Denotational Semantics Supervision 1

Ian Orton (rio22)
Based on an exercise sheet by Ohad Kammar (ok259)

1. Solve exercise 2.5.1
2. Solve exercise 2.5.2

3. We say that a chain, \( x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots \), is \emph{eventually constant} if there exists a natural number \( k \) such that for all natural numbers \( n \geq k \), \( x_n = x_k \).
   (a) Show that every eventually constant chain has a lub.
   (b) Hence, explain why every finite poset is a CPO.
   (c) Show that every monotone function preserves lubs of eventually constant chains.
   (d) Deduce the following result: Let \( D, E \) be CPOs such that all chains in \( D \) are eventually constant. Show that all monotone functions \( f : D \to E \) are continuous.

4. Solve exercise 2.5.3

5. Let \( D, D' \) be domains. We say that a function \( f : D \to D' \) is a \emph{continuous isomorphism} if it is continuous, bijective, and its inverse \( f^{-1} : D' \to D \) is also continuous.
   (a) Show that if \( f \) is continuous and bijective, and \( f^{-1} \) is monotone, then \( f \) is a continuous isomorphism.
   (b) Find an example for a continuous and bijective \( f \) that is not a continuous isomorphism.

6. (Due to Meseguer) Let \( A \triangleq \{ a_0, a_1, \ldots \} \), \( B \triangleq \{ b_0, b_1, \ldots \} \) and \( \{ \infty \} \) be pairwise disjoint sets. Define a binary relation \( \sqsubseteq \) over \( D \triangleq A \cup B \cup \{ \infty \} \) by \( x \sqsubseteq y \) if and only if:
   - \( y = \infty \), or
   - \( x = y = b_n \), or
   - \( x = a_m \) and \( y = b_n \), for some \( m \leq n \), or
   - \( x = a_m \) and \( y = a_n \), for some \( m \leq n \).
   (a) Draw a Hasse diagram for \( (D, \sqsubseteq) \).
   (b) Show that \( (D, \sqsubseteq) \) is a domain. (Least upper bounds may be stated without proof).
   (c) Let \( (E, \sqsubseteq_E) \) be any domain, and \( f, g : D \to E \) two continuous functions. Show that if, for all \( n \), \( f(b_n) = g(b_n) \), then also \( f(\infty) = g(\infty) \).

7. Question 6, Paper 7, 2014

8. (Optional extension) Question 6, Paper 7, 2015