The Stratified Shortest-Paths Problem (Invited Paper)

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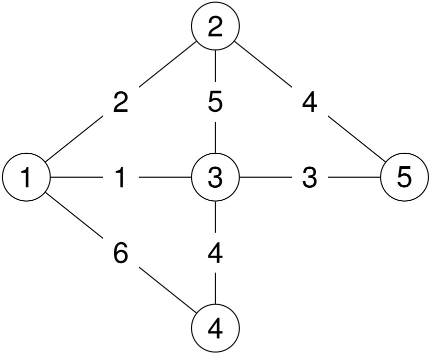
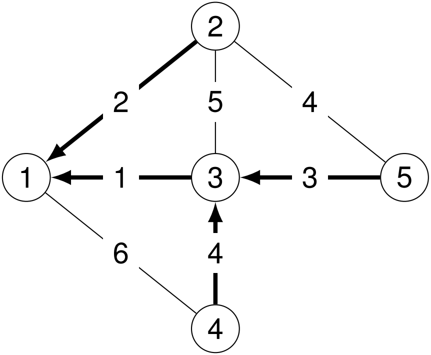
Shortest paths example, $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{bmatrix} \end{array}$$

Shortest paths example, continued



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Semirings (see [Car79, GM84, GM08])

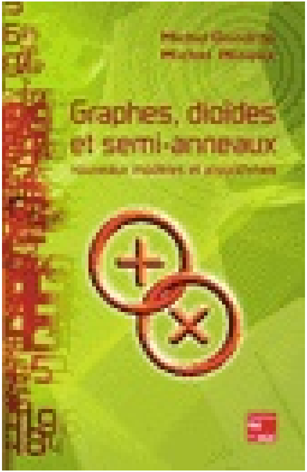
A few semirings

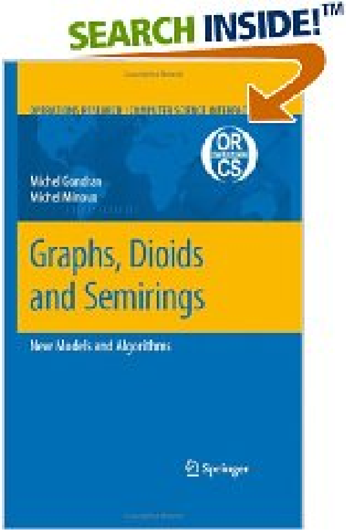
name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^{W}	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^{W}	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

A wee bit of notation!

Symbol	Interpretation
$\mathbb{N}$	Natural numbers (starting with zero)
$\mathbb{N}^{\infty}$	Natural numbers, plus infinity
$\overline{0}$	Identity for $\oplus$
$\overline{1}$	Identity for $\otimes$

Recomended Reading





What algebraic properties are associated with global optimality?

Distributivity

$$\begin{array}{rcrcl} \mathrm{L.D} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c), \\ \mathrm{R.D} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c). \end{array}$$

What is this in $\mathrm{sp} = (\mathbb{N}^{\infty},\ \text{min},\ +)$?

$$\begin{array}{rcrcl} \mathrm{L.DIST} & : & a\ +\ (b\ \text{min}\ c) & = & (a\ +\ b)\ \text{min}\ (a\ +\ c), \\ \mathrm{R.DIST} & : & (a\ \text{min}\ b)\ +\ c & = & (a\ +\ c)\ \text{min}\ (b\ +\ c). \end{array}$$

Local Optimality?

Say that $\mathbf{R}$ is a locally optimal solution when

$$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{R}(q,\ j) = \bigoplus_{q \in N(i)} w(i,\ q) \otimes \mathbf{R}(q,\ j),$$

where $N(i) = \{q \mid (i,\ q) \in E\}$ is the set of neighbors of i.

In other words, $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all neighbors q of i.

With Distributivity

$\mathbf{A}$ is an adjacency matrix over semiring S.

For Semirings, the following two problems are essentially the same — locally optimal solutions are globally optimal solutions.

Global Optimality	Local Optimality
Find $\mathbf{R}$ such that	Find $\mathbf{R}$ such that
$\mathbf{R}(i,\ j) = \sum^{\oplus}_{p \in P(i,\ j)} w(p)$	$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$

Without Distributivity

When $\otimes$ does not distribute over $\oplus$, the following two problems are distinct.

Global Optimality	Local Optimality
Find $\mathbf{R}$ such that $\mathbf{R}(i,\ j) = \sum_{p \in P(i,\ j)}^{\oplus} w(p)$	Find $\mathbf{R}$ such that $\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$

Global Optimality

This has been studied, for example [LT91b, LT91a] in the context of circuit layout. I do not know of any application of this problem to network routing. (Yet!)

