Skew monoidal structures on categories of algebras

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11th July 2018

Skew monoidal categories

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Recently studied in some detail: Uustalu (2014), Andrianopoulos (2017), — MFPS paper, Bourke & Lack (2017, 2018), Lack and Street (2014) ...

Captures some old examples (Alternkirch 2010) and can be better behaved than the monoidal case (Street 2013) monoidal

$\mathbb{T}$ monoidal

reflexive coequalizers in T + preservation conditions

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Our contribution: universal description as a list object with algebraic structure.

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Abstract syntax = free such structure

= a list object with algebraic structure.

A unifying framework for many diverse examples of list objects with algebraic structure

- Notions of natural numbers in domain theory,
- The monadic list transformer,
- Abstract syntax with binding and metavariables,
- Algebraic operations,
- Instances of the Haskell MonadPlus type class,
- Higher-dimensional algebra.

This talk

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- extends datatype of lists

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- described by $A.(I + XA)$.

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Gives a *concrete* way to reason about free T-monoids.

Gives an algebraic structure for T-list objects.

Past work: list objects in CCCs (Joyal, Cockett)

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that is initial: given any $(1AX \times A)$, there exists a unique iterator

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & (X) \arrow[d, dashed, "\mathrm{it}(,)"] & X \times (X) \arrow[l] \arrow[d, "X \times \mathrm{it}(n,c)"] \\
1 \arrow[r] & A & X \times A \arrow[l]
\end{tikzcd}$$

List objects in a monoidal category $(, ,)$

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$$\begin{tikzcd}
P \arrow[r, "P"] \arrow[d] & (X)P \arrow[d, dashed, "\mathrm{it}(,)"] & X(X)P \arrow[l, "P"'] \arrow[d, "X\mathrm{it}(n{,}c)"] \\
P \arrow[r] & A & XA \arrow[l]
\end{tikzcd}$$

List objects in a monoidal category (, ,)

Remark

If each $(-)P$ has a right adjoint, parametrised initiality is equivalent to the non-parametrised version:

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, equal] & (X) \arrow[d, dashed, "\mathrm{it}(,)"] & X(X) \arrow[l] \arrow[d, "X\mathrm{it}(n,c)"] \\
{} \arrow[r] & A^P & XA^P \arrow[l]
\end{tikzcd}$$

List objects in a monoidal category (, ,)

Connection to past work

- Closely connected to Kelly's notion of algebraically-free monoid in a monoidal category.
- The list object () is precisely a left natural numbers object in the sense of Paré and Román. *E.g.* the flat natural numbers $A.(1 + A)$ in **Cpo**.

List objects are free monoids

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Definition

A monoid in a monoidal category (, ,) is an object () such that the multiplication is associative and is a neutral element for this multiplication.

List objects are free monoids

Lemma

1. *Every list object (X) is a monoid.*

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2. *This monoid is the free monoid on* X*, with universal map*

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taking $x \mapsto (x, *) \mapsto (x, []) \mapsto x :: [] = [x]$.

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We can reason concretely about free monoids by reasoning about lists.

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Lemma

If $(,,)$ *is a monoidal category with finite coproducts* $(0, +)$ *and* ω*-colimits, both preserved by all* $(-)P$ *for* $P \in$*, then the initial algebra of the functor* $\big(+ X(-)\big)$ *is a list object on* X*.*

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Remark

This result relies on a general theory of parametrised initial algebras.

The story so far

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Rest of this talk

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...and instantiate this for applications

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Definition

A strong monad T is a monad on a monoidal category $(,)$ that is equipped with a natural transformation $_{A,B} : T(A)B \to T(AB)$ satisfying coherence laws.

List objects with algebraic structure

T-list objects

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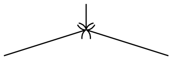
such that for every structure



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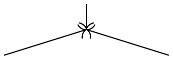

there exists a unique mediating map $(,,):() \rightarrow$

T-list objects

such that

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, "\cong"'] & () & () \arrow[l] \arrow[d, "(,,)"] \\
{} \arrow[ur, no head] & & {} \arrow[ul, "(,,)"']
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and

$$\begin{tikzcd}
(()) \arrow[r, "(),"] \arrow[d] & (()) \arrow[r, "((,,))"] & {} \arrow[dll] \\
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T-list objects

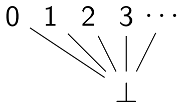
Remark

Every list object is a T-list object.

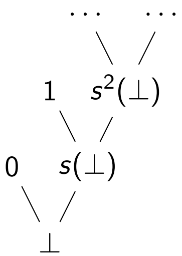
If every $(-)P$ has a right adjoint, the iterator $(,,)$ is a T-algebra homomorphism.

