preceq$ *leaves*$(T_2)$ in the Egli-Milner ordering.

We now use this to induce a pre-order $\preceq$ on the finite non-deterministic D-computations and observe that if $T_1$ is the full subtree of $T_2$ up to a given level, then $T_1 \preceq T_2$. Furthermore, if $T_1 \preceq T_2$, then we can find a tree containing $T_1$ as a full subtree and congruent to $T_2$ in the pre-order. This tells us that if $\mathbf{T}$ is the collection of all non-deterministic D-computations, then we can extend $\preceq$ to $\mathbf{T}$ in such a way that $\mathbf{T}$ becomes the algebraic completion of the poset of finite computations.

Winskel's description of the power-domain now follows immediately:

4.1 THEOREM Let D be an algebraic domain. Then the Plotkin power-domain Pl(D) is isomorphic as partially-ordered set to the set $(\{V(T) \mid T \in \mathbf{T}\}, \subseteq)$, and as pre-ordered set to $\mathbf{T}$ with the continuous extension of $\preceq$.

This relates to the approach given above via the sets $V(T)$. For a given proposition $\Box s$, $\{V(T) \mid \Box s \in V(T)\}$ is a Scott open in the powerdomain, and is of course the one corresponding to the element $\Box s$ of $V^+(\Sigma D)$. Recalling that if $\mathcal{F}$ is any filter then $a \wedge b$ is in $\mathcal{F}$ iff both $a$ and $b$ are, we see that each $V(T)$ gives a presentation of a prime filter on $V^+(\Sigma D)$ (in fact completely prime, due to compactness), and we can read off various implications between Winskel's modal expressions from this.

This technique of course works only when we already know an equivalent of the modal expression about which we are concerned in which the modal operators are nested one deep (and there is perhaps no real need to consider formulae of greater complexity when one is interested only in simple powerdomains). Winskel, however, does provide for iteration of his modal operators, an essential difference between his approach and ours, and so has to introduce further rules

$$\begin{gathered}\Diamond(\Diamond a) \dashv\vdash \Diamond(\Box a) \dashv\vdash \Box(\Diamond a) \dashv\vdash \Diamond a;\\ \Box(\Box a) \dashv\vdash \Box a;\\ \Box(a \vee (\Box b)) \dashv\vdash \Box(a \vee b);\\ \Box(a \vee (\Diamond b)) \dashv\vdash (\Box a) \vee (\Diamond b).\end{gathered}$$

He then states a normal form theorem which reduces him to the case where modalities are nested at most one deep. Since the Vietoris construction uses a language which includes conjunction, we must at least add the absorption law for $\Diamond$:

$$\Diamond(a \wedge (\Diamond b)) \dashv\vdash \Diamond(a \wedge b).$$

In our set-up, we note that the left-hand sides of all these axioms live naturally in $V(A \times V(A))$ (or $V^+(A \times V^+(A))$). It would be interesting to show that the rules present $V^+(A)$ as a closed sublocale (in fact a retract) of $V^+(A \times V^+(A))$. This calculation, however, at the moment defeats me. It would also be interesting to extend this to find out the connection with the full locale of all modal propositions.

As a closing remark, however, let me note that the use of iterated operators in such contexts as Hennessy-Milner logic (where it is essential) seems to arise from the quite different consideration of taking solutions of recursive domain equations, such as

$$D \longleftrightarrow State + Pl(D^{labels}).$$

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