Local Optimality

At a very high level, this is the type of problem that BGP attempts to solve!!

From $(S, \oplus, \otimes, \overline{0}, \overline{1})$ to $(S, \oplus, F, \overline{0}, \overline{1})$

- Replace $\otimes$ with $F \subseteq S \to S$,
- Replace

$$\text{L.D} \quad : \quad a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

with

$$\text{D} \quad : \quad f(b \oplus c) \quad = \quad f(b) \oplus f(c)$$

- Path weight is now

$$\begin{aligned} w(p) &= w(v_0, v_1)(w(v_1, v_2) \cdots (w(v_{k-1}, v_k)(\overline{1}) \cdots)) \\ &= (w(v_0, v_1) \circ w(v_1, v_2) \circ \cdots \circ w(v_{k-1}, v_k))(\overline{1}) \end{aligned}$$

What are the conditions needed to guarantee existence of local optima?

For a non-distributed structure $S = (S,\ \oplus,\ F,\ \overline{0},\ \overline{1})$, can be used to find local optima when the following property holds.

Strictly Inflationary

$$\text{S.INFL} : \forall a \in S,\ F \in F \ :\ a \neq \overline{0} \implies a < f(a)$$

where $a \leq b$ means $a = a \oplus b$.

Important properties for algebraic structures of the form $(S,\ \oplus,\ F,\ \overline{0},\ \overline{1})$

property	definition
D	$\forall a, b \in S,\ f \in F\ :\ f(a \oplus b) = f(a) \oplus f(b)$
INFL	$\forall a \in S,\ f \in F\ :\ a \leq f(a)$
S.INFL	$\forall a \in S,\ F \in F\ :\ a \neq \overline{0} \implies a < f(a)$
K	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies a = b$
$K_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies (a = b \vee f(a) = \overline{0})$
C	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b)$
$C_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) \neq f(b) \implies (f(a) = \overline{0} \vee f(b) = \overline{0})$

Stratified Shortest-Paths Metrics

Metrics

$$(s,\ d)\ \text{ or }\ \infty$$

- $s \neq \infty$ is a *stratum level* in $\{0, 1,\ 2, \ldots,\ m - 1\}$,
- d is a "shortest-paths" distance,
- Routing metrics are compared lexicographically

$$(s_1,\ d_1) < (s_2,\ d_2) \iff (s_1 < s_2) \vee (s_1 = s_2 \wedge d_1 < d_2)$$

Stratified Shortest-Paths Policies

Policy has form $(f,\ d)$

$$\begin{array}{rcl} (f,\ d)(s,\ d') & = & \langle f(s),\ d + d' \rangle \\ (f,\ d)(\infty) & = & \infty \end{array}$$

where

$$\langle s,\ t \rangle = \begin{cases} \infty & (\text{if } s = \infty) \\ (s,\ t) & (\text{otherwise}) \end{cases}$$

Constraint on Policies

$$(f,\ d)$$

- Either f is inflationary and $0 < d$,
- or f is strictly inflationary and $0 \leq d$.

Why?

$$(\text{S.INFL}(S) \vee (\text{INFL}(S) \wedge \text{S.INFL}(T))) \implies \text{S.INFL}(S \vec{\times}_{\overline{0}} T).$$

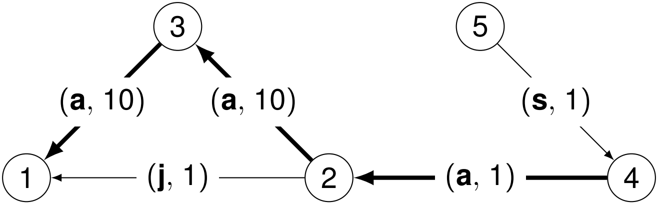
All Inflationary Policy Functions for Three Strata