Natural numbers in **Cpo**, revisited

Flat natural numbers, $A.(1 + A)$:



Lazy natural numbers, $A.(1 + A)_\perp$:



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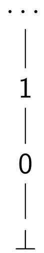

Natural numbers in **Cpo** as T-list objects on the unit

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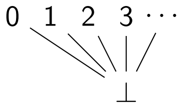

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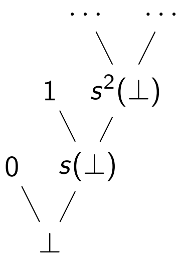

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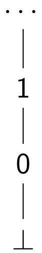

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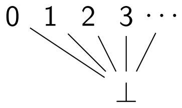

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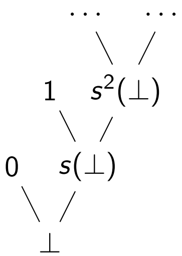

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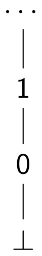

T-list object with $(\times, 1)$ structure and monad $T =$

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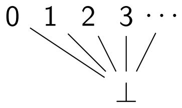

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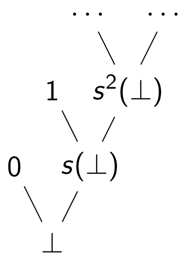

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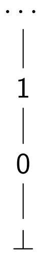

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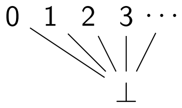

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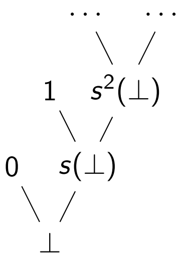

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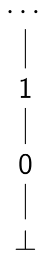

T-list object with $(+, 0)$ structure and $T := (-)_\perp$ the lifting monad

Monoids with compatible algebraic structure

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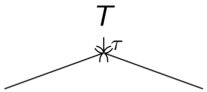

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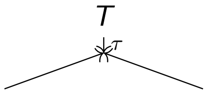

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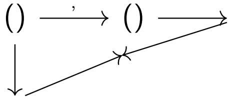

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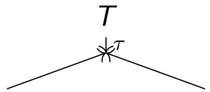

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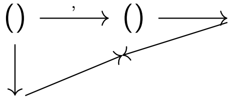

Remark

T-monoids generalise both monoids and T-algebras.

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Lemma

For every monoid the endofunctor $T := (-)$ *is a monad, and* $T \simeq ()$.

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Example

In particular, a T-monoid for the endofunctor $T := S(-)$ is precisely an algebraic operation with signature S in the sense of Jaskelioff, and can be identified with a map $S\eta(S) \to$ interpreting S inside .

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Example

Thinking of a Lawvere theory as a monoid L in $\big('(1), \bullet\big)$, we can identify Lawvere theories extending L with T-monoids for $T := \bullet(-)$.

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We can reason concretely about free T-monoids by reasoning about T-lists.

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For a strong monad $(T,)$ on a monoidal category $(,)$,

Lemma

If every $(-)P$ preserves binary coproducts, and the initial algebra exists, then $A.T(I + XA)$ is a T-list object on X.

Theorem

Let be a strong monad on a monoidal category $(,,)$ with binary coproducts $(+)$. If

1. *for every $\in$, the endofunctor $(-)$ preserves binary coproducts, and*
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Remark

Thinking in terms of T-list objects makes the proof straightforward!

Technical contribution

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T-list object

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T-list object $\rightsquigarrow$ free T-monoid

Technical contribution

$$A.(I + XA) \rightsquigarrow \text{list object} \rightsquigarrow \text{free monoid}$$

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Remark

A natural extension: algebraic structure encapsulated by Lawvere theories or operads. This gives rise to a notion of near-semiring category, which underlies many of the applications.

Applications

Applications

T-NNOs

In a a monoidal category $(,)$:

$$\text{NNO} = \text{list object on}$$
$$T\text{-NNO} = T\text{-list object on}$$

In **Cpo**: gives rise to the *flat*-, *lazy*- and *strict* natural numbers.

Applications

Functional programming

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- In the category of endofunctors over a cartesian category: the $\mathrm{MonadPlus}$ type class $\mathrm{Mp}(F)X := A.\mathrm{List}(X + FA)$ of Rivas is a List-list object.
- In the category of endofunctors over a cartesian category: the datatype

$$\mathrm{Bun}(F)X := A.(1 + X \times A + F(A) \times A + A \times A)$$

is an instance of Spivey's Bunch type class that is a T-list object for T the extension of the theory of monoids with a unary operator.

Applications

Functional programming

- In the bicartesian closed setting: Jaskelioff's monadic list transformer $\mathrm{Lt}(T)X := A.T(1 + X \times A)$ is just the free T-monoid monad.
- In an nsr-category: the $\mathrm{MonadPlus}$ type class $\mathrm{Mp}(F)X := A.\mathrm{List}_*(X + FA)$ is a List_*-list object.
- In an nsr-category:

$$\mathrm{Bun}(F)X := A.\big(J + (I + XA + A) * A\big)$$

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Applications

Abstract syntax and variable binding (Fiore)

In the category of presheaves with substitution tensor product

$$(P \bullet Q)(n) = \int^{m\in} (Pm) \times (Qn)^m$$

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abstract syntax is a list object with algebraic structure

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Remark

This relies on a slightly more general theory, in which the strength $_{X,I \to P} : T(X)P \to T(XP)$ only acts on pointed objects.

Applications

Higher-dimensional algebra

The web monoid in Szawiel and Zawadowski's construction of opetopes is a T-list object in an nsr-category.

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A journal-length version is in preparation.