	0	1	2	D	K_∞	C_∞		0	1	2	D	K_∞	C_∞
a	0	1	2	$\star$	$\star$		**m**	2	1	2			
b	0	1	∞	$\star$	$\star$		**n**	2	1	∞		$\star$	
c	0	2	2	$\star$			**o**	2	2	2	$\star$		$\star$
d	0	2	∞	$\star$	$\star$		**p**	2	2	∞	$\star$		$\star$
e	0	∞	2		$\star$		**q**	2	∞	2			$\star$
f	0	∞	∞	$\star$	$\star$	$\star$	**r**	2	∞	∞	$\star$	$\star$	$\star$
g	1	1	2	$\star$			**s**	∞	1	2		$\star$	
h	1	1	∞	$\star$		$\star$	**t**	∞	1	∞		$\star$	$\star$
i	1	2	2	$\star$			**u**	∞	2	2			$\star$
j	1	2	∞	$\star$	$\star$		**v**	∞	2	∞		$\star$	$\star$
k	1	∞	2		$\star$		**w**	∞	∞	2		$\star$	$\star$
l	1	∞	∞	$\star$	$\star$	$\star$	**x**	∞	∞	∞	$\star$	$\star$	$\star$

Almost shortest paths

	0	1	2	D	K_∞	interpretation
a	0	1	2	$\star$	$\star$	$+0$
j	1	2	∞	$\star$	$\star$	$+1$
r	2	∞	∞	$\star$	$\star$	$+2$
x	∞	∞	∞	$\star$	$\star$	$+3$
b	0	1	∞	$\star$	$\star$	filter 2
e	0	∞	2		$\star$	filter 1
f	0	∞	∞	$\star$	$\star$	filter 1, 2
s	∞	1	2		$\star$	filter 0
t	∞	1	∞		$\star$	filter 0, 2
w	∞	∞	2		$\star$	filter 0, 1

Shortest paths with filters, over $\mathbf{INF}_3$



Note that the path $5, 4, 2, 1$ with weight $(1,\ 3)$ would be the globally best path from node 5 to node 1. But in this case, poor node 5 is left with no path! The locally optimal solution has $\mathbf{R}(5,\ 1) = \infty$.

Both D and $K_{\overline{0}}$

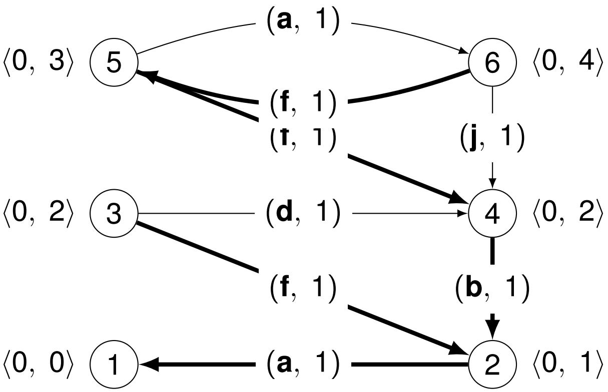
This makes combined algebra distributive!

	0	1	2
a	0	1	2
b	0	1	∞
d	0	2	∞
f	0	∞	∞
j	1	2	∞
l	1	∞	∞
r	2	∞	∞
x	∞	∞	∞

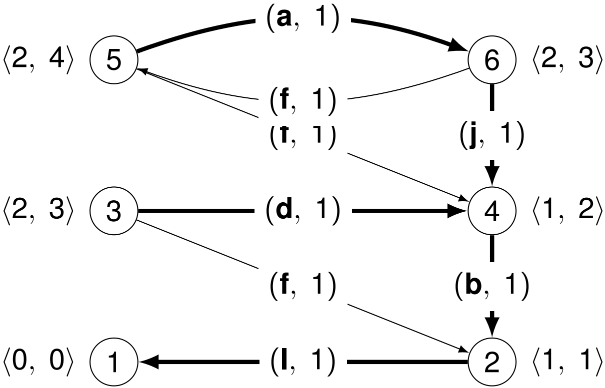
Why?

$$(\mathrm{D}(S) \wedge \mathrm{D}(T) \wedge \mathrm{K}_{\overline{0}}(S)) \implies \mathrm{D}(S \,\vec{\times}_{\overline{0}}\, T)$$

Example 1



Example 2



BGP : standard view

- 0 is the type of a *downstream* route,
- 1 is the type of a *peer* route, and
- 2 is the type of an *upstream* route.

	0	1	2
f	0	∞	∞
l	1	∞	∞
o	2	2	2

“Autonomous” policies

	0	1	2	D	K_∞
f	0	∞	∞	$\star$	$\star$
h	1	1	∞	$\star$	
l	1	∞	∞	$\star$	$\star$
o	2	2	2	$\star$	
p	2	2	∞	$\star$	
q	2	∞	2		
r	2	∞	∞	$\star$	$\star$
t	∞	1	∞		$\star$
u	∞	2	2		
v	∞	2	∞		$\star$
w	∞	∞	2		$\star$
x	∞	∞	∞	$\star$	$\star$

Open Problems

- Complexity
- Applications

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